Linear Systems Theory
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Module - 06
Lecture - 01
Stability analysis

Hello. Welcome to this week 6 lectures on a Linear Systems Theory. So, last week we had done lots of analysis about equilibrium points. We did not defined properly the notion of stability, but several characteristics of those equilibrium points via the phase space gave us some information whether trajectories around the equilibrium point where coming back to the origin or going away from the origin or the equilibrium.

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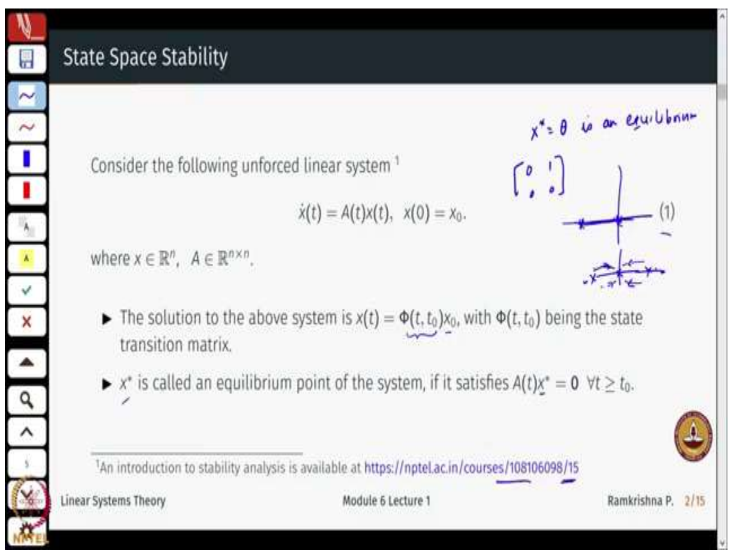

So, today we will give a formal notion of stability. Also of course, we will do mostly in the linear setting and in general in the time varying setting, ok.

I will start with the following unforced system. I will not talk of inputs as yet from $\dot{x} = A(t)x(t)$ with the certain initial conditions and x is in general coming an n dimensional vector.

In the previous control course, you would remember stability or one way of verifying stability is look at the poles of the system, and then you had several characterization of the poles being to the left to the right, on the imaginary axis repeated on the imaginary axis and so on. All this came from a bit of input output notion of stability, right. And the condition to verify or the way we derive the pole based condition for to verify stability comes actually from what is called as the absolute integrability of the impulse function of the impulse response of the transfer function.

We will not revisit those things. If you are interested you can just go back to one of my earlier lectures that is listed here, just to get a little warm up on the b bone notion of stability, what was the bounded input, bounded output notions of stability. I will not elaborate that. But here what we will essentially focus is on the state space stability, more related to the kind of solutions we computed for the system of linear equations starting from the concept of a state transition matrix, ok.

So, for this class of systems what we know is as a solution is well given by x(t) is Φ, which is the state transition matrix and initial condition. What you also know is that x^* is an equilibrium point of the system if it satisfies $Ax^* = 0$. And in most linear cases, the you know the origin x^* the this is the 0 vector is an equilibrium, it is always, even if the a matrix may be in is not invertible say if it is of this form, still 0 is an equilibrium.

There could at some point of time we will multiple equilibrium, say this is an equilibrium point the origin, may be somewhere this is also equilibrium point. So, the line joining these two points is also will this is any point between these two points or the straight line joining these two points will also be in the equilibrium, ok. So, you will have essentially an equilibrium subspace that is what we talked about in one of those conditions in equilibrium in previous weeks lectures, ok.

So, fairly we will just generally denote the notion of equilibrium as x^* and mostly we will deal with the origin, even if the origin is not equilibrium you can shift change the coordinates or shift the origin. Slightly different than the non-linear case because, ok, there we possible will not have an equilibrium subspace, but we will have say isolated equilibrium points, say equilibrium point here, equilibrium point here and so on and each say if may be the phase space goes something like this sorry, ok. It could go this way, right. This and this so this is a stable equilibrium, these two equilibriums could be unstable and so on.

So, we have what we also called as call as isolated equilibrium points. But for us we will just assume that or we know that origin is the equilibrium and that will be the equilibrium of interest. But just to for notational purposes we will just call it generally as x^*, ok.

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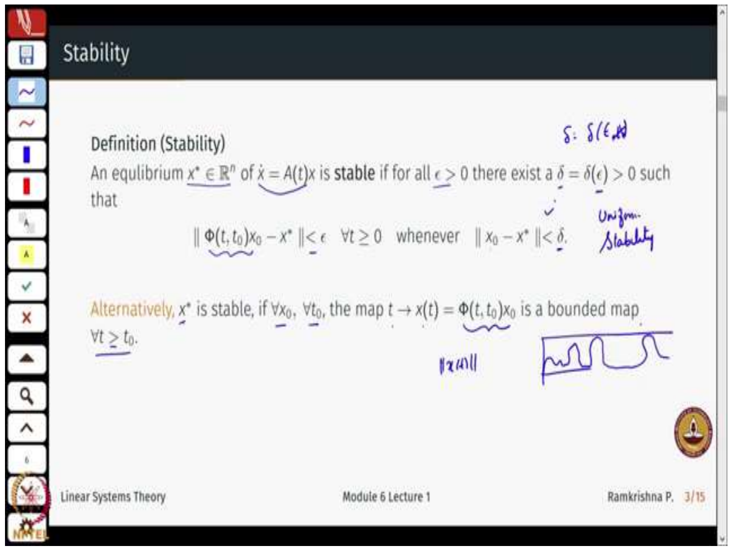

So, the first definition of stability what does it mean? Again, I am looking at stability is always defined with respect to an equilibrium x^*. So, an equilibrium x^* of the system $\dot{x}$= Ax and I just for ease of notation I just draw the time argument for x, is stable if for all ∈ > 0 there exist a δ which is a function of this epsilon such that if I take the norm. So, this is the solution, right; how the solution trajectory moves Φ(t, t_0) x_0 will give me the solution. The norm of this solution minus x^* is less than ∈, whenever $||x_0 - x^*|| <$ δ, ok.

