Linear Systems Theory
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Module - 05
Lecture - 1
Equilibrium Points

Hello everybody. Welcome to this 5th week of lectures on Linear Systems Theory. So, this week will be a little shorter module, but we will focus on some nice qualitative behavior of systems and these are essentially to do with Equilibrium Points of the system.

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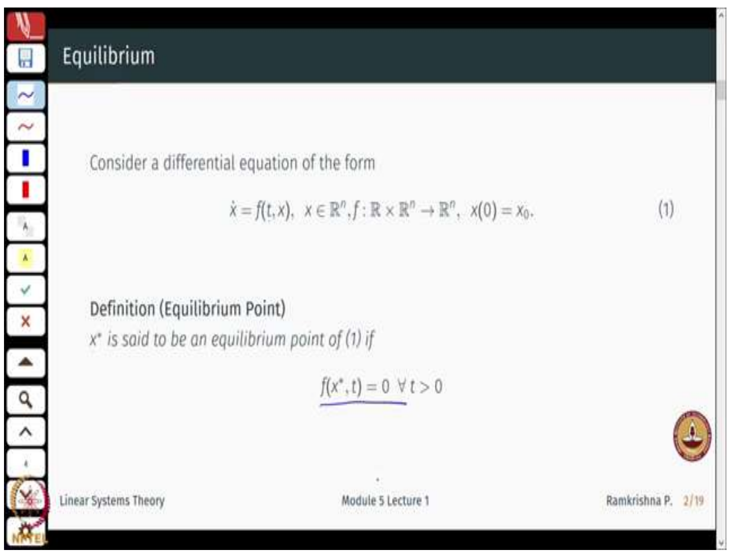

So, well we would have encountered these definitions some times during our previous courses, but we will give it a general setting here of the kind of systems that we will be dealing with. So, if I consider a differential equation of the form. So, this could be non-linear and also I put the t in just to also take into account the time varying nature of it.

So, as usual x comes is an n dimension vector, f is a vector field from R x R^n to R^n with some initial conditions ok. So, the basic definition of an equilibrium points is the following that x^* is an equilibrium point of this system. One if it satisfies this equation right so, f$(x^*, t) = 0$ for all t > 0 ok.

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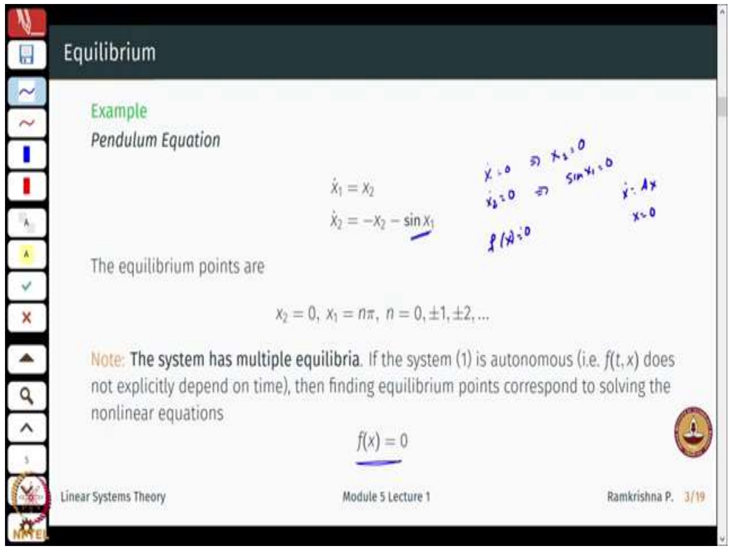

So, what does this mean? So, we will go through few examples to see different kinds of equilibrium that we will encounter. So, in the case of a simple pendulum where everything all the parameters are normalized to 1; I have an equation which looks like this ok. So, equilibrium points are 1 where you know I just say $\dot{x_1}$= 0, $\dot{x_2}$ = 0 and so on.

So, what do I get from the first equation is that so, this will imply that x_2 = 0 which is a kind of to check that the velocity would be 0 at the equilibrium and this would mean that $\sin(x_1)$ is=0 which means it will have a variety of solutions right. So, starting from say π, 2π and so on; so, all this multiples of π.

So, what is a different or a unique about this system is that it has multiple equilibrium. So, if I were just to look at a linear systems, we were essentially looking at $\dot{x}$ = Ax and if A was full rank or invertible, then the origin was the equilibrium. So, this to begin with is a non-linear system, the on nonlinearity appears in this term here and this system is seen to have a multiple equilibrium.

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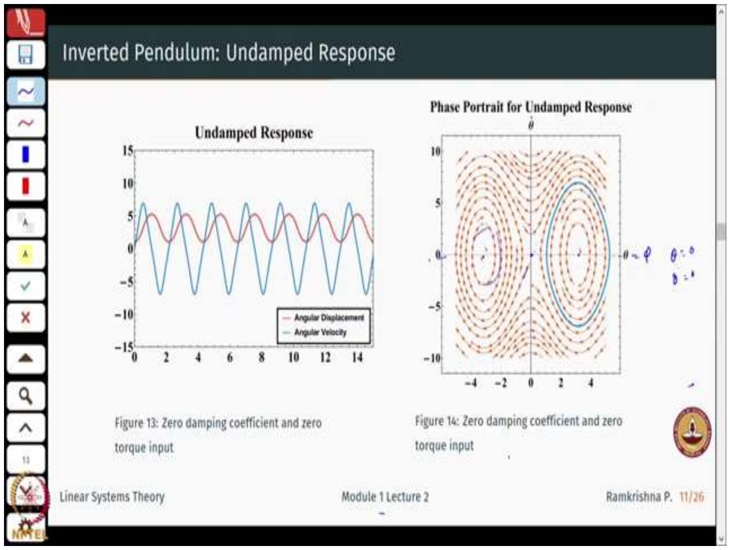

So, if I relate to the phase space that I had drawn earlier for the case of undamped response so, you can see here right. So, this point here is a equilibrium sorry. So, you can see a couple of equilibriums here and one at the origin and so on ok; sorry this is behaving weird ok. So, if I go back to the phase space which I had drawn in one of our earlier lectures, you can see that this actually corresponds to set of different equilibrium point starting point here, you have an equilibrium point here and so on if you keep on progressing to the right and the left.

So, this is a typical case of a system or a non-linear systems which has multiple equilibrium points. What are the nature of this equilibrium points do each of the equilibrium points exhibit the same behavior or not that we will see in the due course of this lecture ok. So, in general so, if so, the we started off with the system f (x,t), but if the system is autonomous that which means that that the system does not explicitly depend on time, then finding equilibrium corresponds to solving just the non-linear equation $f(x) = 0$; same in the case of a pendulum right. So, this was an autonomous system and I was just solving for $f(x) = 0$ ok.

