Linear Systems Theory
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Module - 5
Lecture - 22
Limit Cycles and Linearization

Hello everyone, welcome to this lecture series on Linear Systems Theory. So, continuing on our Week 5 for discussion. Now so, we started off of defining equilibrium points and characterized several notions of equilibrium point depending on whether trajectories around equilibrium, they converge to the origin, they go away from the origin there could be periodic orbits and all this were related to the nature of Eigen values and we did this with quite some detail in second order systems where the eigenvalues could be real; real with both of them stable, one stable, one unstable and several of them including complex eigenvalues and 0 eigenvalues.

Now so, we will today look at a two important concepts; one is which is essentially seen in the context of non-linear systems and also in the linear systems a bit that is the phenomena called limit cycles and then we will go about looking at linearization of non-linear systems ok.

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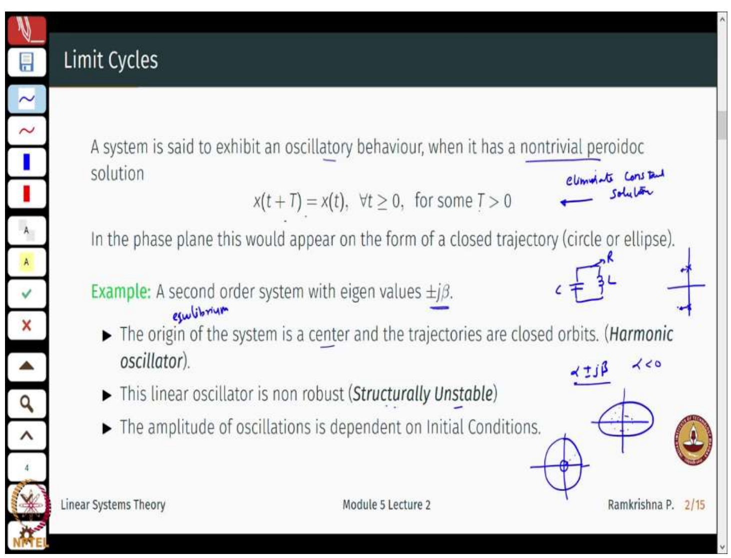

So, just to begin with so of the concept of limit cycles. So, first let us define what is the definition or what is the concept of an oscillatory behavior.

Now so, a system is said to exhibit an oscillatory behavior when it has a non trivial periodic solution that $x(t + T) = x(t)$ for all $T > 0$ and for some time $T >$ zero which is called now the period I am I am calling this non trivial because well which means. So, I would want to eliminate constant solutions even though constant solutions also exhibit this behavior they just constant for all time.

So, they would definitely be satisfying this thing here ok. So, in the phase plane if I am just looking at non trivial periodic orbits, this would so, of non trivial periodic solutions; this would appear in the form of a closed trajectory a circle or an ellipse. So, just recall this just to recall the earlier lecture on of this week. So, this would correspond to system which just had Eigen values of the form $\pm j\beta$ complex eigenvalues where the real part is 0.

So, the origin of the system was defined the origin or we call this the equilibrium earlier. So, the equilibrium of the system is a center and the trajectories were closed orbits and this essentially, we could also resemble like a harmonic oscillator or if we just have a LC circuit. So, this there is some initial conditions, then this will always be oscillating around its equilibrium point.

Now; however, well this linear oscillator is usually a non-robust which means that ok. So, slight perturbations could add some value to the real term in the in the in the complex eigenvalues. And once this is true say if $\alpha < 0$, then the orbits will no longer be periodic, but if $\alpha < 0$, they might just maybe spiral to the origin depending on the values of alpha how large and small they are and so on.

So, a slight perturbation will make it non robust not only that I mean so that. So, this could be in terms of you know some resistances which correspond to the wires which connect the circuit and so on. Not only that its amplitude is also depending on its initial conditions. If I start very close to the origin so, this will just be my amplitude if I start here this will just be the, this will just be the amplitude right.

So, the amplitude depends on the initial conditions. If it is close to the origin the amplitude will just be very small for all times $t > 0$ and similarly if it is far from the origin ok. So, does there exist similar behavior in the non-linear case and in such a way that they could be structurally stable. So, I call this structurally unstable because not because the system is in could be unstable just because the existence of limit cycle can just disappear with some small perturbations right it might just.

So, I am just if I talk in terms of poles and zeros. So, these are my poles a slight perturbation in the system might just push the poles say to the left and once they are slightly to the left or slightly to the right, the limit cycle or the closed orbit no longer exists right so ok.

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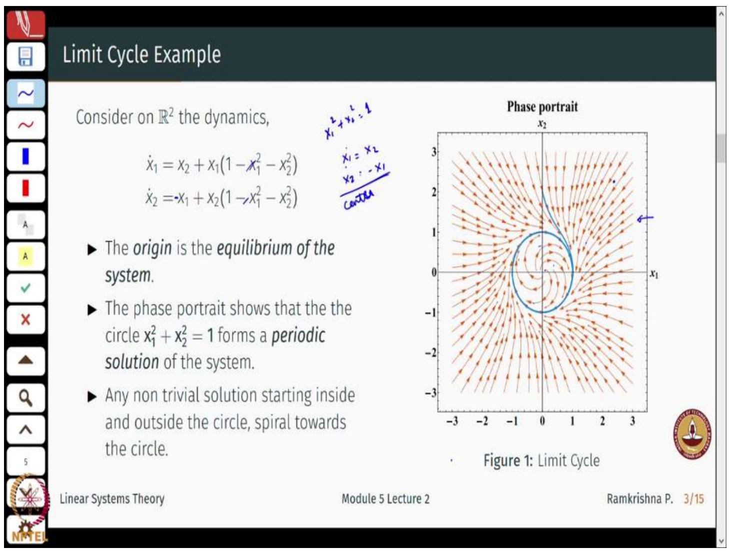

So, what happens when I when I go to a non-linear system ok. So, I have this on R^2. This dynamics ok, I think there should be a minus here ok; sorry for the typo.