We will come to this, right where does the epsilon come from, where does the delta come from, in some cases you also have that the δ is depending on the initial condition. And in this case it is not depending on the initial conditions, so it is usually called as the also referred to as the concept of uniform stability, ok. So, we will drop this $t_0$for the moment and we will say, well it is we are just dealing with uniform stability, ok.

A slightly milder version of this to give an intuition of what this what this statement means or a little understanding, $x^*$is stable if our all initial conditions for all initial times, the map t from x(t), right which is given by the solution this is my solution is a is a bounded map for all times t $\geq t_0$. Essentially, it means that Φ or this solution x(t), ok, so it is a bounded map. So, means the values of x never go unbounded. So, if I were to just draw a little picture well this could be solutions whatever and so on, right. So, these are all bounded, ok.

Now, what does what does this actually mean? So, let us see a pictorial version of it, ok.

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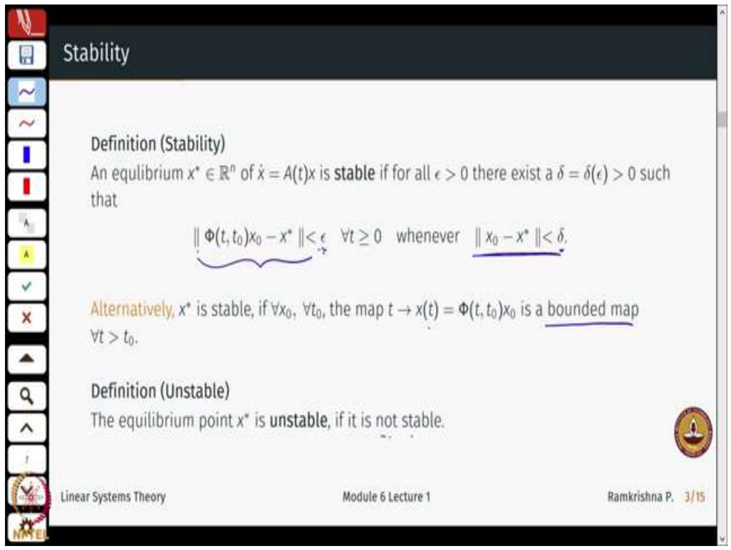

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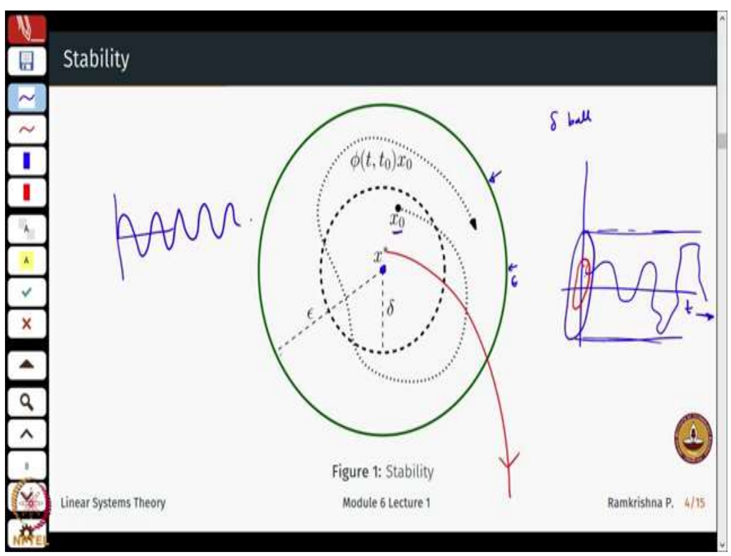

So, I start, with x^* as my equilibrium point here. Let this x_0 come from a δ ball, essentially a δ ball would been a ball centered at the equilibrium and of radius δ. And take any initial condition, so I have this δ ball and take any initial condition within this δ ball this x_0. So, the solution will. So, if I just see this dotted line the way it goes its always confined to this ε ball of radius, you know this is ε ball is again the ball centered at x^* and of radius ε, ok.

So, if I just, move this picture 90 degrees it will look something like this, ok. I will try to draw it as neatly as possible say somewhere I will just use this red color for the δ ball. So, this is my δ ball and, ok, ok, I will come back to blue. So, what does this mean of ok? I start at some initial condition here and as time goes by I am just within the solution is just within this cylinder it could be go here go here, and so on, but it will never breach this cylinder, right. So, it will just move along with cylinder. So, it is a here I am actually even adding the time axis. So, if I tilt it, so these solutions will just move along this cylinder which is in this in this represented in green, ok.

So, essentially it means that the solutions do not go like unbounded. So, for example, if I just say I started initial condition here and just keeps going this way, this is an unbounded solution and this is an unstable behavior, right.