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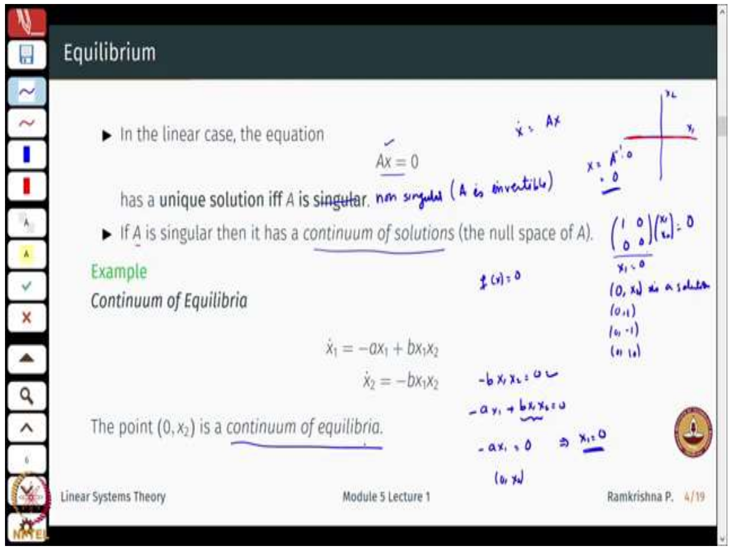

So, is it all very obvious in the linear case that the origin is always the equilibrium point. Well, the answer turns out to be right not very obvious, but we will look at couple of examples. So, in the linear case the equation a x = 0. So, my dynamics are the form $\dot{x}$ = Ax, I said $\dot{x}$ = 0 (Refer Time: 04:55) solving for this equation if Ax = 0. So, this system has a unique solution if and if there is a typo here A is nonsingular or A is invertible.

So, in case when A is invertible, I can just write that x is $A^{-1}0$ that is 0. So, the origin is the unique solution of this equation if and only if the matrix A is invertible. So, on the other hand if say A is singular, what happens if A is singular? So, let us say I take an example like this right x_1, x_2. I am looking for this to be 0 if a solve for this what do I get from the 1 that x_1 = 0 and I do not get any expression for x 2 which means that (0, x_2) is a solution to this.

So, what do I mean by this at any value right, it could be (0, 1) is a solution (0, -1) ,(0, 10) and so on are the solution to this equation. Any value of x_2 with x_1=0 is a solution if I were just were to draw it here x 1 ,x 2. So, x_1 = 0 and x_2 being any values. So, this entire horizontal line is a solution to Ax = 0. In this case or it also means that if A is singular then it has a continuum of solutions right. So, this is a sorry.

So, this entire line here is the continuum of solutions for this for this set of equations or the or for the system which is represented by A of this form ok. So, this is also the null space of A. So, here in the in the case when A was invertible or nonsingular, then the null space is just the trivial point like this that x = 0 is the space. Well this phenomena can also occur in the non-linear case. So, if I have a system which looks like this $\dot{x_1}$= -ax_1.+ b$x_1 x_2$, $\dot{x_2}$ is so on.

So, I just look at the solution of what is the equilibrium just look at the solutions of f(x) = 0. The second equation will give me -b$x_1 x_2$ = 0 first equation is -ax_1 + b$x_1 x_2$ = 0. From here I already know that b x_1 x_2 is 0. Therefore, I am left with just this equation which means that x_1 = 0. So, this is the only thing that I can derive from this equation and therefore, 0 and any x_2 is a solution to this equation of is an equilibrium point for this for this system and therefore, this system also exhibits a continuum of equilibrium.

An interesting case that throws to us lots of insights in to understanding equilibrium points is essentially with second order systems.

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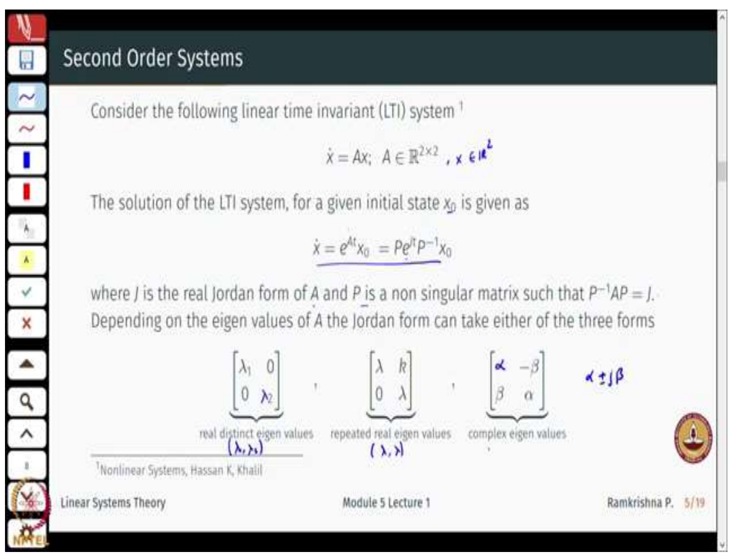

Well let us just say I am dealing with second order linear systems and this linear system could be just be linear by itself or it could come as a process of linearization of a non-linear systems that we will do in the next lecture proceed succeeding this lecture.

So, let me just make things simple here and say I am just dealing with a second order linear system which means $\dot{x}$ = A x, A is in R^{2x2} x is a two dimensional vector ok. So, for any given some initial state x_0, the solution will always take this form and now we know how to compute e^{At}. We can also do it via its Jordan form and so on right. So, e power At and here j is the real Jordan form of a P is a nonsingular matrix which takes it from a given form to its appropriate Jordan form right and P is of course, nonsingular and we also know how to derive this matrix P ok.

So, depending on the nature of eigen values, the Jordan form can take several forms in this case essentially it will take three forms. So, first is when the eigen values are real and distinct. So, this will just be the Jordan form will just be a diagonal also this should be a λ_2 here which means the eigen values are just λ_1, λ_2. In this case, I just realize a nice natural diagonal form.

In case the eigen values are repeated like lambda and lambda are my eigen values, then the Jordan form can take can be something like this where k can either be 0 or one depending on the on the multiplicity of the geometric multiplicity of the eigen values third thing is when I have complex eigen values where $\alpha \pm j\beta$ also ok; listen alpha missing here. So, we will have in this case complex eigen values ok. What do each of these eigen values signify? These are just information on stability or there is a little more of information than that and when do these cases actually occur ok.

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So, first let us start with the case of real eigen values ok.

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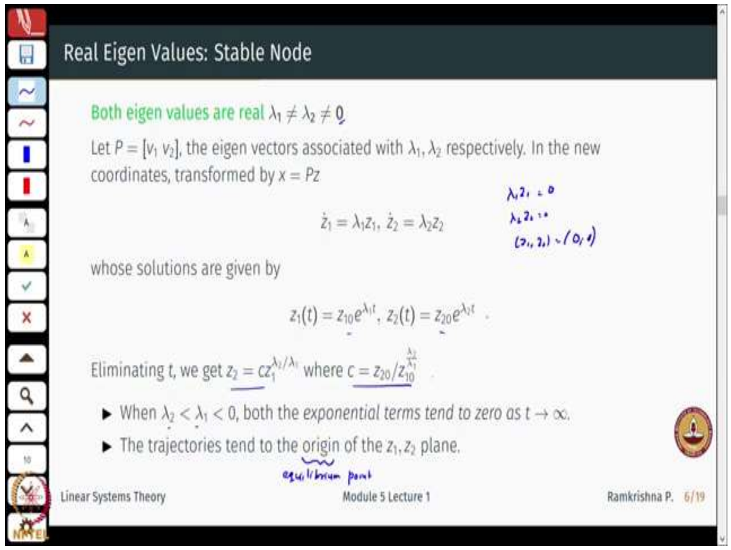

What does it mean both eigen values are real ah? They are both non-zero, we will come to the case of 0 eigen values a little later ok. So, let us say I have this set of eigen values, let me just derive this and I come and come back to this slide.