So, on R^2 I have the dynamics $\dot{x_1}$ is x_2 + x_1(1-x_1^2-x_2^2) and similarly for $\dot{x_2}$ ok. What is interesting to see here is the following. So, when $x_1^2 + x_2^2 = 1$ so, these two terms go away. So, I have $\dot{x_1}$ = x_2, $\dot{x_2}$= -x_1 and this again corresponds to a equilibrium point which is a center right.

So, you will have closed orbits of this form ok. What is the difference between what we were looking at in cases like this and this is it is the following ok. So, I have so, this is how in general the phase portrait would look like. So, I start from here, the phase portraits would go and converge to this to this to this closed orbit or the limit cycle in this case ok.

So, if I am at the origin, I starts tightly away from the origin and I would still converge to this blue circle here which is the limit cycle of the system. So, now, in summary any non-trivial solution starting inside the circle or I start a circle will converge eventually to the circle.

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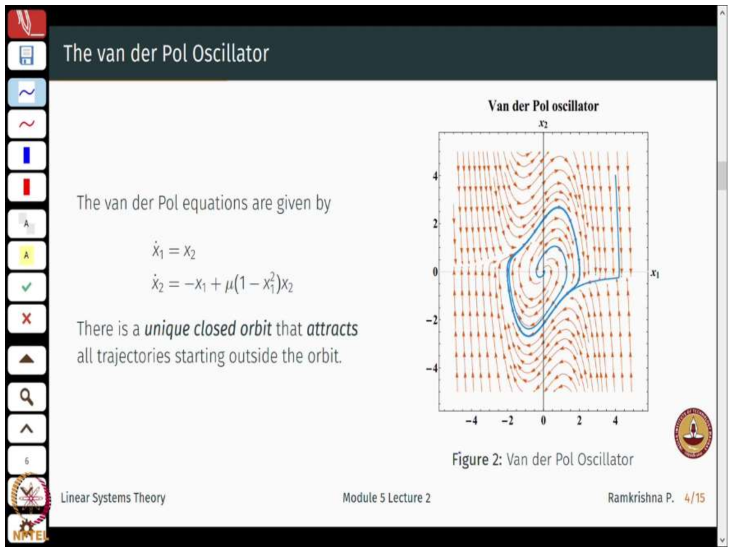

People familiar with circuits would have heard about the Van der Pol oscillator. It is it is a it is a non-linear circuit with dynamics given by the following equation ok. I will not really discuss about the construction of the circuit, but we will be interested in its dynamics.

So, if I draw the phase space, it will still exhibit a unique closed orbit right that attracts all trajectory starting outside the orbit. So, either here or here it will just attract to this blue closed orbit. So just to show that well they need not necessarily be circle or ellipse but they can take any kind of a periodic form. So, this is a very important circuit or a system that has been studied a lot and in physics ok.

So, this is this is just to give you a little illustration of subtle differences between linear systems and non-linear systems apart from the standard definitions of homogeneity and superposition. So, one biggest big difference is in the existence of a of limit cycles. So, here you can just see that there is a unique limit cycle or a unique closed orbit whereas, if I go back to this situation where I have Eigen values $\pm j\beta$, there might just be family of limit cycle.

So, start from initial condition this will be my limit cycle start from another initial condition, this will be my limit cycle; some another limit condition this and so on. Whereas, here give me any initial condition, I will just converge to the bluish circle here or this blue closed orbit in the case of a of a Ven der Pol oscillator. So, that is that is a little distinction when I talk of closed orbits in that linear setting and also in the non-linear setting ok. So, that was about a little qualitative difference between linear systems and non-linear systems apart from the standard definitions that that we would know.

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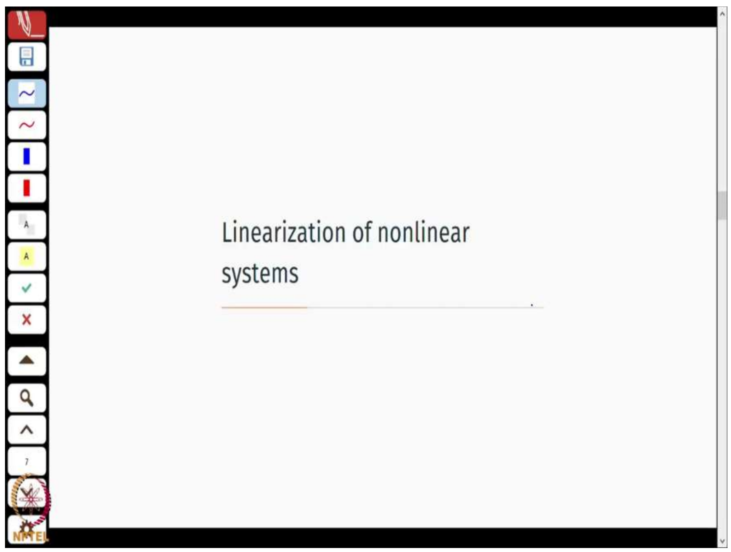

The next concept would be about a linearization of non-linear systems.