So, this is a very basic very weak notion of stability that this could just be doing whatever the solution could just be doing whatever they want. One simple example is the linear oscillator that we did, right. So, it just keeps on oscillating, this always stable, right, ok. So, so that is what this says. So, whenever the initial conditions are in a δ ball the solutions will be confined to an ε ball or that the solution or this map t to this to the solution space is always a bounded map, right. That is what is, it is a very loose or the basic definition of stability.

What is unstable? There is only one definition whatever is not stable is unstable. So, that is the only definition of instability. You do not really check what are the solution going to infinity or they doing whatever and so on. So, whatever is not stable is unstable, ok. So, that is a the only definition of a system being unstable, ok.

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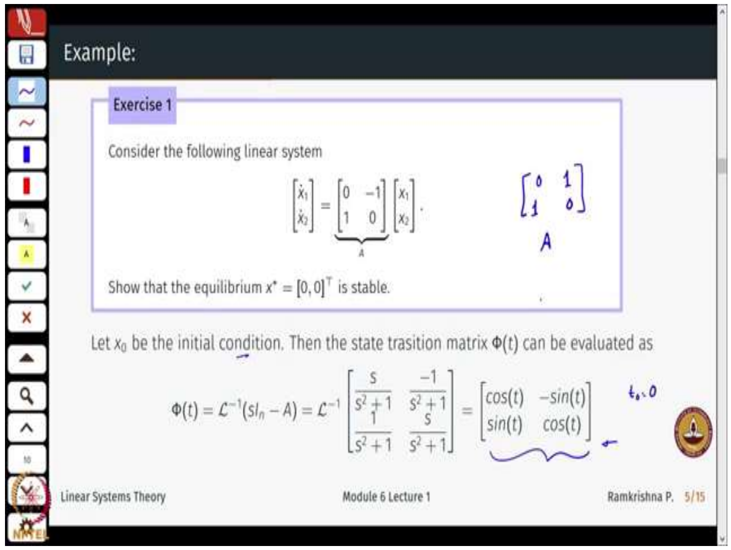

So, just to get give you a little very basic seemingly trivial illustration of what is happening in this in this picture. So, let us start with this second order system $\dot{x_1}$ = -x_2, $\dot{x_2}$ = x_1 and so on. Show that the equilibrium is stable. That is why easy to verify that (0,0) is an equilibrium, ok.

Let us first compute the solution. That I am just blessed for simplicity assume that the initial time is 0. So, let x_0 be the initial condition then the state transition matrix well we know it is computed this way. I do not, I will not go into the steps of this, but by now we know how to compute the state transition matrix.

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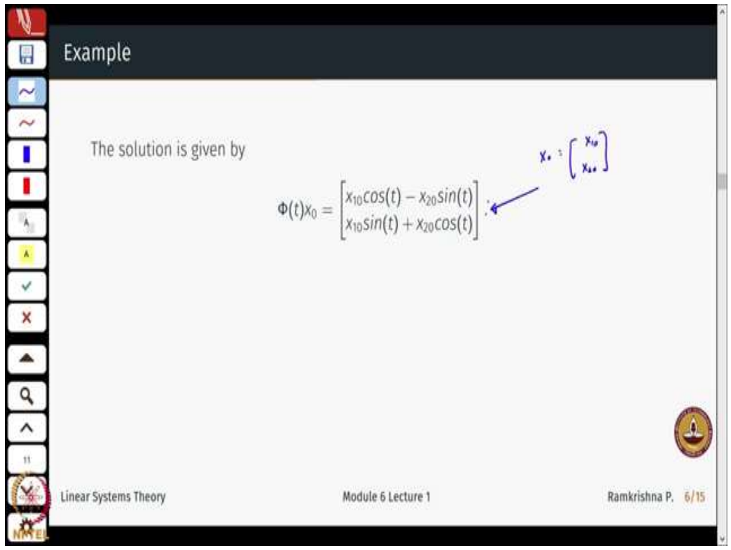

So, given any initial condition x naught which is a two-dimensional vector x_{10} or x_{20}, the solution will look something like this, ok.

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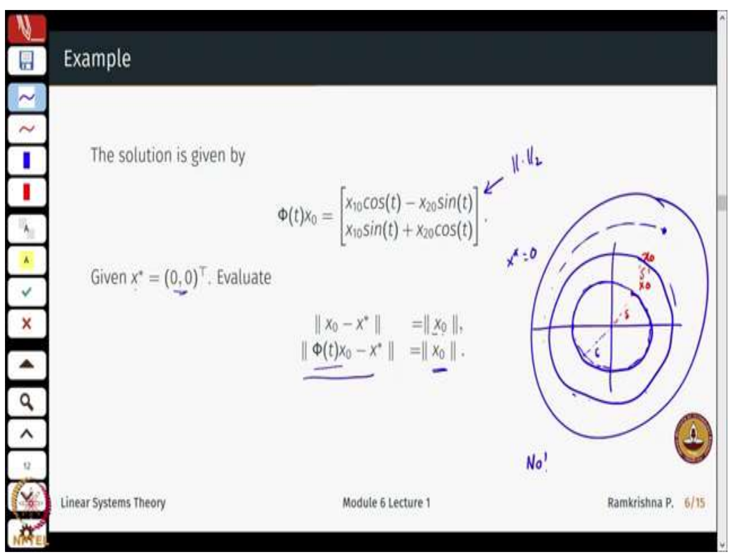

Now, given x^* which is just the origin, what I would want to evaluate is checked for these two inequalities, right. If the initial condition is within a δ ball are the solutions within an ε ball or all the solutions always bounded, ok.