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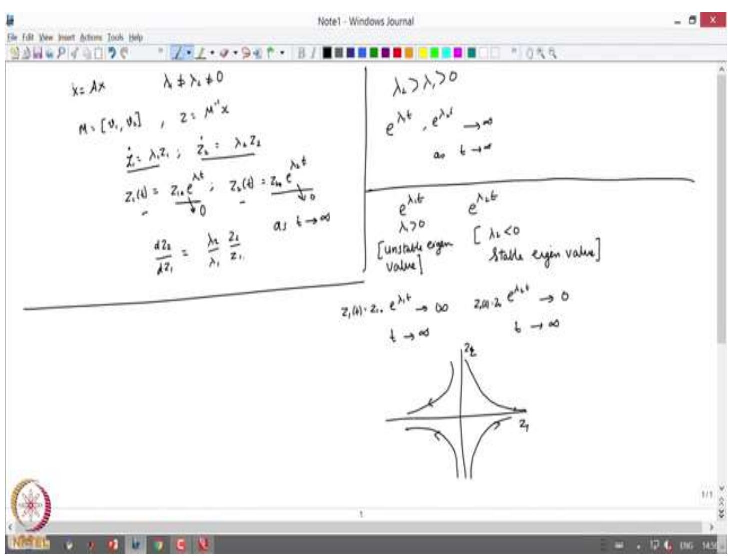

So, I have λ_1 and λ_2 which are not equal to each other and which are also not equal to 0 and again I am looking at a system with $\dot{x}$ = A x.

Now, I know in this case that by taking its eigen vectors v 1 and v 2 and a coordinate transformation which looks like this. M^{-1}x, I can transform this system into a diagonal form and the diagonal form looks like this $\dot{z_1}$ = $\lambda_1 z_1$ and $\dot{z_2}$ = $\lambda_2 z_2$ and this will have solutions z_1(t) = $z_{10}e^{\lambda_1 t}$ similarly z_2 of t is $z_{20}e^{\lambda_2 t}$ ok.

So, this is just how the solutions would look like now depending on the values of lambda whether they are greater or less than 0. These solutions will either increase exponentially or will decrease exponentially and tend to the origin ok. So, back to here so, so, this is what we have now right, the solutions in the diagonal form are just given by this. Now, I can just eliminate t to write my equations of this form this is this a little straight forward to check also how do I eliminate t.

So, I have this two equations right $\dot{z_1}$ = $\lambda_1 z_1$, $\dot{z_2}$ = $\lambda_2 z_2$. So, I just have $\frac{dz_2}{dz_1}$ can be written of the form $\frac{\lambda_2 z_2}{\lambda_1 z_1}$ and I can do all the all the calculus that I know solving integral equations and I just end up with a with a solutions like this with C depending on the initial conditions and so on ok. So, this is this is like easy to check ok.

The first case which we would be interested is when $\lambda_2 < \lambda_1$ and both are negative right. The first observation is since both are negative. So, I will it is easy to check that these will go to 0 this term will also goes to 0 as sorry as t goes to infinity ok. This is this kind of obvious and if I look at in the z_1 - z_2 plane. The trajectories tend to the origin like. So, here z_1 also 0 and z_2 also goes to 0 and of course, in this case the 0 turns out to be the equilibrium point. So, the origin is the equilibrium point.

Just you can substitute ($\dot{z_1}$, $\dot{z_2}$) = (0,0) and end up with equations $\lambda_1 z_1$ = 0 $\lambda_2 z_2$ = 0 and therefore, (z_1, z_2) is (0,0) is the equilibrium point ok.

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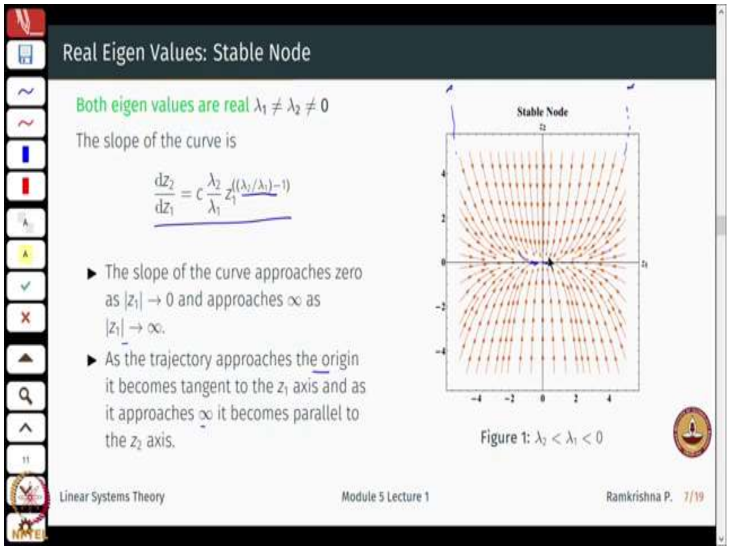

So, what happens in this case? So, if I look at look at so, what I know now is that these in the z_1-z_2 plane, the trajectories tend to 0 as times progresses right or asymptotically ok. So, what how do they actually do that? So, from my equation relating z_1 and z_2, I can compute the slope of this line right how does z_2 change with respect to z_1 right that is what we that that the derivative is essentially the slope of it. So, this is a positive number. The slope of the curve so, it is easy to check from here that the slope of the curve approaches 0 as z 1 goes to 0.

Second is the slope approaches infinity as z_1 goes to infinity right that is what is happening here right. So, as the trajectory approaches the origin, it becomes tangent to the z_1 axis. So, if we look at these things as the approach the origin they are becoming tangent to the z_1 axis and away from the origin, they will be parallel to the z_2 axis right at really at infinity right. And, the second observation is as it approaches infinity, it becomes parallel to the z_2 axis. You can you can just plot this for yourself and check.

So, we have put up already the code to help you draw face portraits of this form. If you have already done this in our week 1 lectures a bit of it starting with phase space. So, this is also a continuation to that there we really did not talk of equilibrium points and the nature of them. But, slowly we will get to understand that the kind of things that we are doing today will essentially relate to stable equilibrium points unstable equilibrium points if I am talking on stable I am looking at an under damped situation, I am looking at an over damped situation, critically damped and so on right ok.

So, this is nice here right. So, I just see that all the trajectories are actually converging to the origin. So, it means that if I am at the origin I will always be at the origin and so, but if I am slightly perturbed here say. So, if say end up at this point here, then I will slowly come back to the origin right.

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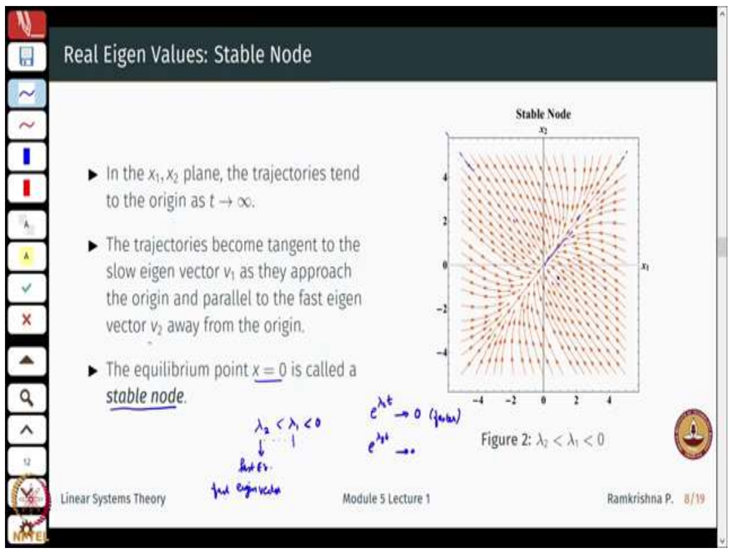

So, let us come go back to the x_1- x_2 plane and check what is happening here ok. In the x_1- x_2 plane, the trajectory is tend to the origin again as t tends to infinity ok. Now, we have here eigen values which are $\lambda_2 < \lambda_1 < 0$. So, we call this as so, if this condition is true, then the term $e^{\lambda_2 t}$ converges to the origin faster than $e^{\lambda_1 t}$ ok. So, this is faster ok. So, we call this is the fast eigen value and then the corresponding eigen vector as the fast eigen vector and similarly the with the slow eigen value and the slow eigen vector.

