Linear Systems Theory
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Module – 04
Lecture – 03
State Transition Matrix

Hello, everybody. So, welcome to this lecture series on Linear Systems Theory. So, we continue with the week course lectures on what we call as solutions to the state space equations and we coin the term called the State Transition Matrix earlier in this in this week's lectures.

So, we just continue around that or answering those questions and specifically in today's lecture we will focus on linear time varying systems. What was the advantage in the time invariant case was that we are nice looking closed form expressions that a matrix exponential e^{At} was essentially the state transition matrix and the properties were easy to infer and even derive. So, let us first because we will just do a little example before we warm up towards the general case.

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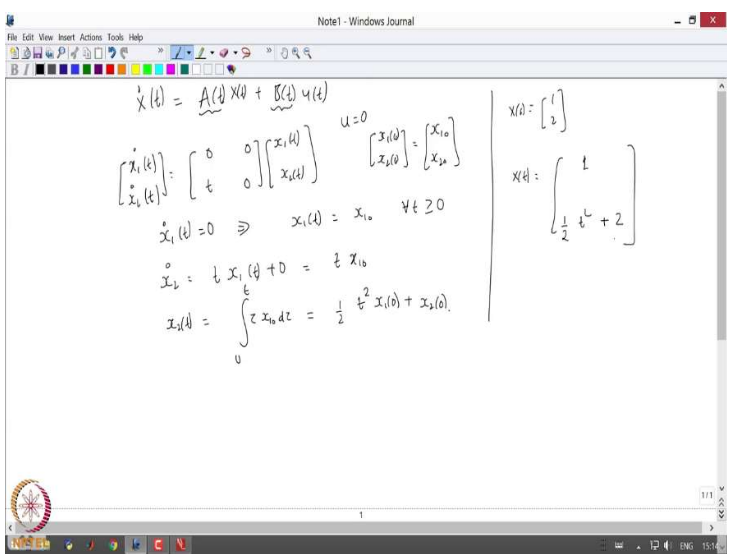

So, I will start with a simple example let us say I have a time varying system. So, essentially I am looking at $\dot{x}$ = A(t)x(t) + B(t)u(t). So, that is the difference is that this matrices A, B also now vary with time ok. So, is a little example; so let us say I have a system which is $x_1(t)$, $x_2(t)$ varying in the following way $\begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$ and $x_1(t)$ $x_2(t)$ and let us assume for the moment that u = 0 and I am actually looking at an autonomous system ok.

How would the solution of this look like? If I write down the first equation that will be $\dot{x_1}$(t) is 0 and let me assume some initial conditions at x_1 at 0 and x_2 at 0 that this could be given by x_{10} and x_{20} ok. So, this is easy to solve that the solution x_1(t) will be x_{10} for all times t ≥ 0 ok.

So, the second equation now. So, I have $\dot{x_2}$ = tx_1(t) + 0 ok. Now, what is x_1(t)? x_1(t) is simply this for all time say this tx_{10} sorry ok. Now, x_2(t) would just be the integral of this right. So, integral $\int_0^t \tau x_{10} d\tau$ and this will be equal to $\frac{1}{2}t^2 x_{10} + x_{20}$ ok. That is the a simple looking equation I can solve easily and let us me just yeah check for you know what if my initial condition is (1 , 2) the solution would be x(t) is 1 and then here I will have. So, x_{20} is 2 + x_{10} is 1, so I will have $\frac{1}{2}t^2$ this is not t 1, this is t here ok. So, that is just in general; so we are again looking at integrating the differential equation ok.

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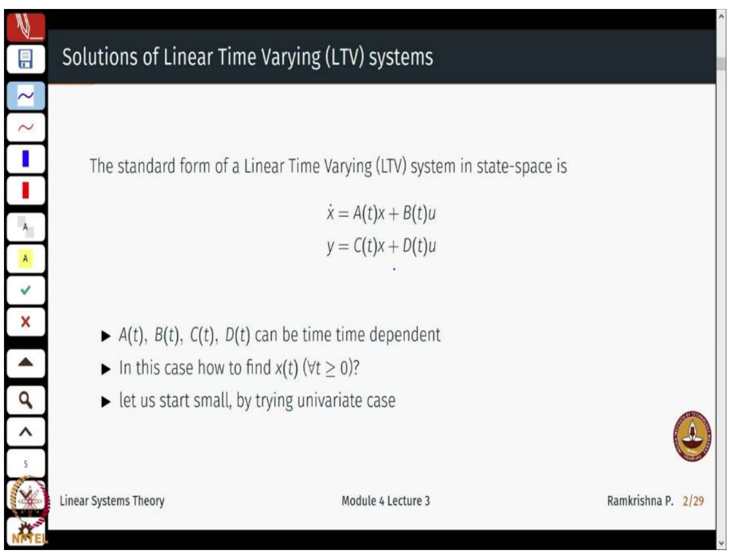

So, so these are the equations that we will be interested in $\dot{x}$ is an A(t)x and so, all this A, B, C and D matrices are now changing with time. So, the question again is the same right. So, given the set of equations with a certain initial conditions how do we find x(t) for all times t ≥ 0.