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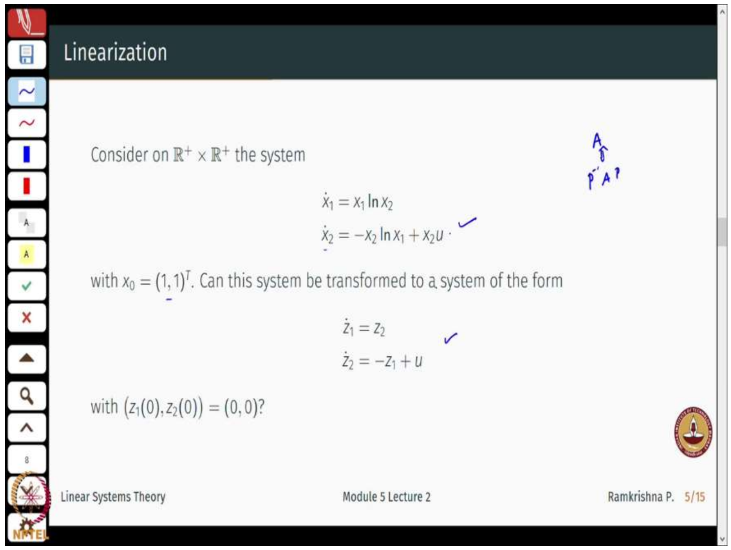

Well if I say I start with this system in R^2, R^+ x R^+, $\dot{x_1}$ = $x_1$log(x_2) and similarly $\dot{x_2}$ = -x_2 log(x1) + x_2u with some initial condition this one.

Now, can I say that well this system is equivalent to this system? In a similar way if I if I take a matrix A, then I do some P^{-1}A P with some similarity transformation I would say that these two systems are similar to each other that their eigenvalues are similar and therefore, there is stability properties and so on are similar.

Now, can I say that this system and this system are the same?

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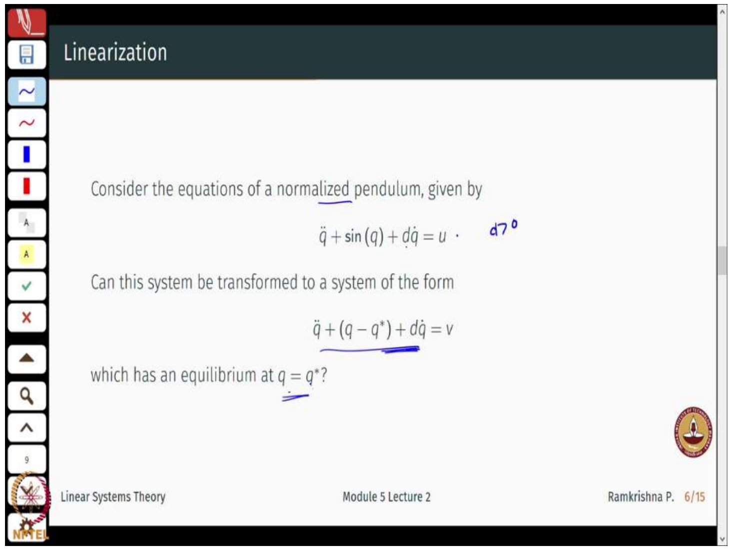

Not only that let me start with the with the example of a non-linear system of this form which is a simple normalized pendulum with the mass the acceleration due to gravity, the length all normalized to one where d is some damping coefficient greater than 0, can I write this system of in this form can I transform this system into this form? Let us say this is this is a linear system right linear system which has a equilibrium at q =q^*.

So, not only not only does this look like a linear system, but I can actually stabilize it to any point in the entire phase plane right starting from 0 to 2π or 0 to-π ok.

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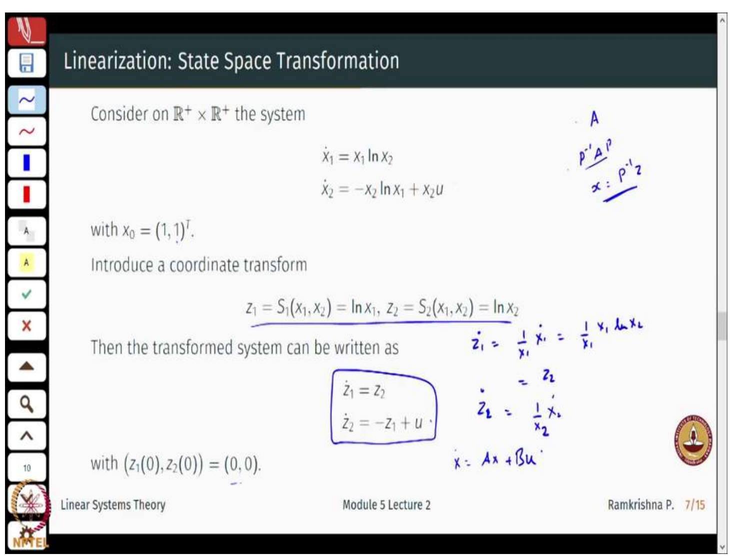

So, come back to the first example right. So,, how we go from a to P^{-1}AP is via a transform x= P^{-1}z or the other way right p times if this is check this.

So, this transformation will transform a system from x coordinates to z coordinates and you can transform it into a diagonal form a Jordan form and so, on ok. Now here I just write look at a non-linear transformation now ok. So, z_1= ln(x_1) z_2 = ln(x_2). Now what is $\dot{z_1}$? So, if I compute this $\dot{z_1}$ = $\frac{1}{\dot{x_1}}$. So, that is $\frac{1}{x_1}$ what is $\dot{x_1}$ = x_1 of x_2. So, this is what is ln(x_2) that is z_2.

Similarly, I compute $\dot{z_2}$ = 1 over $\dot{x_2}$ and I get a very nicely looking linear equation of this form not only that even the initial conditions they transform to the origin. So, one reason one of the reasons that a system can be non-linear is someone wrote it in the wrong coordinates. Now here with a change of coordinates not doing any anything no complicated math or I am not even approximating the system or anything like that I am just doing a coordinate transformation such that the system in the new coordinates looks like a linear system right. This is a very simple linear time invariant system of the form $\dot{x}$ = Ax + Bu ok.