So, as times increases, the trajectories become tangent to the slow eigen vector in v_1 and as they approach and parallel to the fast eigen vector away from the origin. So, you can see roughly here that you know the slow eigen vector should be somewhere around here and the fast eigen vector like somewhere around here right. I think in just quickly check for any example this should hold right.

So,. So, we in the z_1-z_2 plane, it actually looked quite nice of all trajectories are going to the origin and here in this case this will be the slow eigen vector and this will be naturally the fast eigen vector and correspondingly in the x_1-x_2 plane ok. When such a behavior is seen the equilibrium point x_0 is called a stable node because all any trajectory is starting around the origin will actually come back to the origin. So, we will define the notion of stability formally in next week's lectures, but for the movement we can just observe this and call this a stable node right ok.

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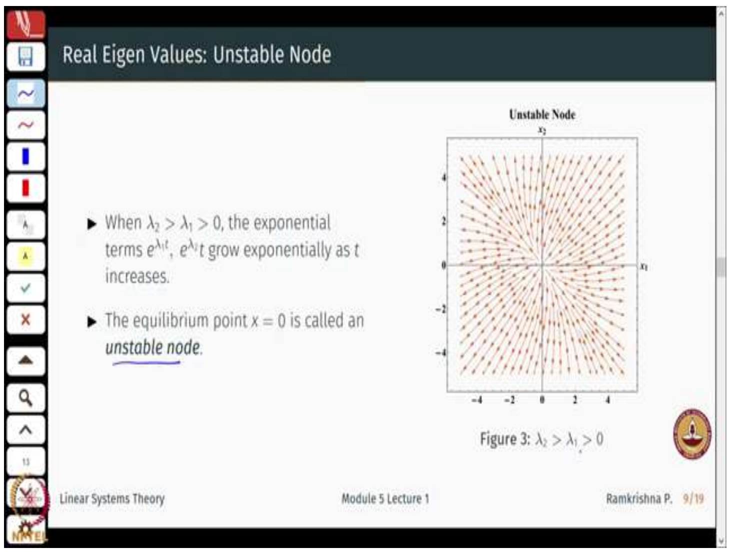

Unstable node its everything is still the same that your λ_1 and λ_2 are right. So, both are positive. So, $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ will both go to infinity as t goes to infinity ok. So, when λ_1 , λ_2 greater than 0, the exponential terms grow exponentially as time increases. It will happen same. So, so if I were to just plot it in the 0 one z_2 plane, it will just be the same except the arrows being reversed right.

So, all trajectory is starting from the origin will or near the origin will tend to go away from the origin whereas, in the stable node case all trajectory starting around the origin will tend to come back to the origin. If your initial condition is the origin, you will always be at the origin ok. So, in this case the equilibrium point is called an unstable node for eigen values which are greater than 0 ok.

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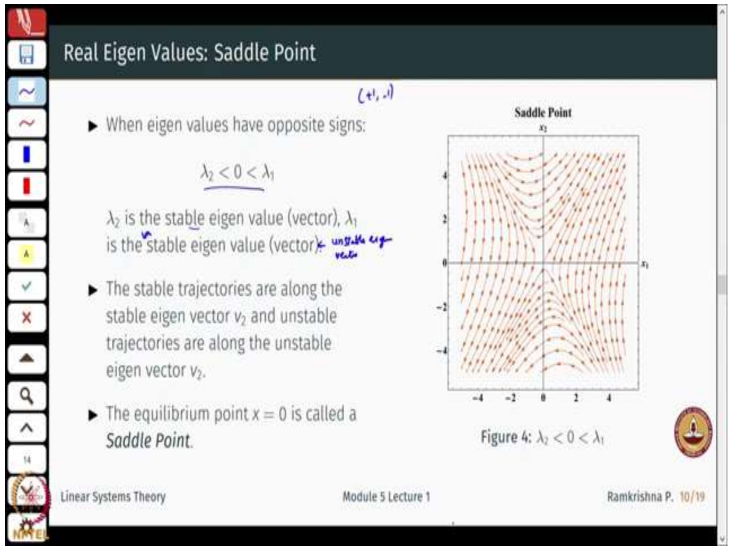

Interesting thing happens when we have eigen values which are real, but which have opposite sign say +1 and -1 of course, this would corresponding to correspond to an un unstable system. So, we call λ_2 which is less than 0 the stable eigen value and of course, correspondingly the stable eigen vector and λ_1. This should be unstable λ_1 is the unstable eigen value and hence the unstable eigen vector.

So, λ_1 is stable and λ_2 is unstable it is a little typo here ok. So, how to understand the behavior of this? Let us again analyze this two terms here. So, I have $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. So, λ_1 > 0 it is, this stable eigen value and λ_2 < 0. So, this is the unstable eigen value sorry λ_1 > 0. So, this is my unstable eigen value and of course, the corresponding eigen vector will be called the unstable eigen vector λ_2 < 0 and I will call this the stable eigen value ok.

So, what would we expect as time progresses that $e^{\lambda_2 t}$ will tend to 0 as t goes to infinity whereas, $e^{\lambda_1 t}$ will tend to infinity as t goes to infinity ok. So, so, just come back to this. So, $e^{\lambda t}$ corresponds to z_1. So, z_1(t) = z_{10}, $z_2(t)$ = $z_{20}e^{\lambda_2 t}$. So, if were to plot this in my z 1- z 2 plane so, this plots would look something like this right. At infinity well the z_2 will tend to 0 and z_1 will go to infinity from starting from any initial condition; this way this way and this way ok. This is how the plot will look in the z_1 - z_2 plane when we have eigen values one of which real eigen values one of which one of which is unstable and the other one is stable ok.

So, in back to the x_1 - x_2 plane, this would look something like this. So, the stable trajectories are along the stable eigen vector and the unstable trajectories are along the unstable eigen vector. So, this could this could so, you can just differentiate the things here or more naturally over here and so on ok. So, in this case the equilibrium point x = 0 is called a saddle point. So, this was about real eigen values.

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What about the case when we have complex eigen values?

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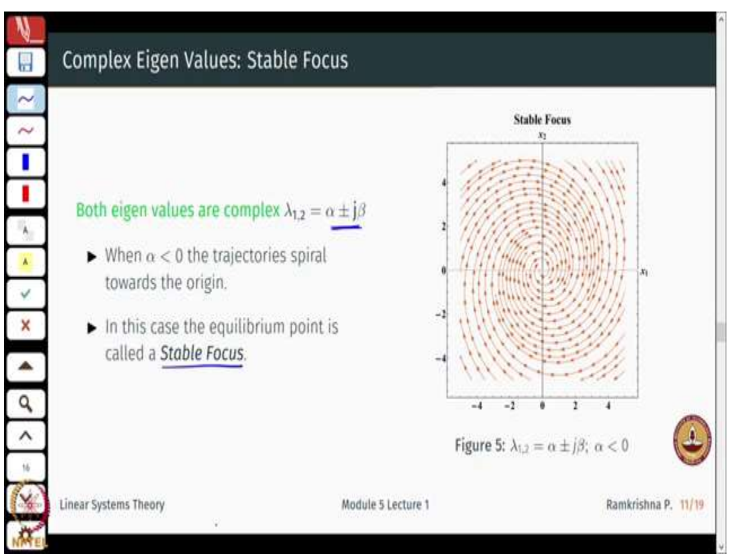

Well that case also turns out to be interesting. So, when both eigen values are complex I have some eigen values, let me call they are α ± jβ ok. So, let us again do this over here ok.