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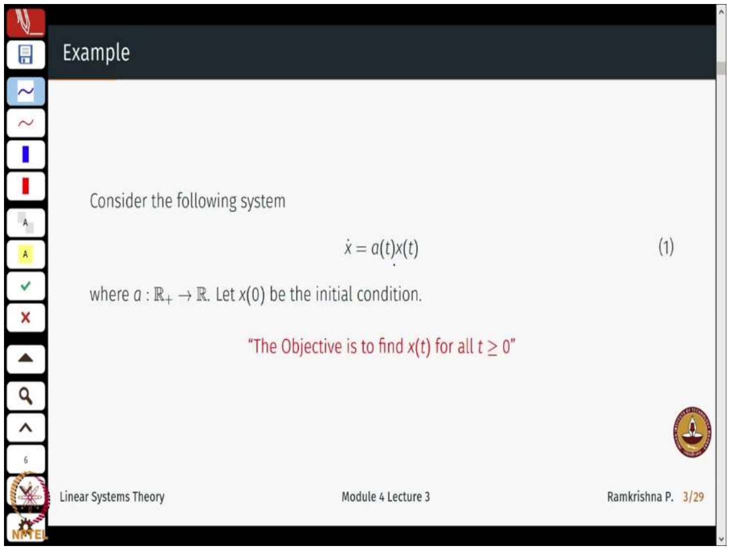

Similar to the to the LTI the time invariant case let us start with a scalar equation or a univariate example ok; a is in R and you cannot and then let us start with some initial condition objective is to find x(t) for all times t ≥ 0 ok.

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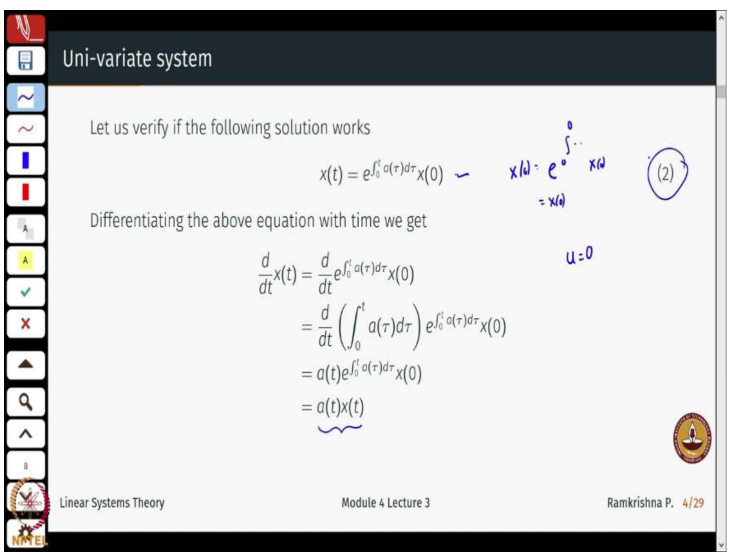

So, let us take some q from the LTI systems and so, we will add some kind of an exponential solution at $e^{At}x_0$. Now, let me propose a solution which is like this ok.

Now, what I would want to verify is this the solution right. So, first is so, it should satisfy two things right as said yesterday or in the in the earlier lectures that it should satisfy the initial condition at x t is a solution to this differential equation here if it satisfies the initial condition and additionally it satisfies the differential equation too. So, let us see first what is the initial condition right. So, x(0) is e power 0 to 0 whatever it is in the integral x_0 and so, the initial condition is like written ok.

Now, does it satisfy the original differential equation? Well, again I am I am talking of the autonomous case where u =0. So, I just differentiate and then ok. So, all the steps are written down here and actually reconstruct or regain my original differential equation and therefore, I can claim that the equation labeled 2 is actually a solution to the scalar differential time varying equation ok.

