**Linear Systems Theory**
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**Module - 06**
**Lecture - 02**
# Stability Proofs

Hi everyone, my name is Ramkrishna from IIT madras. And welcome to this lecture series of week 6 on Linear Systems Theory. So, last lecture we had defined for ourselves few concepts of stability starting with just the notion of a of a stable system where the solutions were uniformly bounded, then you had the notion of asymptotic stability which did not tell me about the rate of convergence of my solution to the equilibrium point, and then we had the exponential stability notion which had some information on how fast my solution approaches the origin and of course, unstable systems the systems which are not stable or unstable systems.

In many of these cases we were talking of solutions converging to the equilibrium point. And in many cases, it may be difficult to compute the solutions explicitly and check whether they are stable or asymptotically stable or not right. Even though we have the beautiful notion of the state transition matrix and computations involving status transition matrix which can give me the expression to the solution explicitly. Now, are there better ways of verifying this, loosely speaking can we find an analogous to the stability verification via the location of poles in when I when I look in terms of a transfer function representation of a system, so that will be the focus of today's lecture. And we will also look at how to not only have conditions to verify stability, but how do we have how do we even prove some notions of stability.

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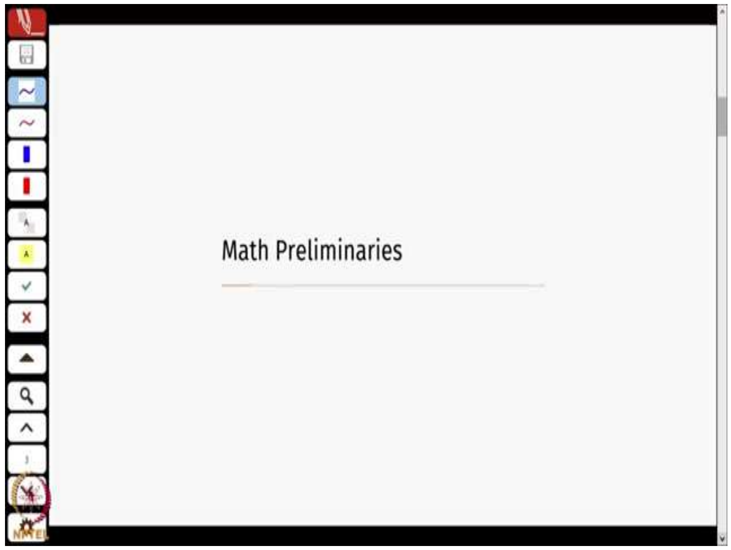


So, so to begin with let us start with a bit of maths preliminaries because we will be dealing a lot with matrices here right.

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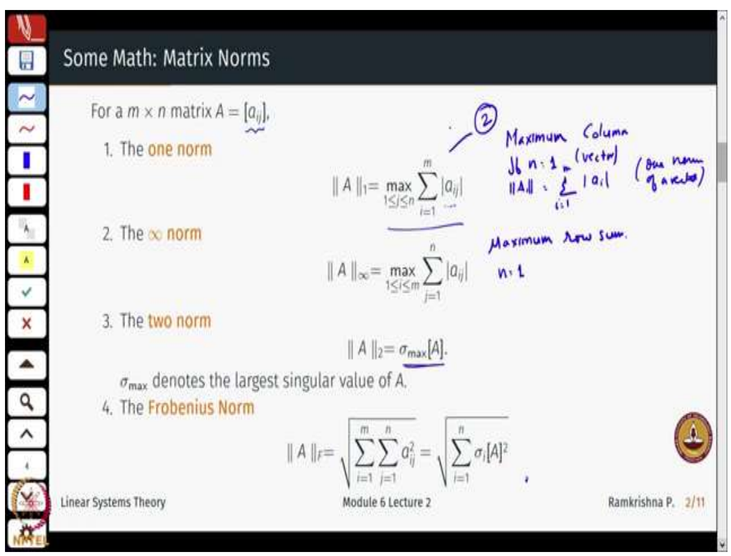


So, I would like to define some notions of norm on matrices. So, just for any m x n matrix A each element of a has this form $a_{ij}$, the one norm is defined in this way. So, I am looking at like in this would could be the summation over the columns, and I am looking at the maximum column sum of a matrix.

So, in if n = 1, I realized the standard vector k. So, I have a vector now instead of a matrix, and the one norm would simply be the summation of all the elements of a, say i = 1 to m like the standard definition of a one norm of a vector ok. Similarly, with the infinity norm, here I am looking at the maximum row sum ok. And similarly I can derive that that for n equal to 1 case, I will have a vector, and this definition will coincide with the definition of the infinity norm for a in case of a vector ok.

Next is the notion of a two norm. So, the two norm is defined in the following way. The two norm of A is just the maximum singular value of this matrix A. And then last notion is that of Forbenius norm defined this way ok. We will not use much of this in this course, but just in just nice to know some, some definitions ok.

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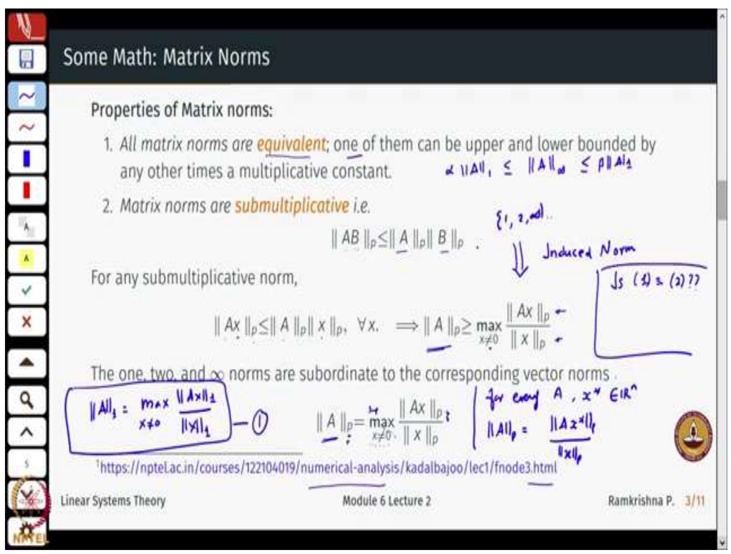


So, what are the properties of norms defined on vector space that will translate to norms on matrix? Well one thing is obvious that all norms are equivalent, like we proved in the in the in the vector space case. What does it mean by equivalence at one of them can be upper and lower bounded by any other times a multiplicative constant? Sy the one norm can be upper bounded, and lower bounded by the infinity norm with some numbers $\alpha$ and $\beta$ here ok.