So, let me compute this norm, right. So, this could be two norm or infinity norm or whatever because we know that all norms in finite dimensions are equivalent. So, give me one norm I can always find a nice relation with the other or you can always bound it with the other norm. So, does not really matter which norm we choose. But for our purposes we will usually look at the two norm, ok. So, this well $x^*$is 0, so this turns out to be just x_0, ok.

Now, the solution Φ(t)x_0 - x^*, so this is computations become easy, right. So, if I evaluate the two norm of this solution, it again turns out to be x_0. So, what does it mean is that well the initial conditions? So, the δ, so this I found a bound to be x_0 where I wanted actually to be equal to x_0, this is also x_0, ok.

So, let us take a trick to verify or to understand what this means. What we know is the phase space the phase space of this systems are concentric circles, ok. So, let me say I take this initial condition x_0. So, so this will be my δ ball, ok. Now, where will the solutions move? Well, the solutions will move along this circle of radius δ, right along all this. Now, this is also equal to ε that is what that is what I mean when I draw these two pictures, right. I start with the initial condition at this point. So, this is whatever is my initial condition then the final conditions will also be or the solutions will just be circling around this again of the same radius.

Further if I say, what if my initial condition is a is here. So, I will have a new δ, let me call this δ' and my solutions starting from here with some initial condition just be circling around here, ok. So, they never go unbounded, right. So, that is the that is the that is the idea here. So, I can just say you take a take a larger ball like this, start from a initial condition here and they will actually the solutions will just be inside this, ok.

So, so very nice and the trivial example, but it gives us a nice understanding of stability. It is easy to check that this system will have a complex Eigen values and so on. As an exercise to check a if the system is or an example for instability just check for this matrix and then try to compute the ε and δ values and you will find that this will not actually not exist, so you will not be able to bound this by some number from the above, ok.

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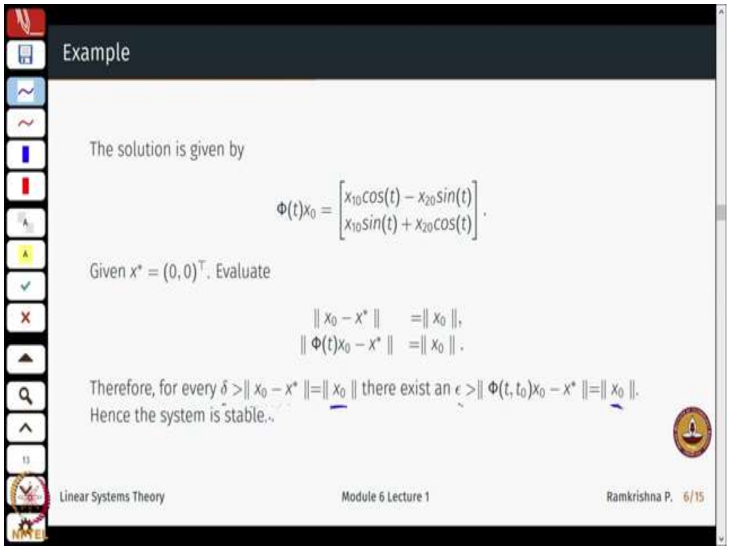

So, I will not really do that worst example of a instability because that that should be easy to check. And this is like, what I proved here is I can just check that for every δ greater than all this which is x_0 there exist an ε, which is x_0 and hence the system satisfies the definition of stability, right.

Now, what next? I really do not what systems which are, I really do not know where they are going, but they are just bounded. I want something stronger, right, so stability I would rather want my system to come back to its equilibrium or its steady state position when it is subject to an initial condition or there is a slight perturbation in the system. Now, how do we mathematically define that notion? And that notion is the notion of asymptotic stability.

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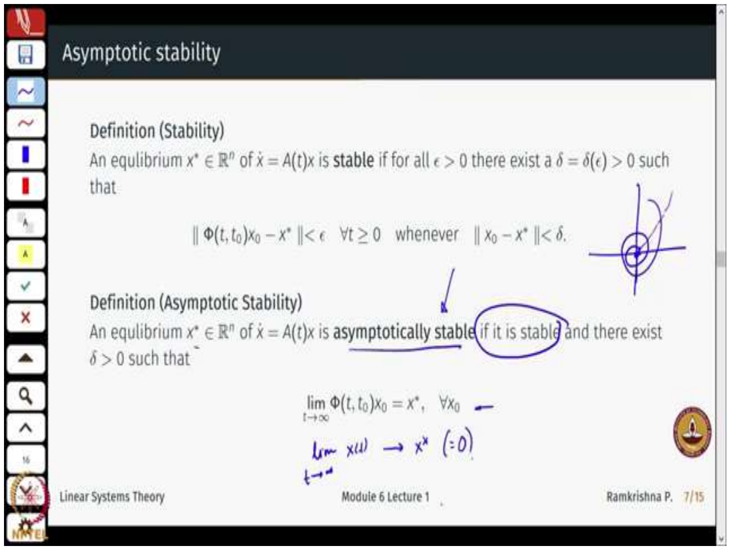

So, the definition states the following. So, we will first read the statement and then try to interpret that. The equilibrium again x^* of this system is stable. If again, so all this ε, δ exist as they would for, so the basic definition of stability should be satisfied. Now, additionally on top of that an equilibrium x^* is asymptotically stable two conditions, one if it is stable and second as t goes to infinity x(t) should tend to x^* which could be the equilibrium in this case for all initial conditions, right.

So, ok, let us see what is what is new here. First is a, so it means that as t. So, suppose I am I am here, so this is my equilibrium, as I will I start from many point I should actually eventually come back to this point. I can come back this way, I can come back this way depending on the nature of the equilibrium point, ok. So, this is this is what it means.