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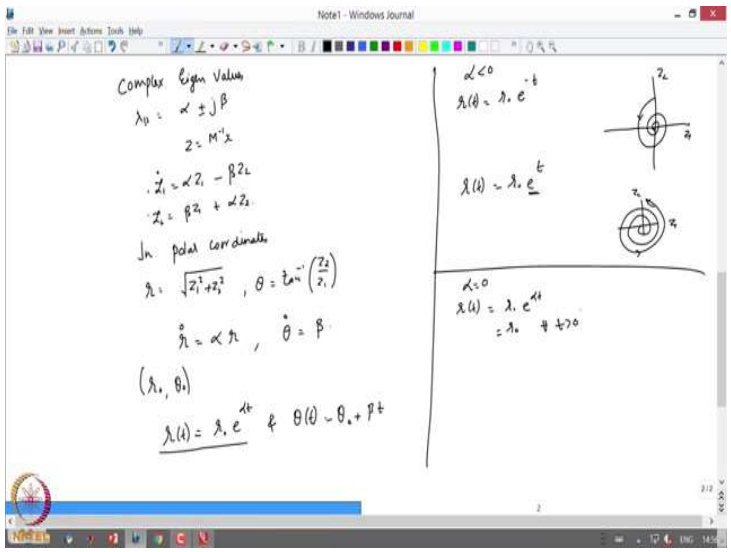

So, when I have complex eigen values, this means my $\lambda_1, \lambda_2$are of the form α ± jβ and of course, I do the usual change of coordinates from z = $M^{-1}x$ to get my system of the following form $z_1 = \alpha z_1$ - βz_2 $z_2 \; = \; \beta z_1$ + αz_2.

So, I just do a little change of coordinates let us say change to polar coordinates ok. What do I have in polar coordinates? r = $\sqrt{(z_1^2 + z_2^2)}$ and the angle θ = $tan^{-1}(\frac{z_2}{z_1})$ of ok. Now if I write this equations in polar coordinates, I will have two coupled first order differential equations that is $\dot{r}$ = αr. The second equation given by $\dot{\theta}$ =β ok.

So, for a given initial state r_0 and θ_0, the solutions are the form r(t) = $r_0 \; e^{\alpha t}$ and θ(t) = θ_0+ βt ok. So, this is interesting here right. So, I see that I have some if I am looking in the polar coordinates, I am looking at the radius of it. So, to speak which is, which has an exponential term depending on t ok.

So, if I just say what happens to the radius when α < 0. So, I have r(t) = $r_0 e^{-t}$t, for α = -1. So, this will mean that in my z_1 - z_2 plane, my trajectories will just spiral to the origin say may be in this way ok; the z_1- z_2 plane right it is also obvious from here right I start with initial radius and it will just go spiraling to the origin ok.

Now, back to here; so, when α < 0 the trajectory spiral to the origin, in this case of complex eigen value; I call my equilibrium point to be a stable focus ok.

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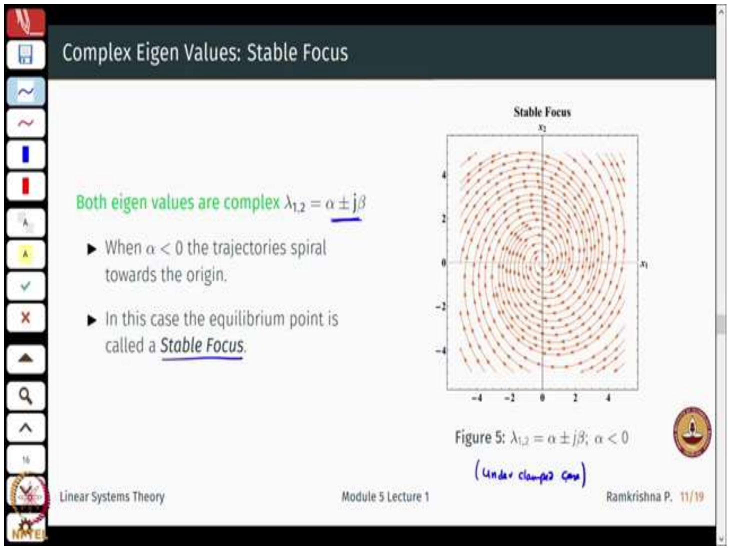

Now, the same thing when α > 0; well I just see from here that when α> 0 r(t) which is r_0 let us say α is 1, you see that the radius actually starting from some initial radius. The radius actually increases with time or it in other words it just spirals away from the origin ok.

So, this is z_1- z_2 the direction of arrows will be away from the origin; in this case, they will be towards the origin ok. In such a case, all the trajectories are going away from the origin the equilibrium point is called an unstable focus ok.

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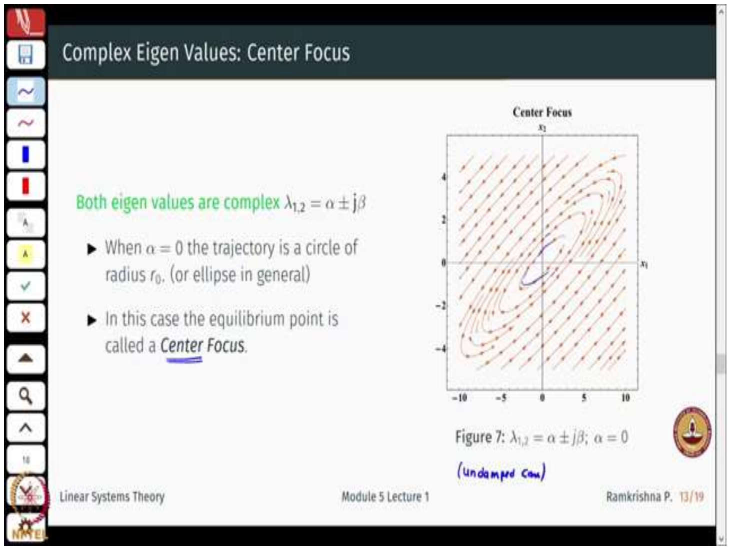

Interesting thing happens when r = 0 when r = 0 in this case sorry α = 0, r(t) = $r_0 e^{\alpha t}$; when α = 0 this is just is just r_0 just be the radius where it actually started from for all times t. This will just be for all times t > 0.

Essentially I am looking at a circle of constant radius r for all times t> ok. So, if I if I come back to my x_1 - x_2 plane so, this could also in general be ellipses right. So, when alpha equal to 0 the trajectory is so, they just they just are in some periodic orbits around the origin like here right or this one. They either is a circle of radius r in this case or more generally, they will look like an like an ellipse.

So, in this case the equilibrium point is called a called a center ok. So, this also if I look at it correspond to few cases that that we learnt earlier right. So, this is usually the undamped case. These things would correspond to the under damped system of course, I am not really talking about what is a damping in this stable in an unstable system that really does not make sense. So, this is this are things that are corresponding to the underdamped case or the undamped system whereas, I go back here this would correspond to something like an over damped system or when λ_1, λ_2 are equal it will correspond to a to a critically damped system. And this are these kind of plots we drew earlier in our week ones lectures.

Now, there we just talked about damping the damping properties of the system, but this is a little more general way of looking at it.

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What happens when we have a repeated eigen values or for the case when we look at a critically damped system?