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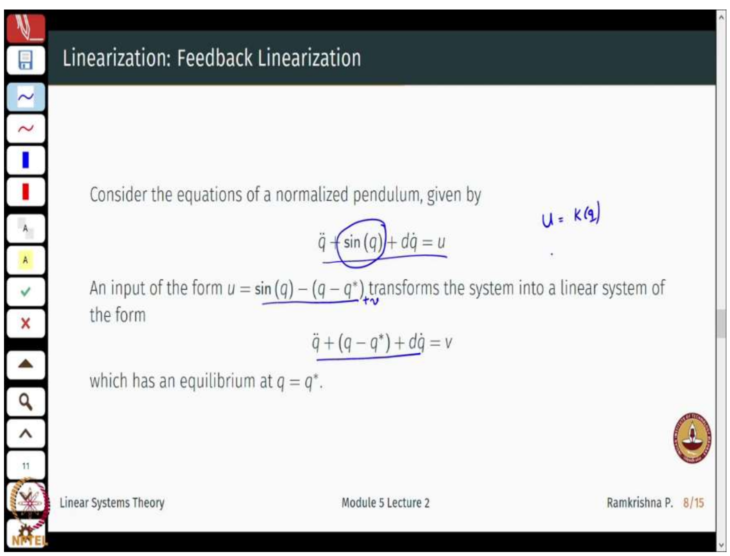

Now, in this example now if I choose a input sin(q) - (q - q^*) maybe just add a plus v here now. So, this will transform the system to a linear system of this form, I can write it in the state space form rights of this. I can convert this second order equation into two first order differential equations ok.

Let us let us go back again to the first example and say ok. The first process of linearization, this is not the first when you start learning linearization right; this is just to give you an idea of linearization processes is via a coordinate transformation I can change. So, this is a non-linear coordinate transformation and it will be invertible and in if I were to use purely mathematical terms this map from X to Z would be a diffeomorphism ok, we will not touch upon those concepts here.

So, I can transform a system from a non-linear system to a linear system via a non-linear transformation. Now does it always exist? Well the answer is no. It exists if and only if certain conditions are satisfied by the system. So, I cannot do this always. So,, I this is not a part of this course of when can I linearize the non-linear system via state transformation that could be a part of some non-linear or geometric control course.

Similarly, here. So, what did I do essentially here is that I used a straight feedback a non-linear function of the state to kill the non-linearity. The non-linearity here comes from the sinusoidal term right. So, this essentially is also called a feedback linearization. Now can I do this all the time? If this was so, simple that I could just kill the non-linearity just by so, mathematically here I am just adding a sinus term to the input and it just the non-linearity disappears. Now can I do it all the time? Again the answer is still no. There are some conditions that my system should satisfy or the structure of the system should satisfy in order to accomplish this feedback linearization. So, there are several non-linear examples in literature which cannot be a feedback linearizable of course, that is not again our point of interest ok.

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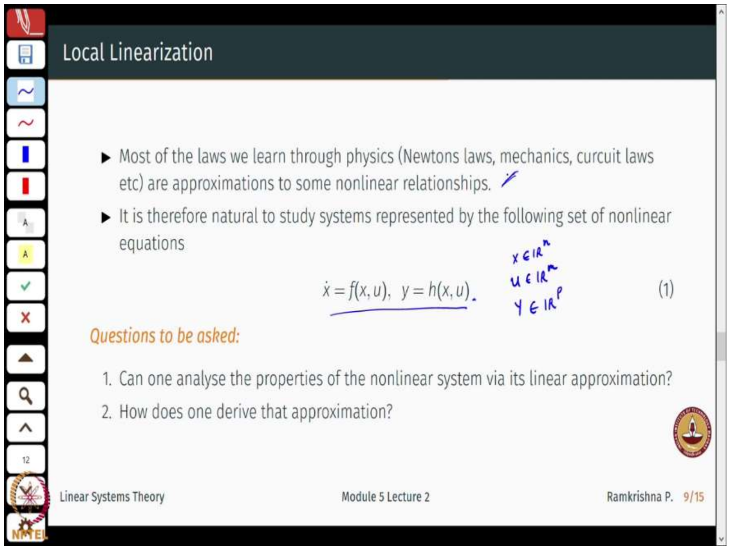

So, what do we in generally learn so far right? When we say start even with mechanics or circuits. So, most laws, we learn through physics right the Newton's laws circuit laws and so on. They are approximation of certain non-linear relationships. So, in the week 1, the first lecture we saw how p = mv was restricted to a certain range only and similarly with inductors and so on.

So, a natural way to study systems is by the following set of equations $\dot{x}$= f (x, u) and y = h(x, u) where x is in see some n dimensional state vector, u is some m dimensional input, Y could be some P dimension output ok. Now so, I so, naturally I know that systems are non-linear even though I learn lot of physics a lot of circuit theory or mechanics through the through a linear approximations of non-linear systems ok.

So, the questions that we will be interested is can one analyze or retain some properties of non-linear system via its linear approximation and how to get that approximation? So, here I am not really doing an approximation I am actually doing a proper transformation, this is not approximation; this is valid so, in a for a larger. This is valid in a in much larger region of the state space than just around a small operating point.

So, these are more systematic processes, but can we do something simpler just to suit out purposes at the movement and to also answer some really basic questions ok. So, let us go back to start with the setting of non-linear setting $\dot{x}$ = f (x,u).