Now, why do I super impose the statement saying that stability is also necessary condition? Why is it not always obvious that at t equal to infinity I am going to the origin, so why should I additionally on top of that check for stability? Ok. Let us let us do an example of that. In the linear case, it might like be obvious, but there might be some classes of systems which would behave very strange, ok. So, let us let us do example of this, ok.

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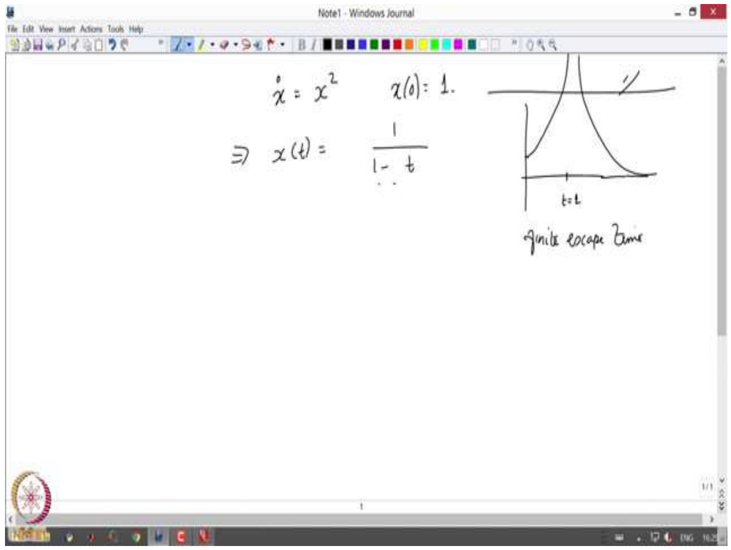

So, I have this system $\dot{x}$=x^2 x square and let me say that x(t_0) is 1, right. Now, what is the solution to this? The solution turns out that x(t) = $\frac{1}{1-t}$, ok. So, some strange thing will is likely to happen. So, at t = 0 my system will say start at say x_0= 1, it will have it will just blow up, right. So, the value it is not defined for t equal to 1. And you might think that well it will actually become back after t = 1 plus and eventually, if I apply the limit rule what happens as t goes to infinity this might actually come back to 0. I just draw slightly better, right, ok.

Now, is this stable behavior? Well this is not, right, because the solutions actually go unbounded. So, what was the definition of stability, that whatever the solutions do, they might just, they should just be bounded by above from some by some number, ok. This typically could happen in some non-linear systems and this phenomena where at t = 1 this system blows up is called the finite escape time, ok. A phenomenon not usually or almost never seen or never seen in the case of linear systems and therefore, to have a general notion of stability we may have to impose this extra condition that first I need to check for stability and then I need to check what happens to the solutions as time progresses, what happens asymptotically to the system, ok.

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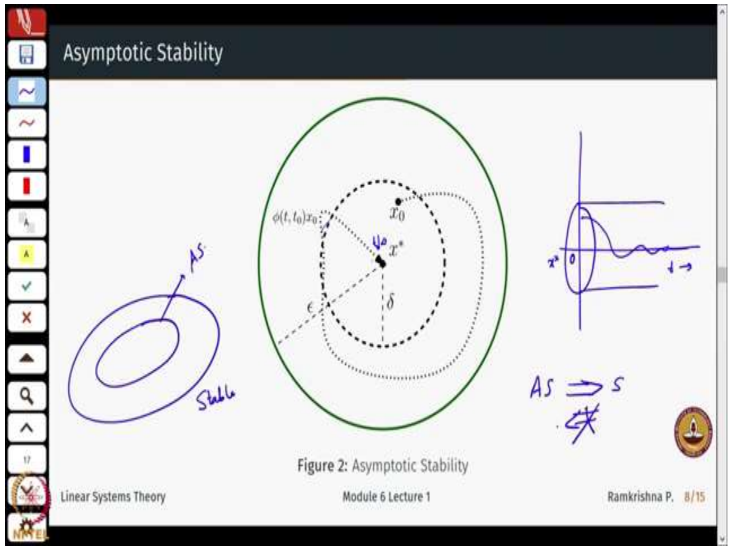

So, what does that mean? Pictorially, I think they should now be like kind of kind of easy to visualize now, right. So, if I go back to drawing the cylinder diagram start from many initial condition, I think that is nice way to look at it is, it just comes back to the equilibrium position. This is the equilibrium or x^*, ok, this is with time, ok. So, so similarly here I start with here and somehow I just say at, so this is a t equal to as t goes to infinity, I come back to x^*, ok. So, that is that is, this is stronger condition to check, right.

So, so for example, is this system asymptotically stable? Well, the answer is no this system is not asymptotically stable. It is stable? Yes. So, so first thing, which is a larger set? The earlier system was stable, but not asymptotically stable. So, asymptotically stable is bigger is a is a restriction on it, right. So, this will be asymptotically, the set of asymptotically stable system, it is stronger condition. So, all asymptotically stable systems are stable systems, but not all stable systems are asymptotically stable systems. So, it is like if I were just to say a asymptotic stable and stable, all asymptotic stable systems are stable, but all stable are not asymptotically stable systems, ok. Just a little counter example to that is what we saw just now, ok.