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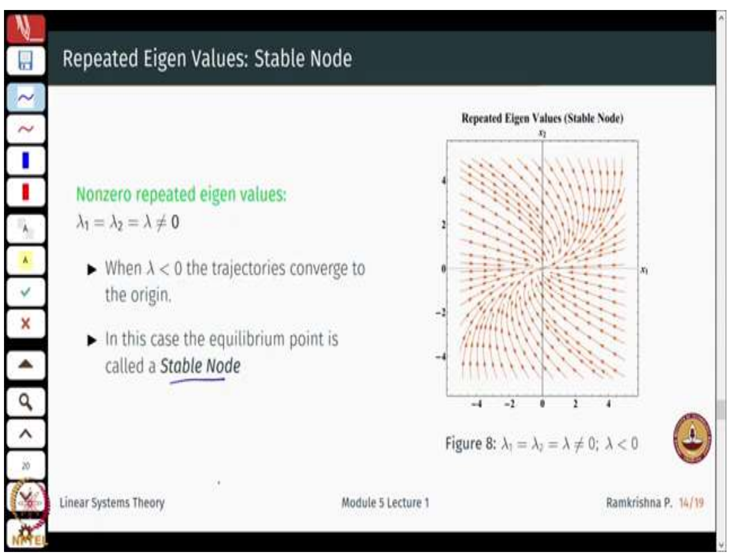

Well when we have non 0 repeated values which means lambda 1 and lambda 2 both are equal to lambda, the trajectory is well as usual they converge to the origin and in this case again the equilibrium is called a stable node ok.

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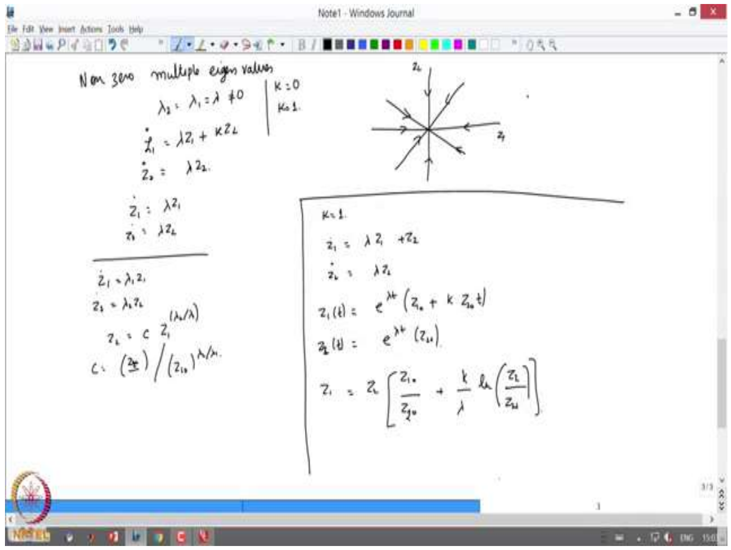

So, let us let us check this in a bit more detail when we have a non 0 multiple eigen values ok; again I will look at ok. So, this means essentially that lambda 2 and lambda 1 are equal to lambda and this is actually not equal to 0. We will come to the case of for the 0 eigen value a little later and with appropriate transformation, I can write this system as z 1 dot is lambda z 1 plus k z 2 z 2 dot is lambda z 2.

So, couple of cases can occur when k is 0 and when k equal to 1 ok. When k equal to 0 I am just looking at these two equations right z 1 dot is lambda z 1', z 2 dot is lambda z 2. So, if I just compare with a first case which I had of z 1 dot is lambda 1 z 1, z 2 dot is lambda 2 z 2 which had the relation between z 2 was given by c times the z 1 power lambda 2 by lambda 1 and then c was z 2 naught by z 1 naught by z 1 naughtt lambda 2 by lambda 1.

So, if I just compare this to the situation where I will have $\lambda_1 = \lambda_2$. It should be easy to check the following right that in if I just draw this in my z 1-z 2 plane. All the trajectories will be coming to the origin this way or this way or this way ok. So, so it is like the little contrast with the slow and fast eigen values and eigen vectors and so on ok.

So, what is also interesting is the case when k = 1. This should be easy to plot I mean you can just check by yourself. So, when k = 1, then I have $\dot{z_1}$ is $\lambda z_1 + z_2$, $\dot{z_2} = \lambda z_2$. So, the solutions would be z_1(t) is e power lambda t z_{10} + kz_{20}t, z_2(t) is e power lambda t z_{20}.

Of course if I just want to write it in terms of z_1 and z_2 that will simply be $z_1 = z_2\ [\frac{z_{10}}{z_{20}} + \frac{k}{\lambda}\log(\frac{z_2}{20})]$ ok. So, I will just no not plot this, but we will just check what this means in the in the x_1- x_2 case. So, it will turn out not surprisingly that that you are actually talking of a stable system because the eigen values are less than 0. We are talking of eigen values being equal to each others. So, we are talking about some critically damped situation where we would naturally expect the trajectories to come back to the origin starting from a neighborhood of the origin.

So, this is how they will look like and will all already we will also call this as a stable node.

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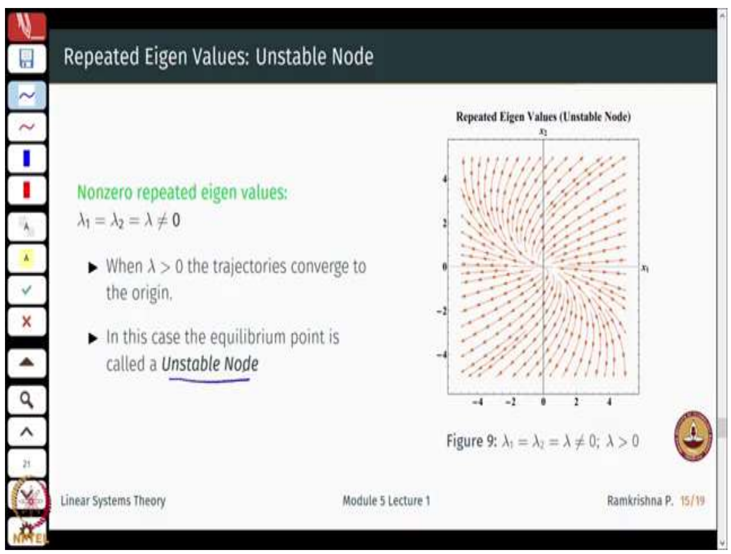

What happens when it is an unstable node, well just the plots will be similar with just the eigen just the direction or of the arrows being reversed. So, naturally I am looking at say eigen value of +1, +1, the system will naturally be unstable and this equilibrium. I will call as an unstable node ok.

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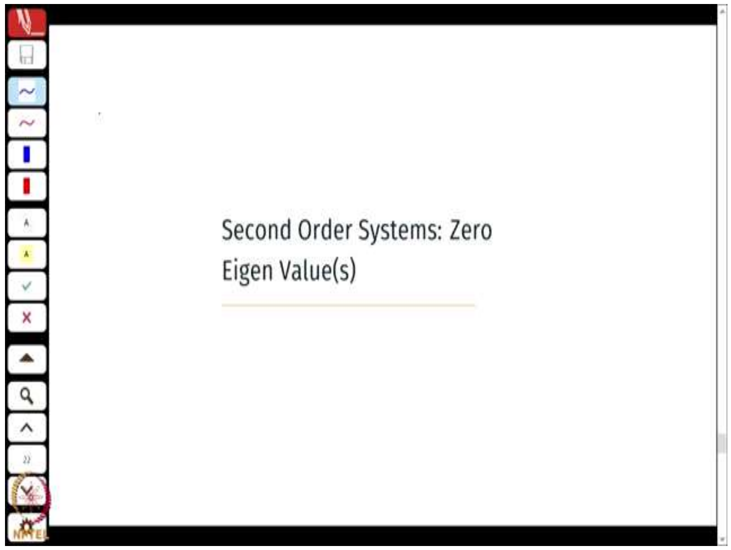

The next thing that we will discuss is an interesting case when there is a possibility of having a 0 eigen value ok.