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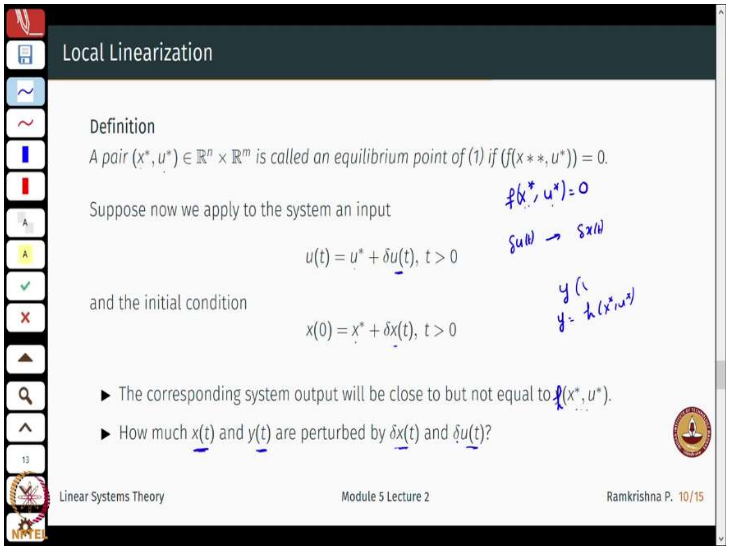

And so, the definitions is the pair x^*, u^* is called an equilibrium point. If f sorry this should read f (x^*, u^*)= 0 or f evaluated at x^*, u^* will just be 0 ok.

Now so, once I am at this point x^*_u^*; I will always be at 0 ok. So, let us say now I apply a slight an input to the system_u^* + δu(t) this is a arbitrarily small perturbation in the input away from_u^* and similarly from the initial condition right. So,, let us say the initial condition is not x^*, but some small perturbation with x^*.

You can also look at it as just applying a small input u of delta u of t and see how it causes a small variation delta x of t ok. So, this is the assumption we make right that that these variations are really small again. So, in this case, the corresponding system output which was y of; sorry it should be x. So, y which was h (x^*, u^*) with these new conditions will be close to, but not equal to the origin output ok.

Now, I need to check how much is the perturbation in X because of these two, what happens to X. So, initially X was just 0 right for all t if I start with at the equilibrium point, I will always be at the equilibrium point. So, when I apply these small perturbations δu. So, what is the change in x and what is the change in y or in other words how much are x(t) and y(t) perturbed by these δx and δu.

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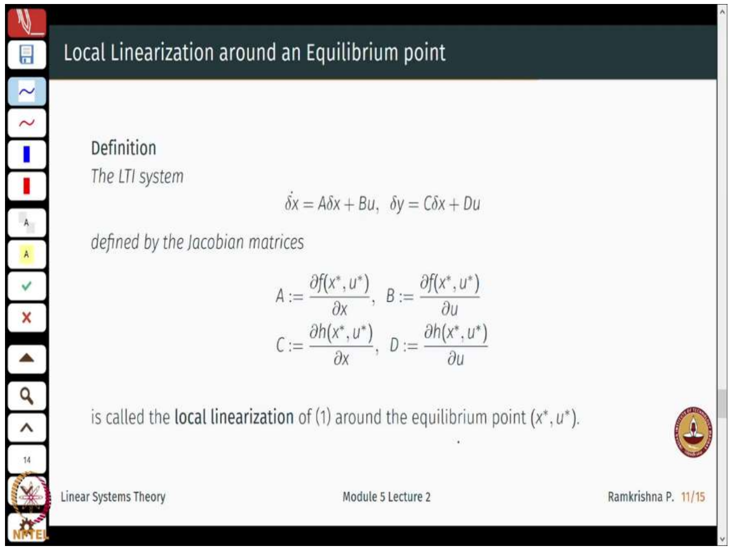

So,, the definition, I will do a quick derivation of this that the linear system $\dot{\delta x}$=Aδx + B$\underline{u}$ δy = Cδx + D$\underline{u}$ defined by these matrices. I will shortly tell you how to derive this. So, this system is called the local linearization of the non-linear system around this equilibrium point. So, whenever we talk of linearization, we will talk linearization around certain operating point right ok.

So, how do we derive for these things?

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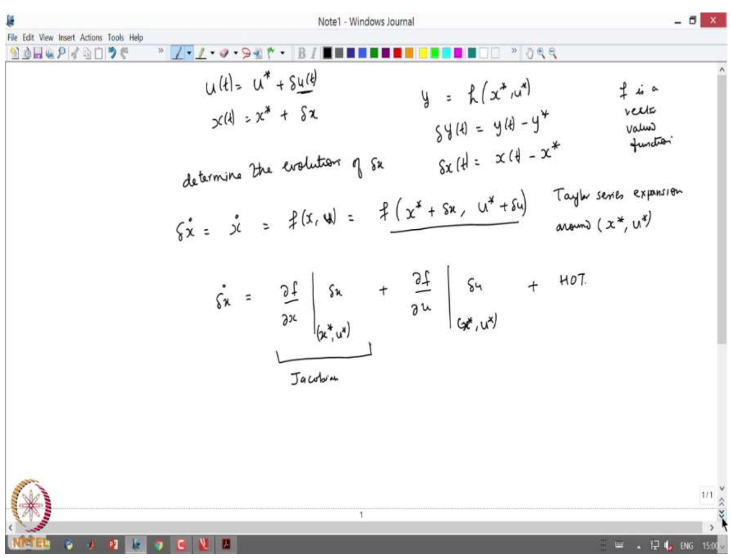

So, the input that I apply is u(t) = u^* +δu(t) right this is very close to the initial condition, but not equal to right. Similarly x(t) = x^* + δx right again this is close to, but not equal to to x^* ok. So, similarly I will have a corresponding change in y = h(x^*, u^*). This is the one at the equilibrium, but my y will be some y + δy.