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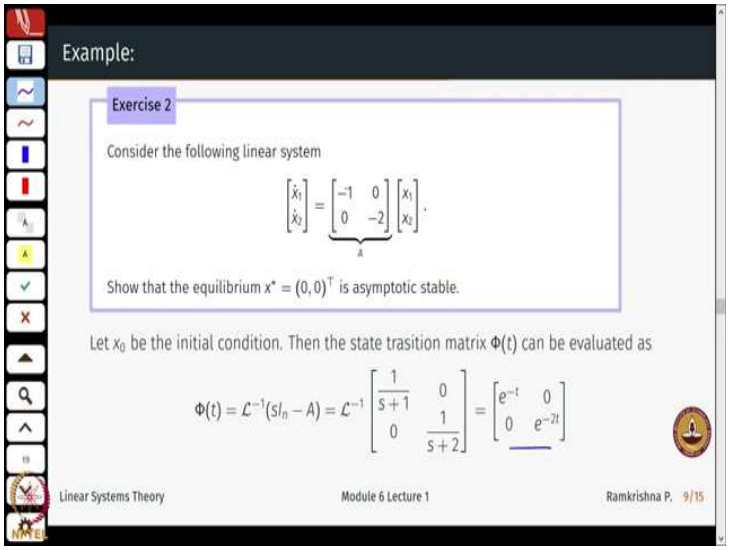

So, let us do again a seemingly simple looking example. If I just where to ask you to compute the state transition matrix solution, it will just be like by-heart, right because it is in the diagonal form and then the e^{At} is easy to compute.

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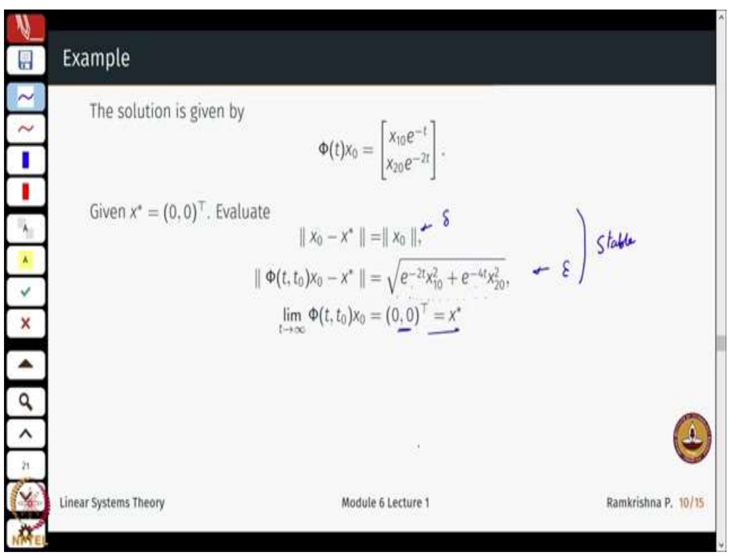

Let us say I start with certain initial condition x_{1o}, x_{2o}, again for simplicity I assumed $t_o = 0$. First, can I check for stability? That will be easy to check, right. So, this is my the δ ball, this will be how the ε ball would look like. So, the first two conditions will tell me that the system is stable, ok. Now, how do the. So, I can actually find the upper limit of this, right. So, I will just leave that as an exercise,.

But what look at this expression here, what happens it has to time two terms which are you know exponentially decaying, right and therefore, if I put the limit as t goes to infinity I will just end up here, right. So, limit t tends to infinity, $\Phi(\mathrm{t},\ t_o)\ x_o$ is the is the origin which is the equilibrium of interest. Again simple example, but good enough to verify what is what is happening here, ok.

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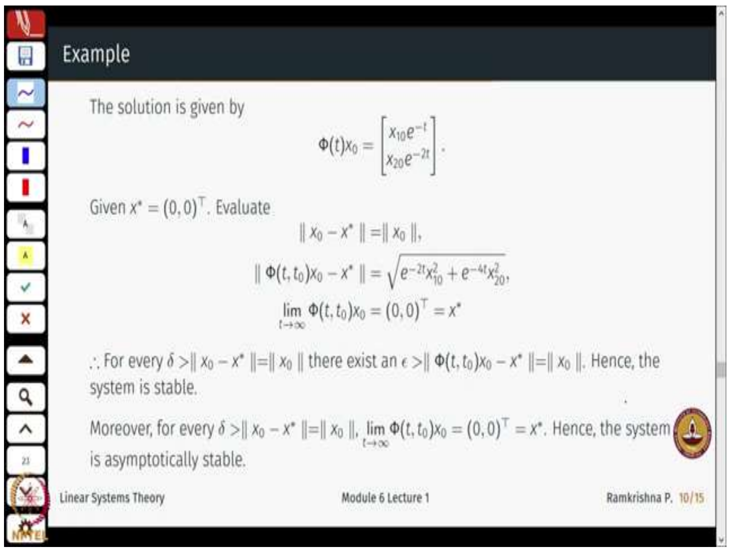

Is that, are we happy when we say this, this is just a statement which concludes that the system is the first that this system is stable? Ok. And second that this system is also asymptotically stable, right. So, so just repeating that and writing down that formally, ok.

So, are we happy with this? Well, one may ask how fast are we upcoming to the origin or how fast are we converging to the equilibrium point. So, whenever I say origin it is also means it also means the equilibrium point. So, this things will be used interchangeable origin, the equilibrium and so on.

So, I would also want my system say for example, it takes forever, right, if it is a if it is a very heavily damped system may be the response is very slow, then therefore, I would also like to see what is the rate of convergence to the equilibrium point. And then therefore, I need to define further a stronger notion or do I can I quantify the rate of convergence and that is where comes in the notion of exponential stability, ok. What does it mean?