Another important property is the sub multiplicative property right. So, if I have two matrices A and B, the sub multiplicative property says that the $||AB||_p \leq ||A||_p||B||_p$, p could be whatever p could be. The one norm the two norm the infinity norm and so on ok.

So, from the definition of the sub multiplicative norm or the property of sub multiplicativity, we can define ok.

What is also in literature if usually referred to as the induced norm ok? So I will quickly run you through what this could be in. So, for any sub multiplicative norm, so here A is a matrix, x is a vector. So, $||Ax||_p$ could be 1, 2 or infinity is less than or equal to the $||A||_p||x||_p$ for all x. This would imply that I can write it rewrite it this way that $||A||_p$ $\geq x \neq 0, \frac{||Ax^*||_p}{||x||_p}$ ok.

A slight little definition here would be or a property would be that this 1, 2 and infinity norms are subordinate to the corresponding vector norms. What does this mean that we have $||A||_p$ is actually equal to the maximum or also refer to this as a supremum sum over all the x. So, I just check for all the x and find what the supremum of this ratio right $\frac{||Ax||_p}{||x||_p}$ ok, this is a little, little typo here ok.

So, where does this come? So, this equality arises from the fact that this sub from the fact that these norms have the property that for every matrix A, I will write it properly. For every matrix A, there exists a vector $x^*$ in $R^n$ for which $||A||_p = \frac{||Ax^*||_p}{||x||_p}$ ok, in p could be 1, 2, and the and the and the infinity norms ok.

So, if I just say just take the case of a one norm, and this would be the maximum of over $x \neq 0, \frac{||Ax||_1}{||x||_1}$ right. So, this, this is what I would get when I just look at the one norm starting from the definition of an induced norm. Now, is this norm which I call it 1, label it as 1 equal to the one norm that I defined here right. So, if I if I go back, so the question is I just write it over here is 1 equal to 2 ok. The, the answer is yes it is a it is a little proof not very complicated. So, I will skip the proof, but you can just follow this link of another nptel lecture on numerical analysis.

So, these notes will give you A very nice exposition to the proof of this. Similarly, you can replace the 1 by 2 here. I am just call this p to be 2, and you can prove that this will actually give you the two norm and so on with the with the infinity norm also right. So, there is a good appearance between the definitions of the one the infinity and the two norm.

And what I derive here why are these sub multiplicative property ok. So, of all these properties, this is what we will be using in this in this lecture.

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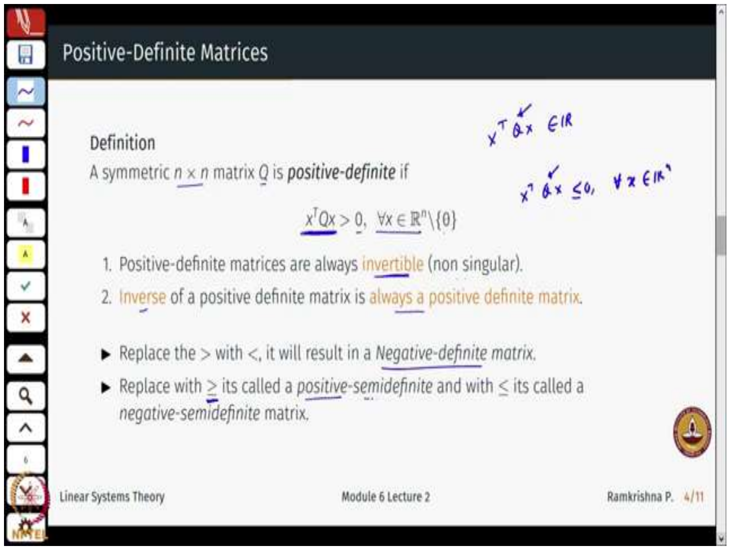


Now, a little to do with the definitions or some properties of a positive definite matrices ok. What is the definition I just take an n x n matrix Q is positive-definite if $x^T$Q x > 0 for all x except the origin or except does the 0 vector all right ok. Now, again how do I test if the matrix is positive-definite or not, well if I go by definitions I may have to check all possible vectors which are in $R^n$, and verify this condition that $x^T$Q x should be greater than 0 ok. So, this $x^T$Q x will be in R that that can be checked or observed quite easily right.

But I would not want to do this for all x and take forever to check if a if a little even a 2 x 2 matrix is positive-definite or not ok. So, before we look at what are the ways to define, what other ways to check if a matrix is positive-definite or not, I just list some properties that such matrices are always invertible. And this is also a good property that also means that they are non singular.

Another good property is that the inverse of a positive-definite matrix is always a positive-definite matrix. Again the proofs might be might be pretty, pretty, pretty simple. So, I just skip. So, if I if I instead write $x^T$Q x < 0 for all x in $R^n$, then this will define for me what I call as a negative-definite matrix. And if instead of the strict greater than sign, if I just replace it by greater than or equal to it will be called a positive-definite so sorry positive-semi definite matrix and with the less than or equal to so in this case, so this will be a negative semi definite matrix. So, I am just talking of the properties of this Q matrix, positive-definite, negative-definite, positive-semi definite and negative-semi definite.

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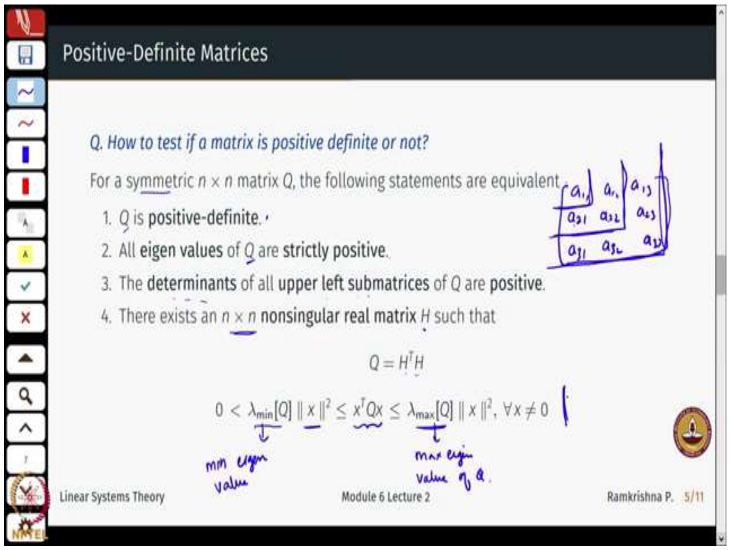


Now, how to check computationally if the given Q matrix, square matrix is it positive definite or not ok. I cannot compute all $x^T$ Q x for all x in $R^n$ that will take me infinitely long time. But I will be just see if there are alternative methods of doing that, so that is what is in the next slide. So, Q being positive-definite is equal to saying that all eigenvalues of Q are strictly positive right ok. Here I am talking of symmetric matrices only right. So, this should be stressed ok.