So, y(t) here would be or sorry that the change δy would be y(t) - y^* at the equilibrium point. Similarly the change in x would be x(t) - x^* ok. So, we will now deter mine the evolution of delta x and similarly we will do for δy. So, $\dot{\delta x}$ is from here is $\dot{x}^*$ will be 0 because it is it is it is a constant this is f(x,u). Now how do this x comma u vary? They vary according to this thing (x^* + δx, u^*+ δu) ok.

Now, I can do our a Taylor series expansion for this ok; taylor series expansion around x^* and u^*ok. So, this will give for me delta x dot is the partial derivative of f with x evaluated at x = x^*, u = u^* times δx plus the partial derivative of f with u, also evaluated at x star u^* and a δu plus all the other remaining terms which I will just call the higher order terms in δx and δu.

So, this is called the Jacobian matrix also. So, you are actually differentiating a vector valued function. So, f here is vector valued function ok. So, how to compute this I will just do a little illustration So, that it is a little easier to understand.

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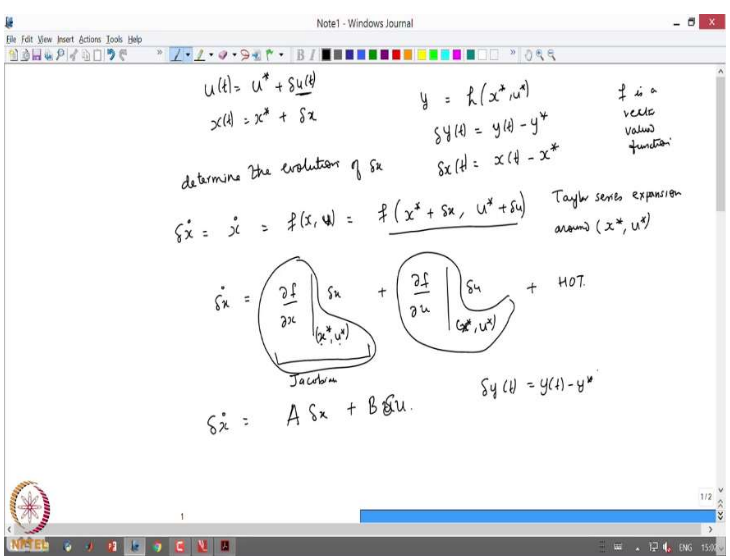

So, let us say I have a system in R^2 would be x1 dot $\dot{x_1}$ = $f_1(x_1, x_2)$, $\dot{x_1}$ = $f_2(x_1, x_2)$ right. So, this x is in R^2 and f is also in R^2 ok.

So,. So, what is; so, this is like a in general we write this as $\dot{x}$ = f(x) right where x is x_1,x_2 and f is f_1 and f_2. So, $\frac{df}{dx}$ will be the following matrix partial of f_1 with x_1 the partial derivative of f_1 with x_2 partial derivative of f_2 with x_1, partial derivative of f_2 with x_2 and similarly if you have u, it will be it will be the same right.

So, this is how we compute the Jacobian matrix or the gradient of a or the derivative of a of vector valued function and you can just extend it to n dimensions right. So, it will it is a simple straightforward extension right. So, if I now go back so, I have $\dot{\delta x}$ = Aδx where A was $\frac{df}{dx}$. So, this that is what you see right. So, $\dot{\delta x}$= $\frac{df}{dx}$ evaluated at A. So, this is A, this entire thing here Aδx. These are all constant right because you are evaluating them at particular points x^* and u^*. So, they just be a constant matrix and similarly here.

So, this will be Bu again its evaluated at x^* and u^* and so, it will be a be a constants it will be δu ok. Similarly I can do with the output equation δy(t) was y(t)- y^* and I can just do the do it very similarly and then find out what is δy. I will I will skip that that little derivation ok.

So, C would just be a partial of h with x and D would be partial of h with u all evaluated at x^* and u^* ok.

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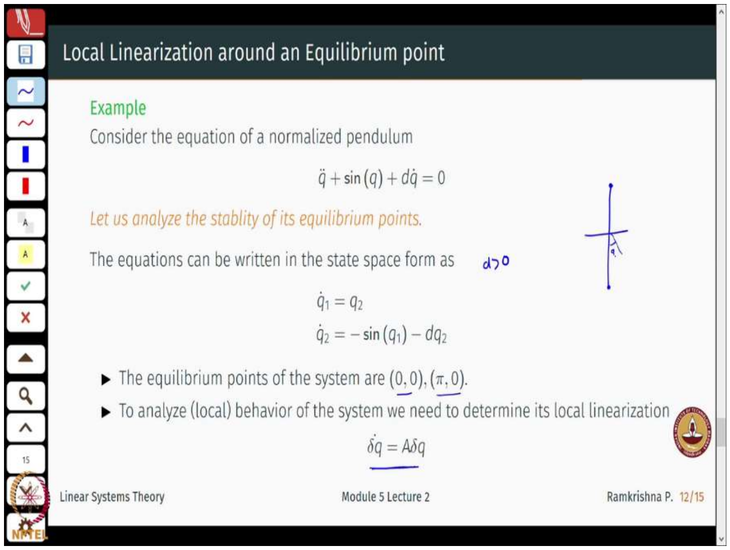

It is well it is it is interesting to see a little example here right. So, first is I just look at the equations of a normalized pendulum and let us see what is a linearization tell us here ok. So, first I write this in the state space form $\dot{q_1}$ = q_2, $\dot{q_2}$= -sin(q_1) - dq_2; again d is a is a damping coefficient greater than 0.

So, the two points equilibrium points of interest are (0,0). So, this the downward position of the pendulum is an is an is an equilibrium point and the upward position is also an equilibrium point and then this will be my q ok. So, any other would be just be a an multiple of these values of 0 and nπ. So, for example, 2π is also an equilibrium, 3π is also an equilibrium point and so on. So, these are the two equilibrium points of practical interest.