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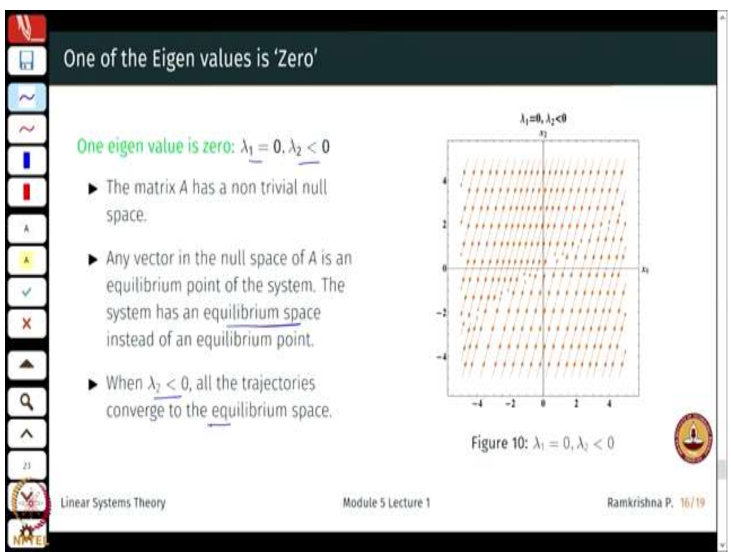

So, let us start with just one of the eigen value being 0 let us say $\lambda_1 = 0$ and then $\lambda_2 < 0$ ok.

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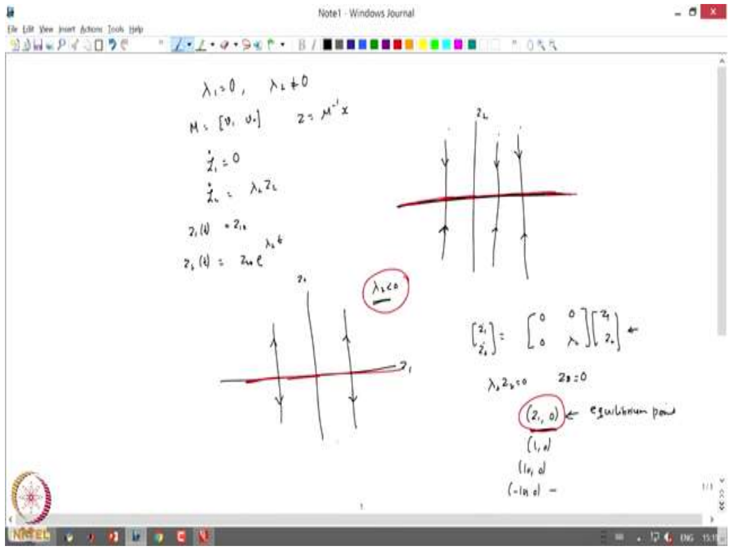

So, how will it how will the system look at in how will the transform system look like? So, when I just write it in this Jordan form. So, λ_1 = 0 $\lambda_2 \neq 0$ and I just again its transformation matrix [v 1 v 2] and via the transformation z = M^{-1}x in my new coordinates, I have $\dot{z_1}$ = 0, $\dot{z_2}$ = $\lambda_2 z_2$. The solutions are pretty straight forward to compute $z_1(t)$ will just be whatever it began with its initial condition z_2 (t) will be $z_{20}e^{\lambda_2 t}$.

So, if I were just to plot z_1 and z_2 say for initial condition over here, z_1 will just be here and if λ_2 <0 the trajectories would just now be behave this way. So, this is the initial condition of z_1, then the trajectories will be here if this is the initial conditions the trajectories will go this way for all λ_2 which is less than 0 ok.

So, what does that mean right? So, first is the matrix a has a non-trivial null space and any vector in the null space of a is an equilibrium point of the system ok, how will we deduce the equilibrium points. So, if I have a system like this $\begin{bmatrix}\dot{z_1}\\ \dot{z_2}\end{bmatrix}$= $\begin{bmatrix}0 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$z 1 z 2. What are the equilibrium points if I just solve for this, I will have $\lambda_2 z_2$ = 0 which means z_2 = 0. Then there is this is the only solution I get from solving this.

So, any point of the form (z_1,0) is an equilibrium or is an is an equilibrium point of the system or in general here I will have an equilibrium space instead of an of an equilibrium point right. So, so any point take any z 1 where z 2 equal to 0. So, (1,0) is an equilibrium similarly is (10,0) (-10,0) and so on ok.

Now, when lambda is less λ_2 <0 all trajectories converge to the equilibrium space. So, what is the equilibrium space here? This is entire z_1 axis right. So, any point in z_1 with z_2 = 0 is the equilibrium space say here or let me just draw it in a red. So, this is my equilibrium space which is obtained by just this one; this is my equilibrium space.

So, when $\lambda_2 < 0$ any trajectory right. So, this let these trajectory these trajectory every each of this trajectory will converge to the equilibrium space ok.

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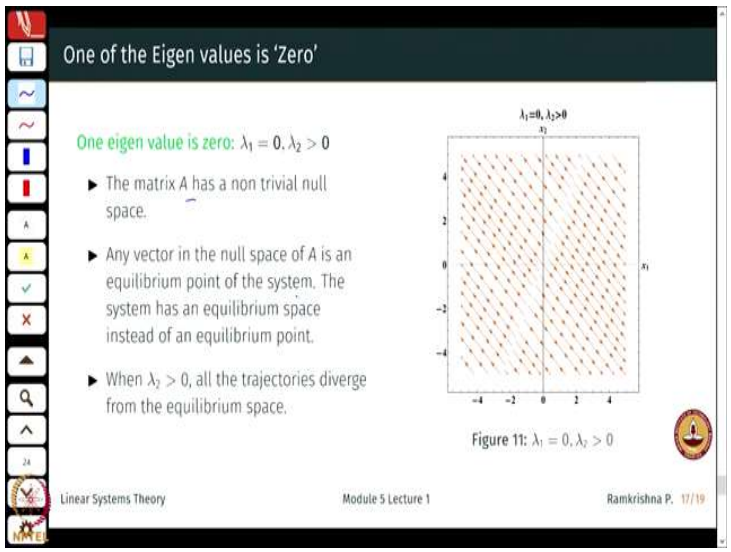

Next what if $\lambda_2 > 0$, everything will be the same. The matrix A will still have a non-trivial null space and any vector of in the null space of A is again the equilibrium point and the only thing that will change is all the trajectories diverge from the equilibrium space.

So, let me just draw it here so, the trajectories. So, this is z_1 this is z_2 and this will just be my trajectory. So, all trajectories will go away from the equilibrium space the equilibrium space is the entire horizontal axis in this case.