For a symmetric n cross n matrix Q, when I whenever I say Q is positive-definite, it also means that all eigenvalues of Q are strictly positive ok. Not only that this is I can another way of testing is a determinant of all upper left sub matrices of Q are positive. So, let us say I am just looking at a matrix of 3 x 3 ok. So, what is what does this mean that first $a_{11}$ > 0.

And if I compute the determinant of the 2 cross 2 elements starting from here this determinant should also be positive. So, $a_{11}$ > zero the determinant of these four elements $a_{11}, a_{12}, a_{21}, a_{22}$ should be positive. And this also should be positive the entire determinant of the matrix. So, these are all the left upper left sub matrices of Q, this one, this one and of course, the original one by itself ok. Is there another test? Well, yes, there exists. So, Q is positive definite. If there exists an n x n non-singular matrix H such that Q can be represented as $H^T$H ok. So, again this H should also be invertible.

A final property which again I will not do the proof, but we will use this extensively is that given a matrix Q, so $x^T$ Q x right so which is again which will be greater than 0, when it is A positive-definite matrix, when Q is a positive-definite matrix. So, this will be lower bounded by $\lambda_{min}(Q)||x||^2$ and upper bounded by $\lambda_{max}(Q)||x||^2$. So, this is the minimum eigenvalues, and this is the maximum eigenvalue of Q ok. This should again be easy to verify, and I will not do the proof of this. So, this is what we will use in our stability proofs ok.

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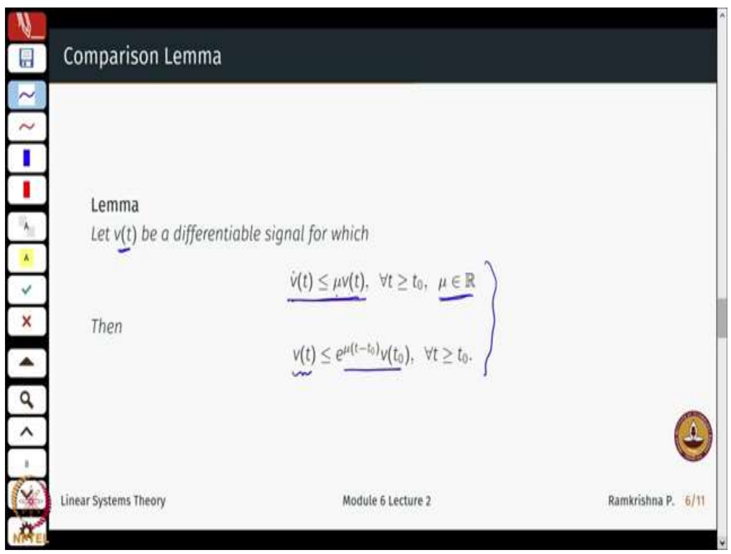


Now, another result that we will use later in the proofs is the following. And I will do this the proof of the comparison lemma in a in a supplementary lecture after I finish the main lectures in this week 6. So, start with a differentiable function v (t) be a differentiable signal for which I have some kind of a differential equation involving v that $\dot{V}$(t) ≤ μV(t) well for all times t> $t_0$, and some mu being some scalar quantity.

If this holds, it turns out that V(t) ≤ $e^{\mu(t-t_0)}$V($t_0$); it sounds intuitively it, it looks A little ok, but we will we will actually prove, prove this later on. But this will be another year result that we will use extensively in this lecture ok.

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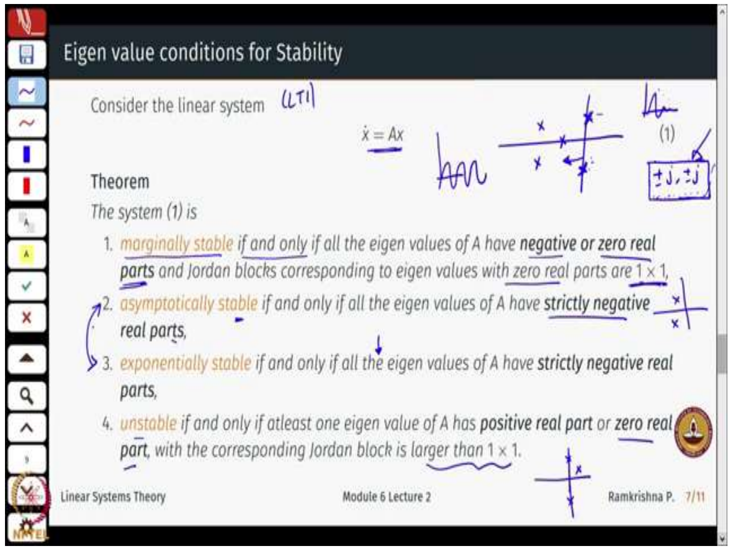


Now, coming back to how do we check for stability. Again the motivation is can I have some better test than by just verifying solutions if they are stable, asymptotically stable, exponentially stable and so on. So, the definitions here would give me some tests based on the eigenvalues or the properties of the A matrix ok. So, we start with the linear system. Again this is LTI system $\dot{x}$ = A x, I do not explicitly consider the influence of B at the moment. But this is give you give us a nice exposition to what is what is coming up.

So, the system 1 is marginally stable if and only if right, so it is it works both ways all eigenvalues of A have negative or zero real parts. So, like very much to do with the poles if the poles are here the eigenvalues are here, here and here, this is all stable ok. And not only this, so, this 0 real part is comes with one more condition that all Jordan blocks corresponding to the eigenvalues with 0 real parts are 1 cross 1.

So, what does this mean? So, if I have a poles of the form ± j, ± j like repeated poles on the imaginary axis, this system shows out to be unstable ok. You can check for this to this condition, these are also in any earlier control lectures of basic undergrad control engineering would tell about this right. If they are repeated poles on the imaginary axis, then that leads to instability.

How does this how to the solutions look like say in this case, so this poles will have exponentially decaying term. So, this will in some sense have some kind of an oscillatory behavior essentially because of the poles at the on the imaginary axis. If there are no poles on the imaginary axis, you might actually find that the oscillations might just die down right depending on what kind of damping the is exhibited in the system ok.

So, back to definitions of stability or how do we verify with eigenvalues, well, the eigenvalue should either be on the imaginary axis or to the left of it. The only caveat here is that if they are on the imaginary axis, I need to do an additional test corresponding to the Jordan block ok. Asymptotically stable is easier this is any that I just have to check if all the eigenvalue should be on the left half plane, no, no, eigenvalues on the imaginary axis.