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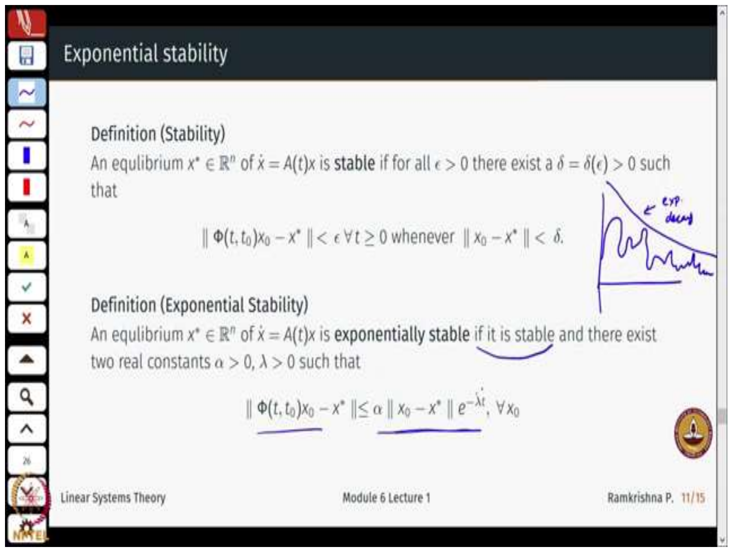

Again, the basic thing is to look at the definition of a stability, ok. When is the system additionally exponentially stable? Of course, first I should check if it is stable or not and then there exist two constants α and λ such that something like this holds, ok. What does this means? Say, suppose I have a solution which starts from initial condition it goes whatever like this like this and may be eventually goes to the origin.

Now, exponential stability would mean can I actually bound this from above by an exponentially decaying curve, right. So, this is this is exponential decay, right. So, if my solutions are always under an exponential curve then this is called an exponentially stable system. Mathematically, well this is an my solutions, so this is my solution is bounded by an exponential function, right. This is exactly what is written here, right. So, this is this will be called the decay rate and so on, right, ok.

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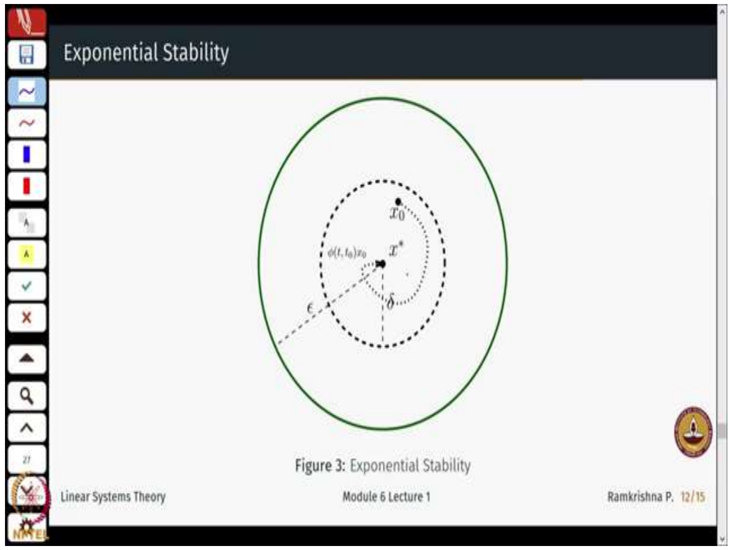

So, I can just write down a picture which looks like this, but I think it is it is it is easier starting from a from what we know earlier, it is now easy to interpret this picture.

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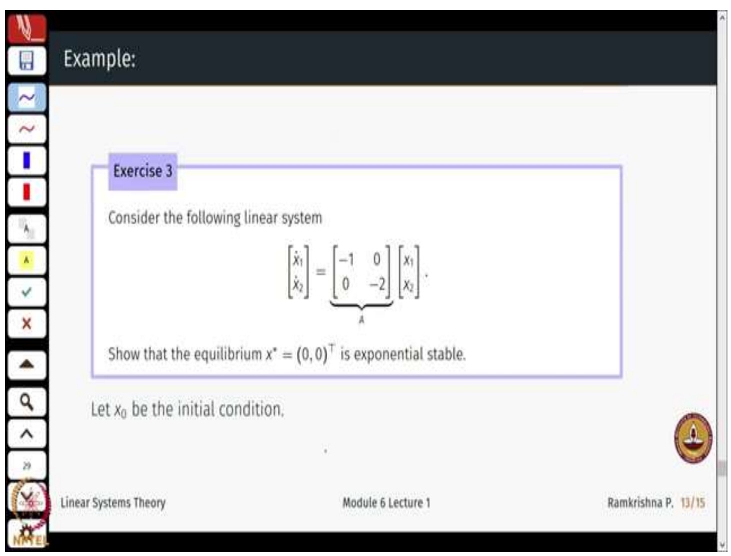

Ok. Again, I come back to this to this system. Now, I want to verify this is exponentially stable and then if it is exponentially stable what could be the values of this α and λ, ok.