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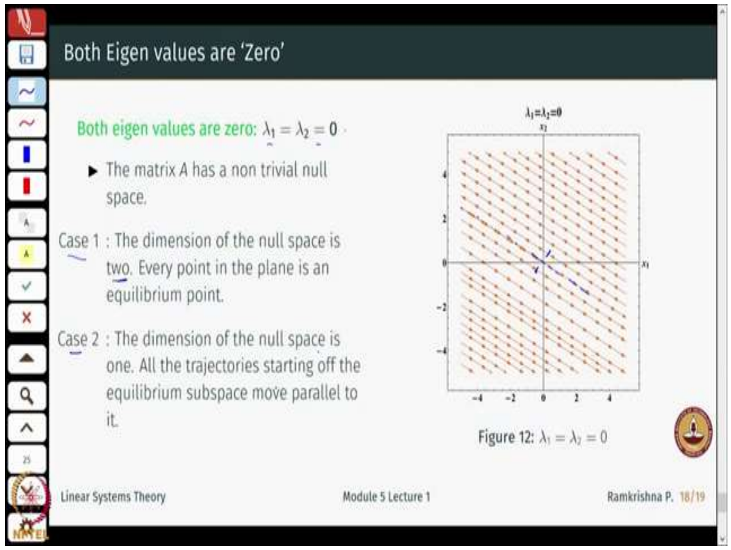

The last case that we will look at is the case when both the eigen values are 0 not surprising to note that the a matrix will still have a non-linear sorry the a matrix will still have a non-trivial null space. So, in this case when both eigen values are 0, we can potentially look at two cases.

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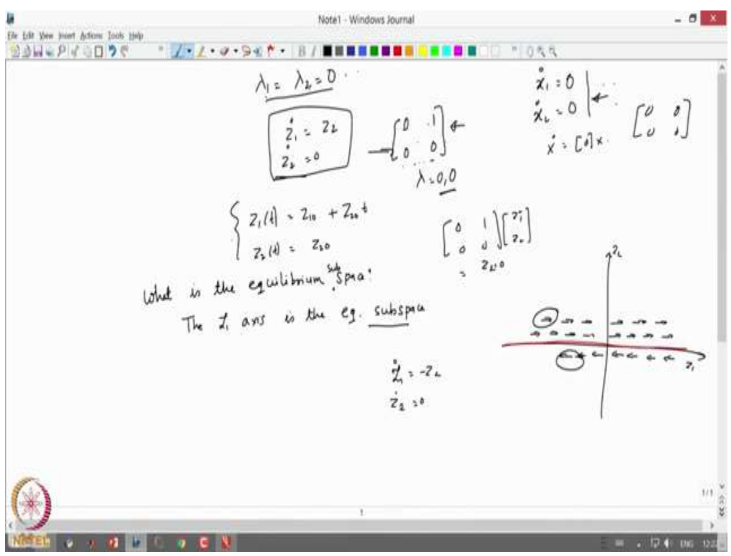

So, I am looking at both eigen values λ_1 and $\lambda_2 = 0$. So, one one case could be that I am looking at systems of the form $\dot{x_1} = 0$, $\dot{x_2} = 0$ or in other words, $\dot{x} =$ [0]x ok. So, this will have both eigen values to be 0. This would correspond to the case when the null space is of dimension 2 and not only that very point in the plane will correspond to an equilibrium point ok.

So, this case is it may not be too interesting for us to look at the phase plane, but what is interesting is the case when both eigen values are 0 and the dimension of the null space is 1 ok. In how does that that happen that could happen in cases when well, I have systems of the form $\dot{z_1}$ = z_2 and $\dot{z_2}$ = 0. So, in this case the a matrix is of form 0 1 0 0 and if you compute the eigen values, they will turn out to be (0 , 0).

So, the a matrix is not completely 0 here. It has a it non-zero entry here, but still both of the eigen values are 0. You can compute check this as the simple exercise ok. So, it may not necessarily be in this form all the time, but via some appropriate transformations. You can write the systems to be in its form. So, when does this case happen and when does this case happen, it again is it is the same that you are looking at a certain Jordan form. So, if you compute the Jordan form of this form, you will you can check its algebraic and geometric multiplicity and check it is all algebraic and geometric multiplicity and you will have the appropriate Jordan form.

So, this is the Jordan form when the algebraic multiplicity is 2 and the geometric multiplicity is one whereas, here it will be a slightly different case. I will I leave that as an exercise ok. So, this case is a little interesting to draw the phase space. So, what can I see directly even without worrying about the solutions is that z_2 is a constant and $\dot{z_1}$ varies positively with z_2 or with the sign of z_2. If it is -z_2, then it will vary negatively with increasing z_2 and so on ok.

So, how does the solution look like so, I will have z_1(t) is some initial condition z_{10} + z_{20} t and z_2(t) = z_{20} ok. Now what is the equilibrium space? So, in this case well, you can you can check easily right. So, I am just look at the solutions to this one for z_1 and z_2. So, this will give me that z_2 = 0 and therefore, the entire z_1 axis is the equilibrium sub space this is called as equilibrium sub space.

So, if I were to plot on the on the z_1- z_2 plane. So, this entire z 1 axis is my is my equilibrium subspace on the z_1- z_2 plane ok. What happens close to the equilibrium it is a easy to check that my phase curves will just be parallel to the equilibrium or the equilibrium sub space. So, here they just be the reverse sign now. So, there could be another cases when $\dot{z_1}$ = -z_2, $\dot{z_2}$ = 0. What will happen to the phase curves will be exactly the same just with the directions of arrows here and here being reversed ok.

So, that is like to different cases when λ_1 and λ_2 both are equal to 0 and the equilibrium sub space depends again on the on the Jordan form right. So, when the algebraic multiplicity is equal to the geometry multiplicity the Jordan form simply has this form and the equilibrium subspace will have dimension two whereas, in this case the Jordan form takes a form like this and you will have solutions for the phase space given by this set of equations.

So, to summarize when both λ_1, λ_2 = 0; case 1 the dimension of the null space is two again depending on the Jordan form and the case two that dimension of null space is one and if I were to plot in general in an in an x_1- x_2 plane all. So, the equilibrium sub space could be somewhere like here passing through the origin. So, all trajectory starting of the equilibrium subspace either here or here. They will just move parallel to it similarly to what we say in the of the plots in the in the $z_1 z_2$ plane and equivalently in the $x_1 x_2$ plane. They will look something like this right ok.

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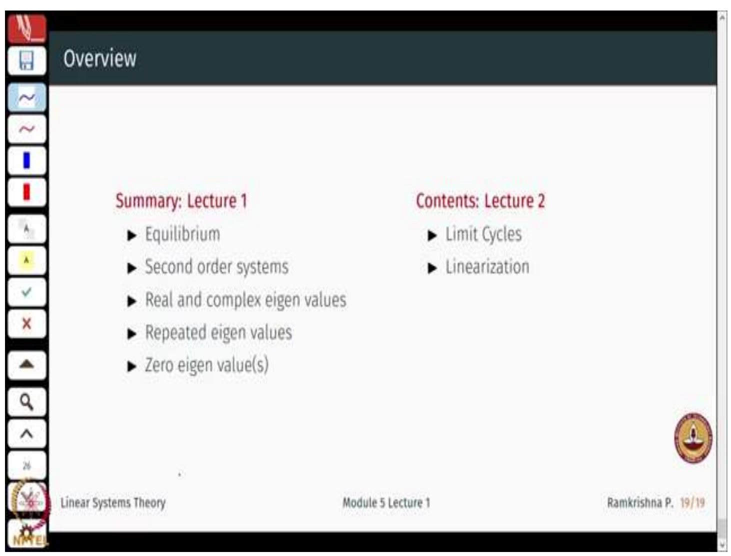

So, just to conclude we have defined the notion of an of an equilibrium point. We did a lot of analysis qualitative analysis for several equilibrium points of second order systems. We had real eigen values complex eigen values repeated eigen values what if one or both of the eigen values go to 0 that is contains a bit of information of what will be useful for us in stability analysis.

So, just to conclude this week's topics we will deal with limit cycles which is a very interesting property of linear systems sorry of non-linear systems which not necessarily exists in linear systems. And, then we will look at couple of or few methods of linearization of how do we start from a non-linear system and end up with a linear system. So, that will be in the next lectures.

Thank you.