So, the definition of the theorem says that that the system is asymptotically stable if and only if all eigenvalues of A have strictly negative real parts similar thing holds also for the exponential stability case in the LTI case, so that is what we had claimed not claimed last time. But we had I had mentioned to you that in the case of linear time invariant system, asymptotic stability is also equal to exponential stability. It may not be true in other cases of non-linear systems, but we are not interested in that, but we will slowly prove this in this lecture.

Last thing about unstable system, well, system is unstable if and only if well if there is one eigenvalue on the right half plane, then its unstable. Again if there is there are imaginary eigenvalues or eigenvalues with zero real parts, then the corresponding Jordan block should be of size larger than 1.1 for example, in this case right, so that is the only thing that we must be careful of. What we will prove is not are not things ready to and in stable or unstable systems, I will just prove conditions related to stability.

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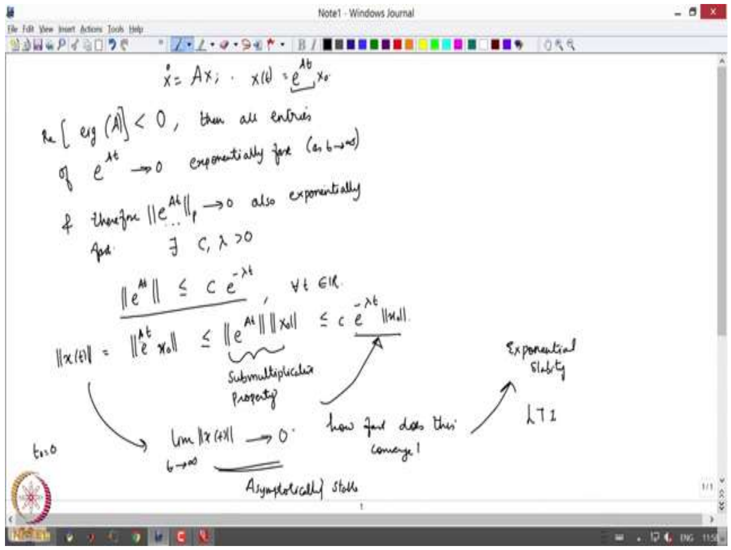


So, let us first begin by showing that for LTI systems, asymptotic stability is equal to exponential stability ok. So, I start with again $\dot{x}$= A x with solutions of the form x(t)= $e^{At}x_0$ to the $x_0$ being the initial condition ok. So, well, lot of it depends on $e^{At}$. And what we know that if the eigenvalues of A have strictly negative real parts right. So, the real part they say the real part of the eigenvalues are strictly less than 0, then this would mean that all entries of $e^{At}$ converge to 0 exponentially fast right so as t goes to infinity.

So, this is the entries converge exponentially fast, and therefore $e^{At}$ converges to 0 also exponentially fast sorry ok. So, it could be for any, any, any matrix norm one, norm two norm or the infinity norm. And therefore, this would mean that there exists constants c, λ > 0, such that $e^{At} \leq ce^{-\lambda}$ again for all t ok.

Now, how does the solution look like, ||x(t)|| is $e^{t}$ let me assume $t_0$ =0 just for simplicity, $e^{t}x_0$. So, this will be less than or so $\leq e^{At}x_0$. So, here I am using the sub multiplicative property of matrix norms right. And now with this expression I know that this is $ce^{-\lambda t}x_0$ok.

So, what does this mean that, the solution, ok, now check what happens to x(t) as t goes to infinity ok. First thing I can obviously say that this converges to 0 as t goes to infinity again assuming that all the eigenvalues of A are strictly less than 0, which means $e^{At}$ converges exponentially fast and so on ok. Now, how fast does this converge? So, even if

I even before I ask this question, so this is already telling me that the system is asymptotically stable ok.

How fast this is converts and I go to the right hand side, and I see that the solutions can always be upper bounded by some exponential curve here, exponentially decaying function here. And therefore, this also leads to the equivalence between asymptotic stability and exponential stability for a linear time invariant system, now these two are indeed the same, same concepts right. So, again the steps are, are, are pretty, pretty neat to follow.

So, what we showed here is the following that well that asymptotic stability would mean that all eigenvalues of A have strictly negative real parts. Not only that whenever A has eigenvalues which are strictly negative in the real part, this also means that they are naturally exponentially stable right, so that is the little proof of the equivalence between asymptotic stability and exponential stability for LTI systems.

All the time again computing eigenvalues may be computationally difficult for me or computationally expensive even though I am doing it on a computer say via matlab for example. For a large size matrix, it might take me a very long time. Now, are there another or better kind of conditions that I can check which are which are maybe computationally efficient for me to verify if the stability if the system is asymptotically stable or not. Now, yes, I am interested in asymptotic stability or I will just use one word either asymptotically stable or exponentially stable, and they would mean the same.

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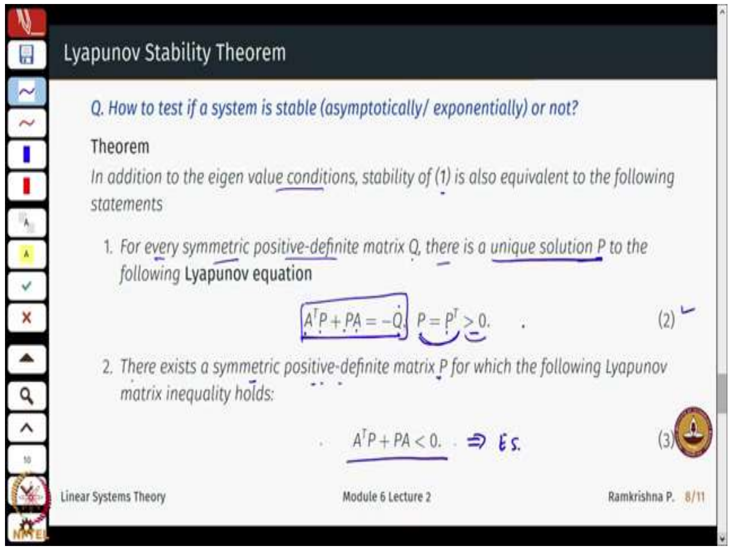


So, these are two match looking conditions that in addition to those eigenvalue conditions stability of the of the of the LTI system is also equivalent to saying the following that give me any matrix Q like for every right. So, for every symmetric positive-definite matrix Q, stability would mean that there exists unique solution P to this following Lyapunov equation right. So, given Q, A is the system matrix can, I find a P. So, if a P exists this and this then this system, so the solution to this if I if a, if a solution exists to this equation, then the system is a exponentially or asymptotically stable. Moreover the P is such that it is symmetric and positive-definite right. So, we will we will prove this. Second condition says that there exists again a matrix P, which is symmetric positive-definite for which this inequality holds ok.