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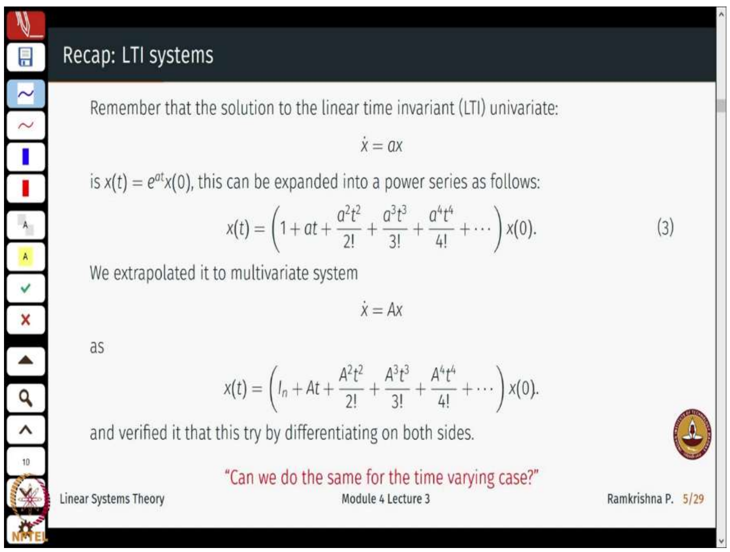

So, what did we do how did we extend this to the n-dimensional case in the LTI case right. So, we had x(t) is $e^{At}x_0$ as the solution and we derived this also as a power series right. So, if you remember those derivation and loosely we could kind of extrapolate this to the multivariate case in my cursor in a very crude way saying just replace the small a with capital A and things will work out. And, then everything instead of being a scalar is now a vector and the one becomes an identity and so on right.

Of course, then we did actually we just do it do it blindly, but we actually verified that this was actually a solution to the vector differential equation and by again verifying it with the initial condition and also that this solution actually satisfies the differential equation ok. And, now can I just do the same for the time varying case ah.

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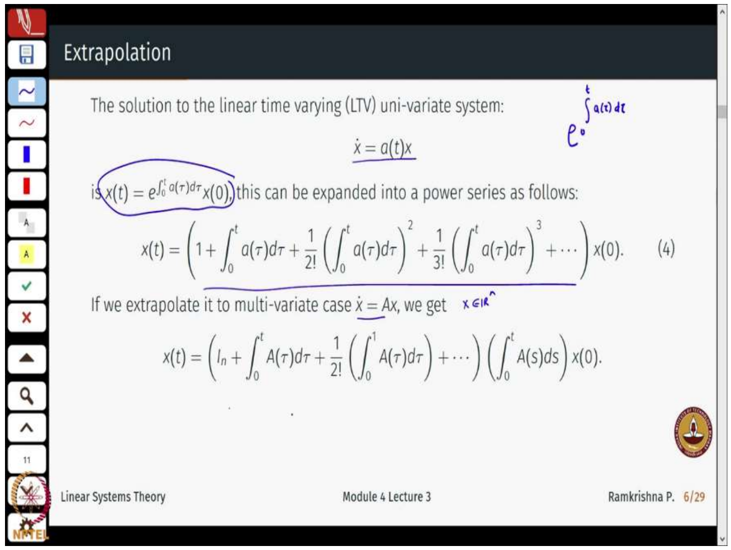

So, so let us start again with the set $\dot{x} = e^{At}x$ where x was of this form ok. Now, I can look at this exponentially right. I have another exponential which with an integral sign $e^{\int_0^t a(\tau)d\tau}$, now I can always expand this right. So, with the formula of the exponential and I get something like this ok. Now, I can extrapolate all right. Let me try and extrapolate this to the multivariate case where $\dot{x}$ = A x with x being in R^n instead of R ok.

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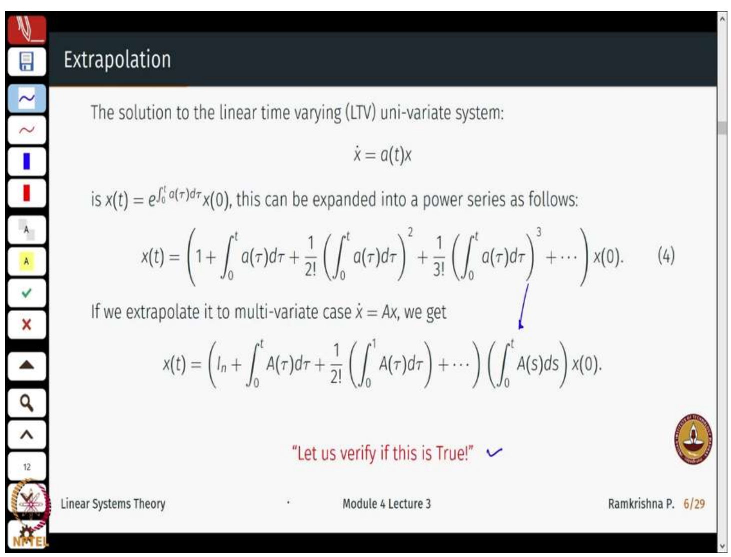