So, to analyze the local behavior we first look at its local linearization that is I look the equations in terms of $\dot{\delta q}$ is Aδq ok.

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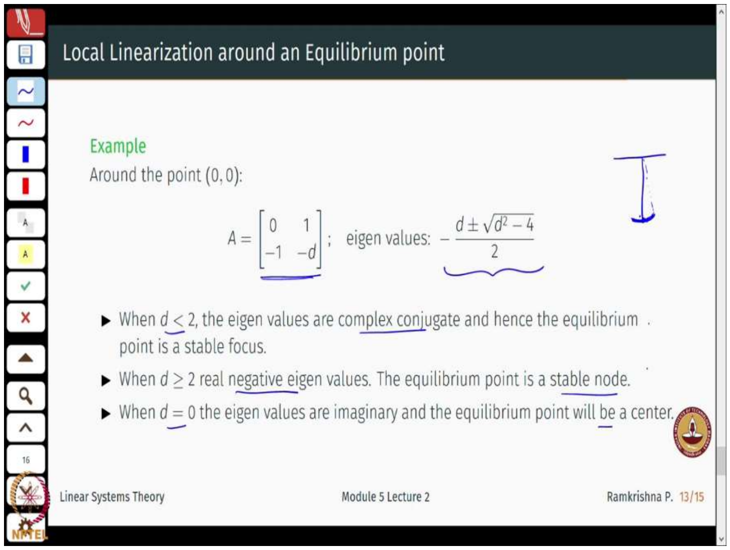

So, I just do the linearization of this. So, I will just quickly run you through this.

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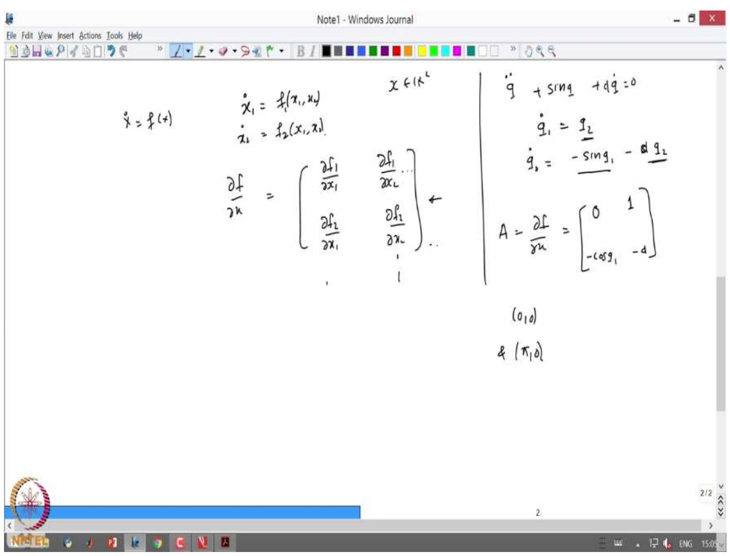

So, the equations were $\ddot{q}$+ sin(q) + d $\dot{q}$ = 0 or if I write it in the first order differential equations, $\dot{q_1}$ = q_2, $\dot{q_2}$= -sin(q_1) - dq_2 ok. So, this what is here ok.

Now, let me compute the A matrix for this A is $\frac{df}{dx}$ evaluated at the equilibrium point. So, d f by d x here is it is a $\frac{df_1}{dx_1}$. So, $x_1 x_2$ or the q_1 q_2 here; so, this will be 0. So, I am looking at the at the partial derivative of this term with respect to q_1 that is 0 with respect to q_1 that is 1. Partial derivative of this term with respect to q_1 will appear here as $\cos(q_1)$ partial derivative of this term with respect to q 2 will appear here -d and now I compute this at (0, 0) and (π, 0).

So, these are these are my two equilibrium points ok. So, at (0,0) I get my equilibrium point. So, my a matrix to be something like this and its corresponding Eigen values are can be computed to be like this. So, when d < 2, the Eigen values are complex conjugate and hence the equilibrium is the stable focus ok. So, similarly when d ≥ 2, there will be real and this Eigen values both of them will be negative. The equilibrium point here would be called a stable node when d = 0 the Eigen values are imaginary and the equilibrium point will be a center.

So, physically this would mean that I just have a pendulum which is stable at its downward position, I just perturb it slightly it will re come back to its original configuration as long as d is greater than 0. How fast or how slow will it come back will again depend on the amount of damping in the system right. So, this is a clear relation to what we studied and also physically the this is also intuitive right that the downward position is always the stable position and d = 0 which means there is no damping a small perturbation, I will just keep on oscillating around the equilibrium point right ok.

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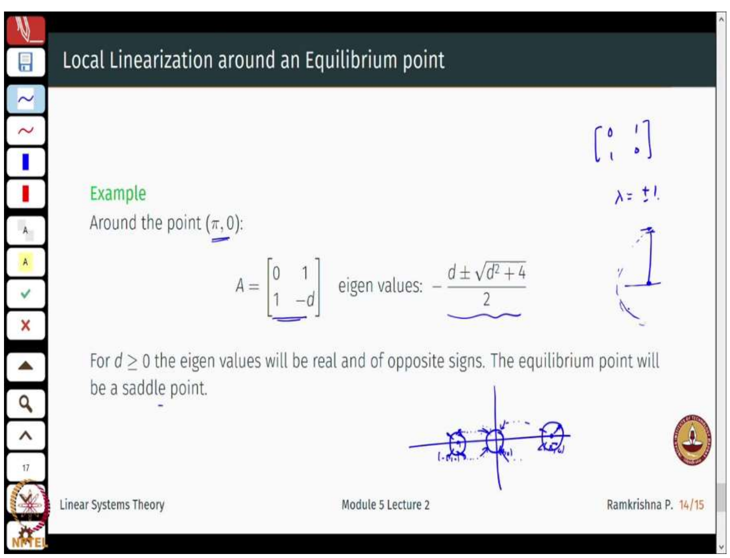

Now, interesting thing will happen when I am looking at in linearization around the point (π, 0) in which case my A matrix takes this form and Eigen values are like this ok. So, for

$d \geq 0$, the Eigen values will be of real and opposite signs and the equilibrium point will be a saddle point. So, I can just quickly compute for 0 1 0 where $d = 0$ the Eigen values would be a ± 1.