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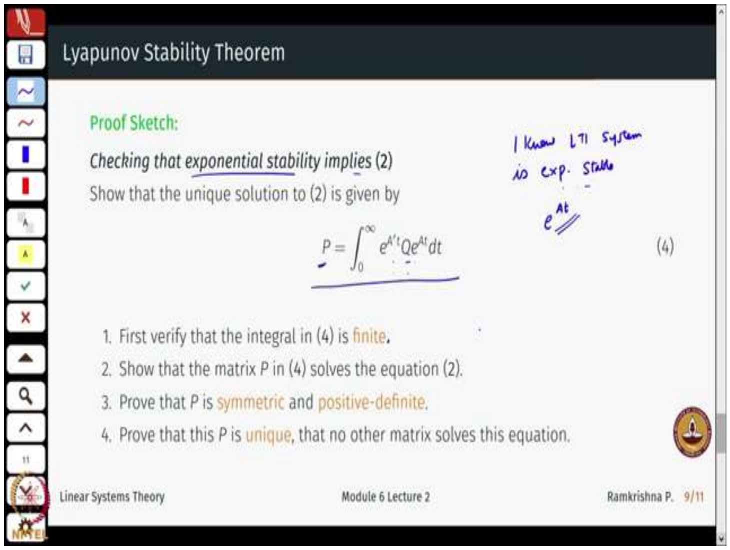


I will give you a little gist of the proof, and then we will go into the details of it. So, first we need to understand, what is given to us and what is to be proven. So, the part 1 of the proof goes in the following way. So, we will check that, when the system is exponentially stable it actually implies this equation ok. So, we will begin by showing that the unique solution is given by this indefinite integral from 0 to infinity, and blah ok. So, I know that the system is exponentially stable right that that the system the LTI system is exponentially stable ok.

So, first I need to I need to check the properties of this integral that this I integral actually converges that this, this is actually finite ok. Second I need to show that if the system is exponentially stable which means $e^{At}$ converges exponentially fast and so on, that if this is true, then this P actually solves this equation ok, not only that we also should show this extra properties at this P. Whenever it solves that equation is actually symmetric and positive-definite. Lastly, we also should be able to show that this P is unique because I am looking here at a unique solution P to the to this what I called as the Lyapunov, in Lyapunov equation ok. So, what does stability mean, given a matrix Q, can I find P which solves this equation such that this P is unique, it is symmetric and positive-definite ok. Let us, so let us do this steps one by one.

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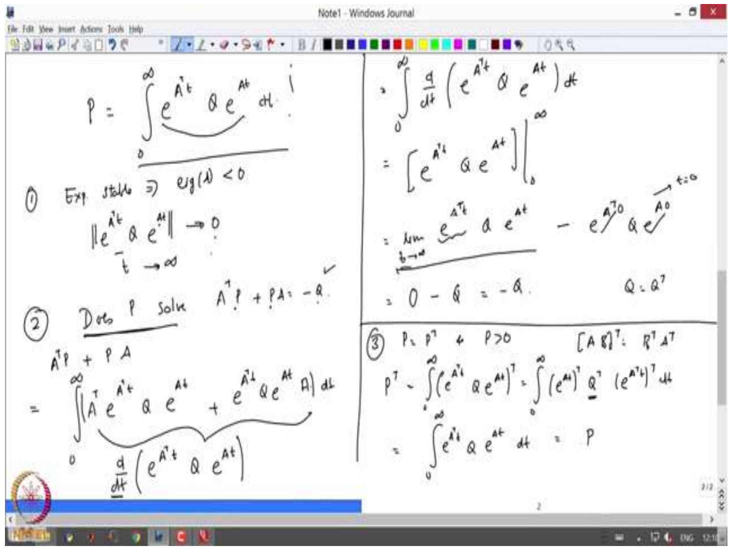


So, I have P as $\int_0^\infty e^{A^T t} Q\, e^{At} dt$ ok. First is the integral well-defined or does this integral actually exist that at 0 to infinity of some function should not give me infinity ok. So, what do I know, I know that the system is exponentially stable, which also means that I the eigenvalues of A are strictly less than 0. I am talking of the real parts of eigenvalues of A being strictly less than 0. And therefore, if I look at this quantity $e^{A^T t}$Q$e^{At}$ ok, so this will converge to 0 exponentially fast as t goes to infinity right. And therefore, I can say that this integral is absolutely convergent or this limit actually exists right.

That is so what I am a making use of the property, I am making use of the property that A is an exponentially stable matrix and that is what gives me. So, conversely if A is unstable, then this would not go to 0 right, this quantity would possibly blow up and go to infinity as t goes to infinity and in that case this integral is not well-defined ok. So, so this is the step number 1 is done ok. Step number 2, now does P solve this equation $A^T$P + P A = -Q ok. So, $A^T$P + P A is ok, I substitute this in this equation. So, I will substitute for P at 0 to infinity $A^T$.

So, what is P? P = $\int_0^\infty e^{A^T t} Q\, e^{At} dt$ P. So, P is again $e^{A^T t}$Q$e^{At} A dt$ ok. So, closely look at this quantity inside the bracket. So, this is essentially $\frac{d}{dt} e^{A^T t}$Q$e^{At}$ ok. Now, what am I left with, so if I replace with the $\frac{d}{dt}$ I have $\int_0^\infty \frac{d}{dt} e^{A^T t} Q\, e^{At} dt$ ok.

Now, this is this is looks simple now . So, I have $e^{A^T t}$Q$e^{At}$ with limits from 0 to infinity ok. So, this is what. So, the first one would be, so I am looking at the limit as t goes to infinity $e^{A^T t}$Q$e^{At}$ - ok. So, when t = 0, this will be $e^{A^T(0)}$Q$e^{A(0)}$ when I am just looking in this in this the second term as t equal to 0, and the first term as t goes to infinity ok.

Now, look at this term carefully, what happens to this term as Q goes as t goes to infinity, I know that A is a stable matrix exponentially stable matrix. So, this entries will go to 0 as t goes to infinity. So, I am left with a 0 here minus, so what is this, this is the identity, this is the identity, so I am just left with a minus Q ok. So, what I have shown here is that this P solves this equation right I substitute for P here and I get Q right. So, I know now that P which I defined it in this way actually solves my equation. Now, is that enough? Well, not really, third step would be to check if P = $P^T$, and if P is positive-definite ok. The first step should be easy to check start from here. What is P transpose, P transpose is 0 to infinity, I just have this three matrices and I just invoke the transpose formula.