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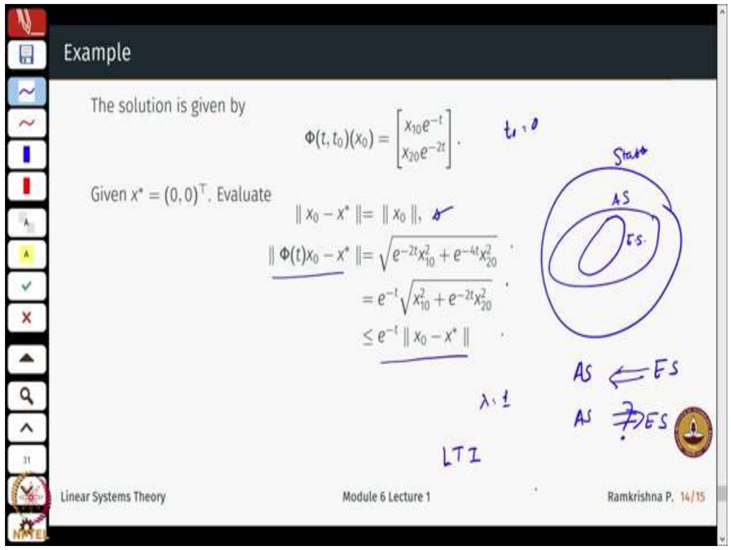

I will again skip the initial, skip the you know little computations. Again, t_o could just be $t_o = 0$, ok. So, again this is this is from previous. So, what does the solution satisfy? $\Phi(t)x_o - x^*$ is just bounded by e^{-t}. So, it decays at a rate of $\lambda = 1$, ok. So, this 3 steps can be can be easily verified. I am just computing the two norms here, so that should be easy.

Now, so far well we know that this is stable, something inside could be asymptotically stable, it is a smaller subset and exponential stable is even harder condition to check, right, so therefore, this is the strongest condition to check would be that of a exponential stability. So, stable is just is a is a bigger set, asymptotically stable systems are stable, exponentially stable systems are asymptotically. So now, relation between asymptotically stable and exponentially stable all exponential stable systems are asymptotically stable, but are all asymptotically stable systems exponentially stable. Well, it turns out in the LTI case, which means the linear time invariant case exponential stability and asymptotic stability would actually like coincide, right and we will do a proofs of this.

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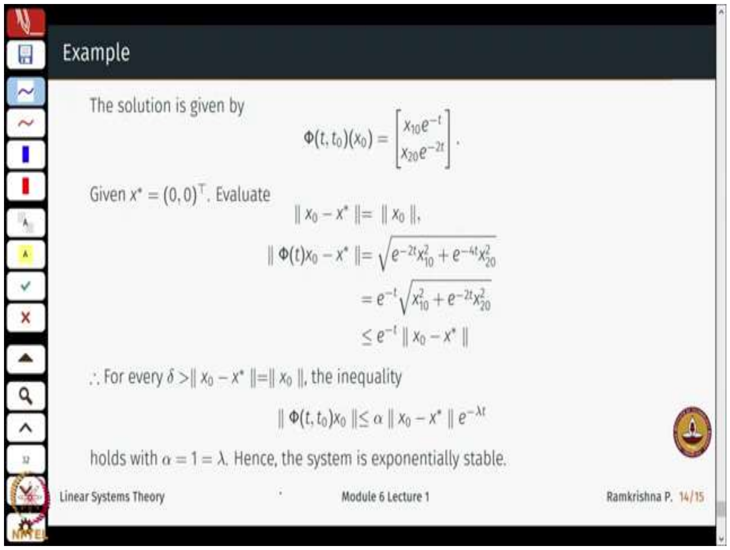

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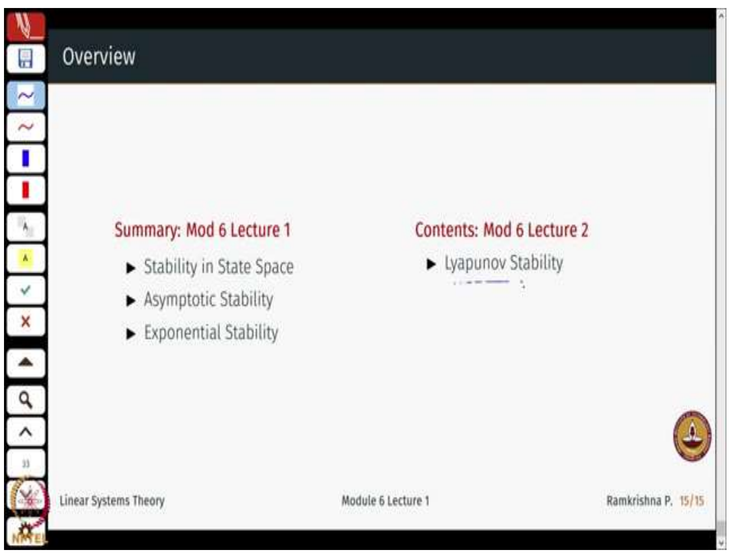

So, I just conclude today by just introducing to you the 3 notions of stability. The basic definition of stability, asymptotic stability which is the stronger version, and something which also tells us the rate of convergence is the is exponential stability.

Now, given a system now how would I verify stability? Say I am given system inR^4 Now, can I all the time compute the solutions, compute this constants α, λ and so on or compute various norms? That may not always be easy. Now, are there effective computational tools which will help us verify stability of systems? Similarly, in the transfer function case, we do not really compute the check if the if the impulse response is absolutely integrable or not, right, we just translate that to an easier way of verifying which is with respect to the poles.

Similarly, in the state space methods we will next look at the notion of lyapunov stability and identify tools which will help us verify stability certificates for LTI systems. Again, much of the tools that we will use will come from week 2 and week 3 lectures of linear algebra and of course, part of week 4 and week 5. So, that will be coming up in the in the next lecture.

Thanks for watching.