Now, let us let us so, what it turns out is a $(\pi, 0)$ is a is an is an unstable equilibrium because it will have Eigen values which are plus and minus one and this is a is a stable Eigen value. So, if I roughly plot the phase space here. So, this would be a (0, 0), $(\pi,0)$,$(-\pi, 0)$ and so on.

So, around here the territories will tend to the origin in this way either they will spiral or depending all on the values of g, around here the territories will diverge away from the origin, around here also; territories will diverge away from the origin they will meet here and then come back here trajectories from here will go from the origin and again soon. So, locally what I see is that if I compare with the non-linear phase space, locally around the equilibrium point the behavior is the same right and this is also a little intuitive right. So, if I take a pendulum on the upward position and I give a set perturbation it will just come back here right so; that means that this is an unstable equilibrium.

So,, the definition of stability which we will do in the next weeks lectures more formally is a slight perturbation from the equilibrium point does it if the system comes back to its original configuration, then it is a its a stable equilibrium; otherwise it is an unstable equilibrium. Also stability we will talk essentially in terms of equilibrium points only ok.

So, this was a little hint about our little introduction to linearization around an equilibrium point a local linearization. So, you can for just exercise as an exercise check these two examples here which we begin with. So, here origin is an equilibrium point, just check the linearization around the around the equilibrium or the or the origin and check what kind of what is the characteristic of this equilibrium in terms of a center or saddle point or stable node, unstable node and so on.

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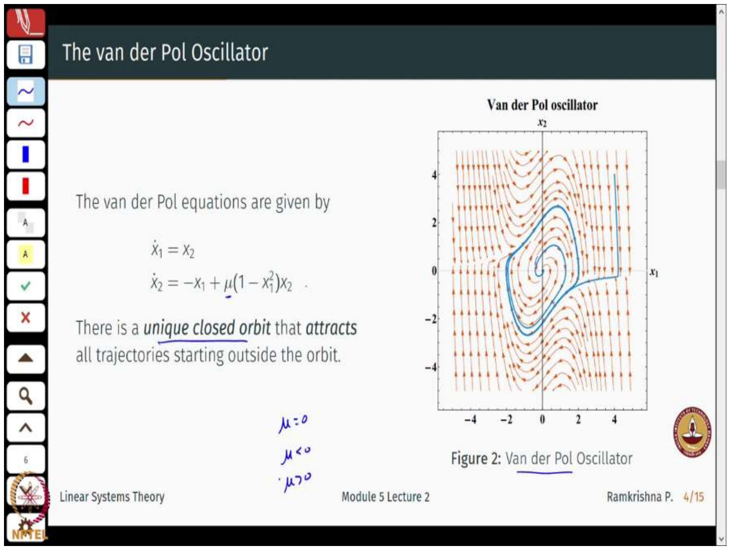

Similarly, you can do for this you can also play around a bit with μ = 0, μ < 0 or μ > 0. So, it is just a very basic mechanical exercise. So, I will not really do problems on this, this is just a computing basic derivatives and then and computing those Eigen values and mostly we will just deal with second order systems ok.

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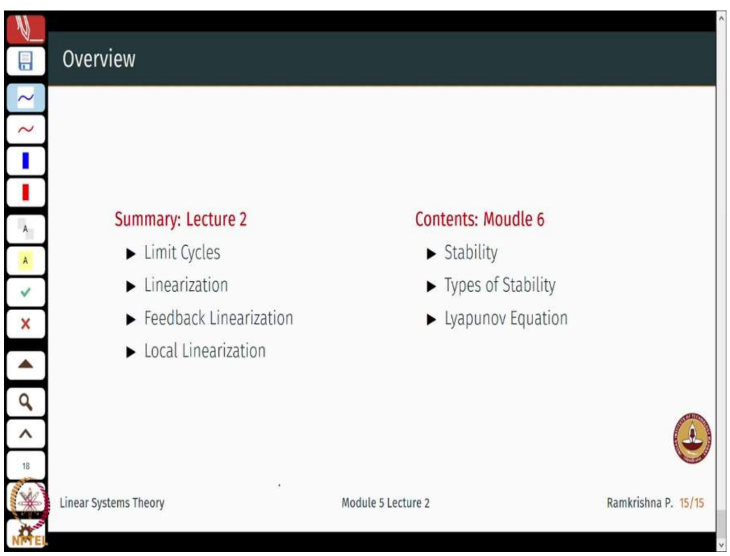

So, that that concludes the lecture on linearization. In the next week thus that will be a very comprehensive discussion on stability, types of stability from asymptotic stability to just stability to exponential stability and so on. And also look at some how do we verify stability of system when I look at transfer functions, I just say well complete the poles. If the poles are to the left, then the system is stable if the poles are on the imaginary axis and there are certain conditions that that need to be checked and so on. So, are there techniques; what are the techniques that we encounter while we study about stability of systems in the state space domains both in the continuous time and discrete time. So, that will be coming up in week 6.

Thanks for listening.