So, I think if I write it correctly $(AB)^T = B^T A^T$ ok. So, this will be $e^{A^T t}$Q e ok, I will just maybe write a little more elaborate steps. I have $\int_0^\infty (e^{A^T t} Q e^{At})^T \, dt$ of this. So, this will be $\int_0^\infty e^{A^T t} Q^T \, (e^{A^T t})^T dt$ ok. So, the first term would be $e^{A^T t}$, Q is symmetric, symmetric is what Q = $Q^T$. So, I will just write it as Q, and the third term will be $e^{At} dt$, and $P^T$ = P ok. Now, this is this is done.

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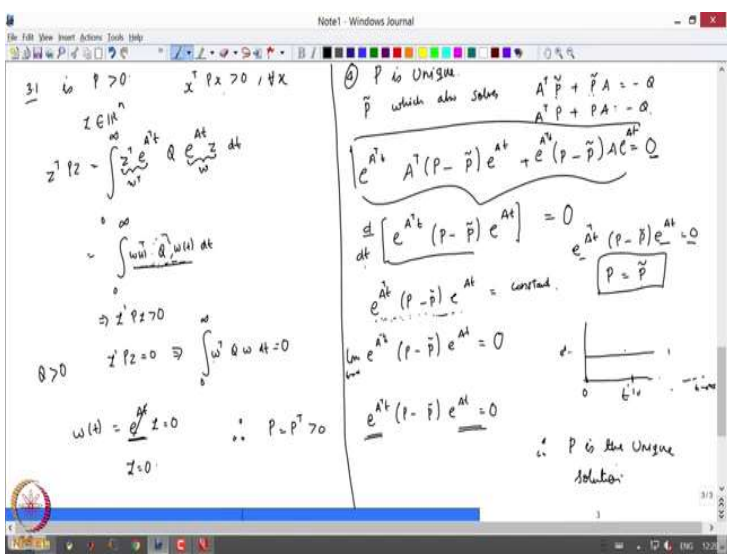

Next is say 3.1 is P positive-definite matrix. What is the definition of a positive-definite matrix that $x^T$Px ≥ 0 for all x ok. I do not really do an eigenvalue test here, but I just do by definitions ok. Just take an arbitrary vector z in $R^n$ ok, and then compute this quantity $z^T$P z that is $\int_0^\infty z^T e^{A^T t} Q\, e^{At} z dt$.

Now, this is some vector right you call it w, and then this therefore, will be $w^T$ I have sorry 0 to infinity some vector $w^T$ this is again depending on time Q w(t) ok. Now, what do I know, I know that Q is greater than 0; and therefore, if Q is greater than 0, then this quantity should be greater than 0 ok. And therefore, I have that z transpose p z is actually greater than 0 given that Q is also positive-definite, and then I just invoke the definition right. So, if I have a vector transpose Q times the same vector that should be greater than 0, and this will hold for all z. So, I am not I am not defining any particular z here, but this will hold for all z.

So, when whenever $z^T$Pz = 0, this would imply that this thing is 0 $w^T$Q w =0 ok. Now, when can this happen, this can happen if w(t) = 0 right from the definition of the of the positive-definite matrix and this will be 0. So, so what was the definition if we if we recall go back right here, so anywhere apart from the origin, it is greater than 0. So, at when x=0, it is obviously, equal to 0 which means that this should be 0 that is $e^{At}$z = 0. And I know because A because A is exponentially stable matrix that this guy will not be 0. So, the only option left is z =0 ok.

And therefore, we have proven that p is not only symmetric, but is also positive-definite right. So, we are done now with this properties that in the step. So, I just verified that the integral is finite not only that I also showed that P solves equation two right the which was essentially this, this equation. Third step I proved that P is symmetric, and it is also positive-definite. Last one is to show that this p is actually unique that you cannot have some other $P_1, P_2$ or $P'$ which solves this equation ok.

So, how do we go about proving this? And much of proof sometimes are also by contradiction. So, I assume or I begin with assuming that let there will be some other matrix p 1 which also solves this, and see what happens if there is any other candidate solution let me compare that with my original solution. And see does that actually exist another candidate solution to this that is what how we will prove that P is unique ok.

Now, let me prove just consider any other $\tilde{P}$ which also solves this thing $A^T\tilde{P}$ + $\tilde{P}$A = -Q right. What I already know from the first three points here that P is also a solution ok. So, this, this guy $\tilde{P}$ comes from nowhere and claims that ok, I am also a solution. Now, I will just verify what does this mean. So, I just do some manipulations here. So, this is will turn out to be $A^T$P ok, I just write it a little ok. $A^T$(P- $\tilde{P}$) + (P- $\tilde{P}$)A = 0 let us by subtracting this equations ok.

So, let me just do a little trick here. I will just multiply this on the left by $e^{A^Tt}$ on the right by $e^{At}$. Similarly, here so I am multiply to the left by $e^{A^Tt}$ on the right by $e^{At}$ this will still be 0, the right hand side will not change ok. So, this turns out the quantity inside that $\frac{d}{dt}$($e^{A^Tt}$ (P- $\tilde{P}$)$e^{At}$) right which is essentially what is above and that actually is equal to 0 d by dt of this entire quantity expands to this big expression here and what do I know of this expression that this actually is 0 when d by dt of something is 0 I know that $e^{A^Tt}$ (P- $\tilde{P}$)$e^{At}$ = constant ok.

Now, if I show that this constant is 0, then things will be easier for me, because if this goes to zero, if $e^{A^Tt}$ (P- $\tilde{P}$)$e^{At}$; so this is p A transpose here, $e^{A^Tt}$ (P- $\tilde{P}$)$e^{At}$= 0. Then P will actually be turn out we will turn out to be $\tilde{P}$ that the other guy who comes and claims to be the solution is actually a solution itself that is, because $e^{At}$ is always invertible right ok.

Now, how do I show this that this is actually 0, ok. Look at this expression carefully; right this is actually a constant. So, if I were to just plot this with time, so this function of whatever this is right, so e power A blah; so this is constant ok. So, if this is constant whatever value it holds for t equal to 0, should hold for say t equal to 10, to t equal to 100 and all the way as t tends to infinity, ok.

So, if this holds at infinity what happens to this quantity at infinity, $e^{A^Tt}$ (P- $\tilde{P}$)$e^{At}$, what is the value of this as t goes to infinity, because e is exponentially, because e is exponentially stable that is what I am I am being exploiting all the while. So, this will be 0 right, because at infinity I know it is 0, so it should if it is constant a valued function, then the same value should also hold at t equal to 0, same value should also hold a t equal to 10 and so on.

And therefore, this number by itself e for this matrix by itself $e^{A^T t}$ (P- $\tilde{P}$)$e^{At}$ is 0 again, because the system is exponentially stable; I started without assumption, let the system be exponentially stable and then I prove the condition number 4, right. Now, this is 0, if this is 0, then I know that P=$\tilde{P}$ and therefore, p is not only the solution, it is also the unique solution. In addition to it satisfying the property of it being symmetric and positive-definite ok, so that was about the proof of this ok.

Now, what should I prove last. So, that stability or asymptotic or exponential stability is also equivalent to the to this to this statement that there exists again a P for which the following matrix inequality holds that $A^T$P +PA < 0, ok. So, what are the proof steps is it is a little, might look a little tricky with these steps, but we will do that one by one. So, again define we will show that whenever this holds that whenever this expression holds that this will imply exponential stability ok, so we will assume this to be true ok.

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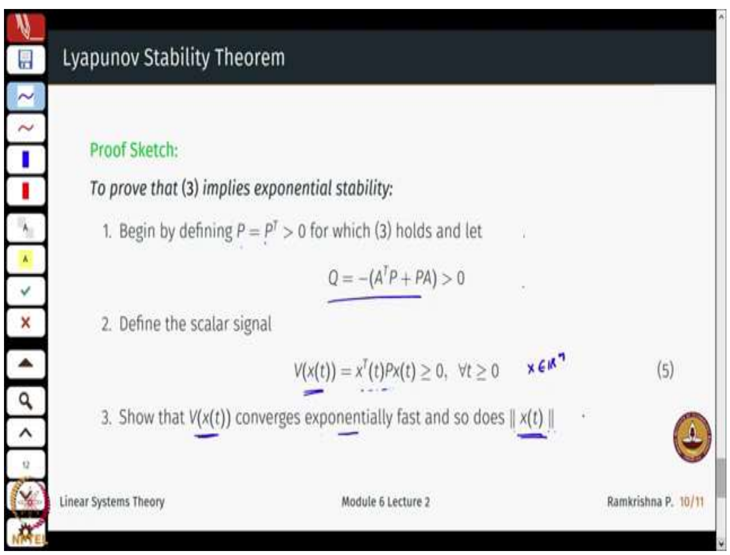


So, I start by defining a P which is again symmetric positive-definite for which three holds right this one and I define this quantity like this, this comes from say some equation number 2, ok. Now, I define another quantity a scalar signal right so x comes from $R^n$, so this will be scalar signal that the if I show that if V(x(t)) converges exponentially fast the solution. Therefore, ||x(t)|| will converge exponentially fast and this will also mean that the solution of the our original system converges exponentially fast resulting in exponential stability, ok. We will write that down one by one and then and then check for ourselves.

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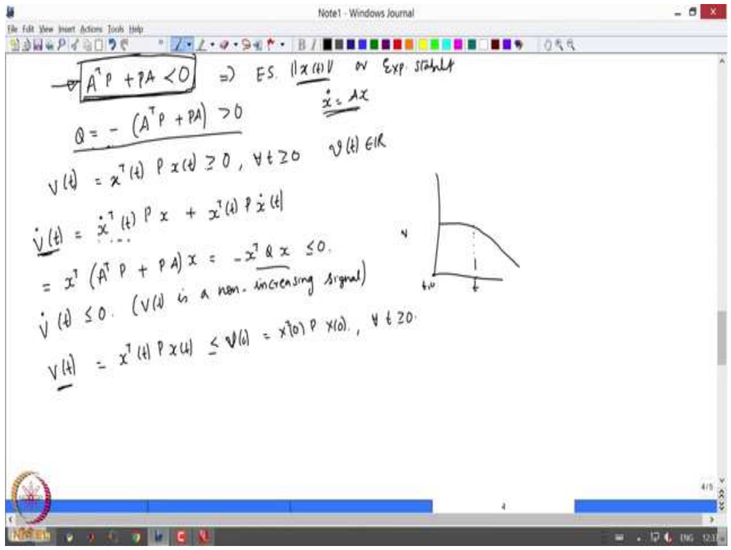

Ok. So, the idea is that I should show that if this is less than 0, then the system is exponentially stable or the solution x(t) converges exponentially fast ok. So, let us say assume that Q is such that minus $A^T$P + PA that this is now greater than 0 ok. So, if a condition like this is holds, then this will be obvious ok.

Now, define a function V(t) is $x(t)^T$Px(t), I slowly tell you what is the physical significance of this function. But for the moment we will stick to the to the proof of what we are supposed to do here, proof of exponential stability. Now, I take the time derivative along the system trajectories. So, this will be I am computing $\dot{x}^T$(t)Px(t) + $x^T(t)$P$x(\dot{t})$ ok. Now, what do what is x dot, $\dot{x}$ = A x ok. So, what I get here is this will be $x^T(A^TP + PA)$x, what is this $-x^T$Qx and this is less than or equal to 0 ok, so this is what I have now, ok.

Now, just look at this $\dot{V}$ its it is a function right. So, $\dot{V}$ so V is a real valued function and if $\dot{V}$ ok, what is $\dot{V} \leq$ 0 ok. So, what does this mean and this means that $\dot{V}$ or V(t) is a non-increasing signal. And if $\dot{V}$ is non- increasing, then V(t) just say I take any arbitrarily time the x transpose t p x of t would be less than or equal to the value of the signal V at 0 is $x(0)^T$Px(0); and this holds for all t $\geq$ 0.

So, what does this mean that V is a non-increasing function. So, V can either it may be constant or it might it might decrease or whatever right. So, the value at any time so this is t equal to 0, so the value of V at t = 0 will always be greater than the value at some other time t or the value of V at some time t will be either less than or equal to its value at t = 0, ok. How did we prove this, we proved this by just taking the derivative of V along the system trajectories and assuming we know that this is true. So, the idea here is to show that satisfaction of this kind of a matrix inequality leads us to exponential stability, next.

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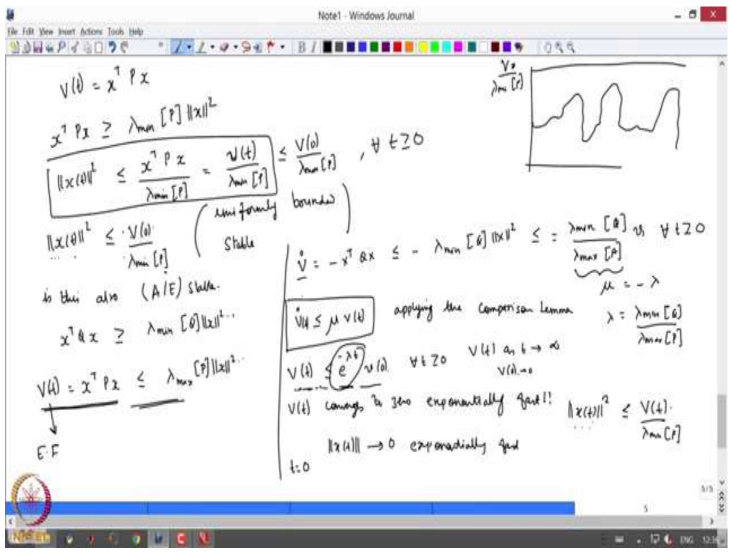


So, V = $x^T$Px, now this $x^T$Px from the matrix properties which we listed earlier in this lecture is greater than equal to $\lambda_{min}(P)$ the minimum Eigen value of P times $||x||^2$ ok. So, from this what I can conclude, first I can conclude that $||x||^2 \leq \frac{x^T Px}{\lambda_{min}(P)}$ ok. What is $x^T$Px, $x^T$Px is the function which I defined earlier V(t), this remains the same ok; this is less than or equal to V of 0, because V is A non-increasing function as proved just in the in the in the previous steps and this is valid now for all times t ≥0 ok.

So, what does this mean, if I just take these two expressions $\frac{V(o)}{\lambda_{min}(P)}$ ok, so this is this is known to me the value of the function at 0, p is also known to me right, because I assume that there exists a p which satisfies this inequality. Now, what does this tell me about the solution, it tells me that the solution is uniformly bounded sorry, that whatever happens the solutions will now maybe they start here, they go here and whatever, they are always be bounded by this number here this guy $\frac{V(o)}{\lambda_{min}(P)}$, so this is the this is uniformly bounded.

And therefore, I can at least now with this step say that the system is stable, marginally stable so to speak ok. Now, next step to show is this also asymptotically stable, asymptotically or exponentially stable ok. Now, let us let us do this thing ok. Now, this matrix Q which was here this guy, here and here and so on, this matrix Q is also such that $x^T$Qx $\geq \lambda_{min}(Q)||x||^2$. V sorry, this V(t) which was $x^T$Px, the value of this is always upper bounded by this $\lambda_{max}(P)||x||^2$.

Now, look at the expressions for $\dot{V}$ right starting from here, what we can say is the following that $\dot{V}$ which is -$x^T$Qx, this will be less than or equal to the negative again of $\lambda_{min}(Q)||x||^2$ ok. This will be less than or equal to, so I just use on top of that this inequality $\frac{-\lambda_{min}(Q)}{\lambda_{max}(P)}$v; and this happens for all times t greater than or equal to 0. So, just a little steps which you can easily verify from all the expressions that we have written here ok.

So, let me call this so, this is again this will be a number right. So, the minimum Eigen value of Q which is which is known in the λ the maximum Eigen value p of p which is known, so this will let me call this as μ. So, I have $\dot{V(t)}$≤μV(t) right, so this and this ok. Now, I have a differential equation now in V, which looks something like this; $\dot{V(t)}$≤μV(t).

Now, there was this thing called the comparison lemma which I stated that whenever V which is a differentiable function, I know that my V there over there was a differentiable function is ≤μ; μ was a ratio of those minimum and maximum Eigen values of Q and p respectively with a negative sign. Whenever this happens, then V satisfies something like this ok.

So, let us exploit that and write down. So, with applying the comparison lemma now ok, what do I have now what can I say about V, this V(t) is now less than or equal to e power minus, so $e^{-\lambda t}$, of I will call this λ, let me just I just say that this μ= -λt. Let us say for simplicity t naught is the is just 0, v of 0 for all times t greater than or equal to 0. So, this λ = $\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$ ok. So, what does this tell me about V, so V is a function this λ will always be greater than 0, because Q and p are positive-definite matrices. So, they are all their eigenvalues will be strictly positive, this negative sign will add up to that and say that this V of t as t goes to infinity ok, we will go to V of 0 right, ok.

Now, V of 0 could possibly be the origin also. So, this and at what rate does it do, this quantity here tells me now that V(t) converges to zero, exponentially fast right ok. Now, if V (t) converges to 0, exponentially fast; how are V and x related, V and x are related via this. So, if V goes to 0 exponentially fast, then x(t) will also converge to 0 exponentially fast, ok.

What is the relation between x and V, so from here I can say this if I just restrict till here, I have that $||x(t)||^2 \leq \frac{V(t)}{\lambda_{min}(P)}$ ok. V converges exponentially fast and therefore, x converges exponentially fast and therefore, where did we start with; we started by assuming this to be true and if this is true, this now implies that x converges exponentially fast or I have proved exponential stability of $\dot{x}$= Ax right. So, this were a little proofs of how all these statements were equivalent.

So, we just invoked a few properties of the matrices that we started off with today. We invoked the comparison lemma, we also invoke the relation between the definitions of positive-definite matrices and their relations with the minimum and the maximum Eigen values, this proofs are nice and very intuitive also. It will be nice for you to just write down the steps for yourself, so that you get an idea of how proofs in general are done in any control literature; so that might be easier for then for you then to understand any other research papers in this in this area, ok.

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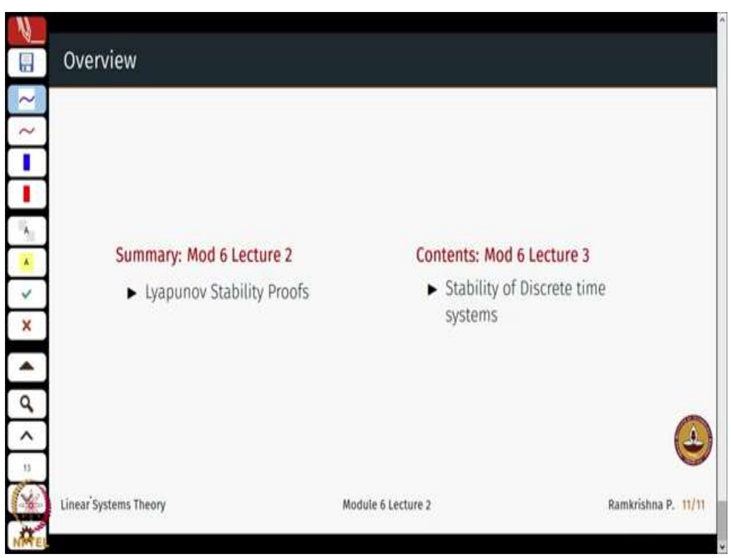

So, just to conclude what we did today was we started off with proving lots of Lyapunov stability proofs, started off with giving Eigen value conditions for asymptotic stability and then we had some kind of an and matrix inequality, like conditions to prove stability. Next lecture will be a little short I will skip all those proofs, because the steps will be exactly the same as what we did in this lecture. So, we will essentially deal with discrete time systems in the in the next lecture and that is coming up shortly.

Thanks for listening.