So, can I just write it directly in this way? So, and also verify if this is true. How do we verify if this is a solution. So, I just try to do this for the for the vector case and just check if this is actually true. So, again what do I need to verify? I need to verify the initial condition, I also need to verify that this solution actually satisfies the differential equation by just differentiating this x(t) ok.

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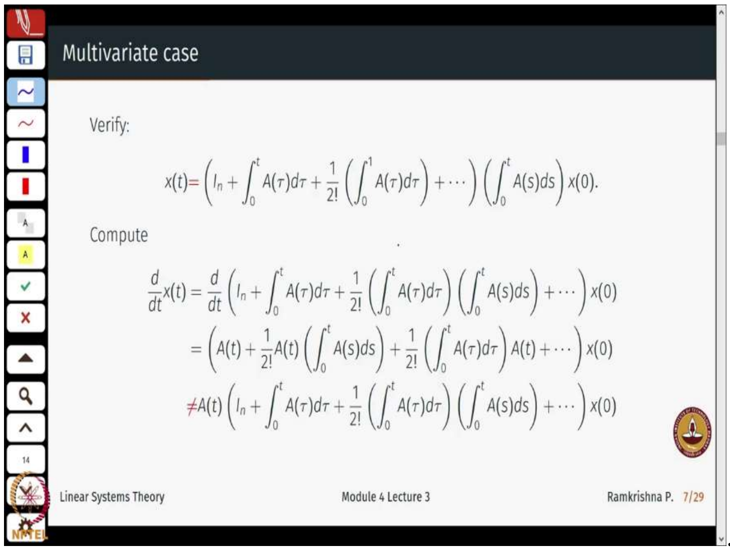

So, well, I just do all the computations, I will skip them, but they are actually written down quite nicely here. So, what we find is that if I just extrapolate this by just replacing small a with a capital A, it turns out that this is actually not the solution right.

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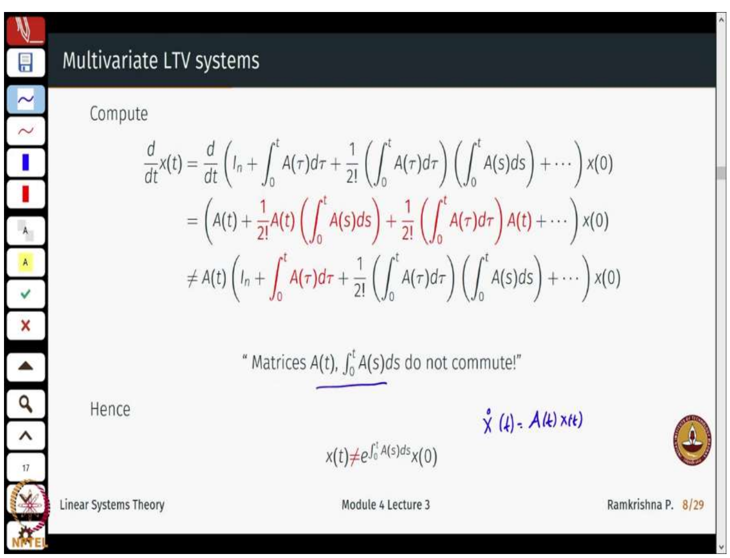

So, then, essentially because these matrices do not actually commute right. So, therefore, the solution to the vector differential equation $\dot{x}(t) = A(t)x(t)$ cannot be directly obtained by extrapolating the scalar or the univariate cases this is the proof is written down quite well here. So, I will skip reading those steps, but you can just write down for yourself. So, the aim is to show that I just cannot do the crude way of extrapolation that I did in the LTI case yes because I cannot I can if I can actually show that this is actually not true ok.

So, there are ways to do this and that is called the Peano-Baker formula. It looks a little ugly and complicated, but it is nice to know how people actually ended up building up theory to solve this kind of equations of when and there were no tools how would people actually think about even arriving at solutions ok.

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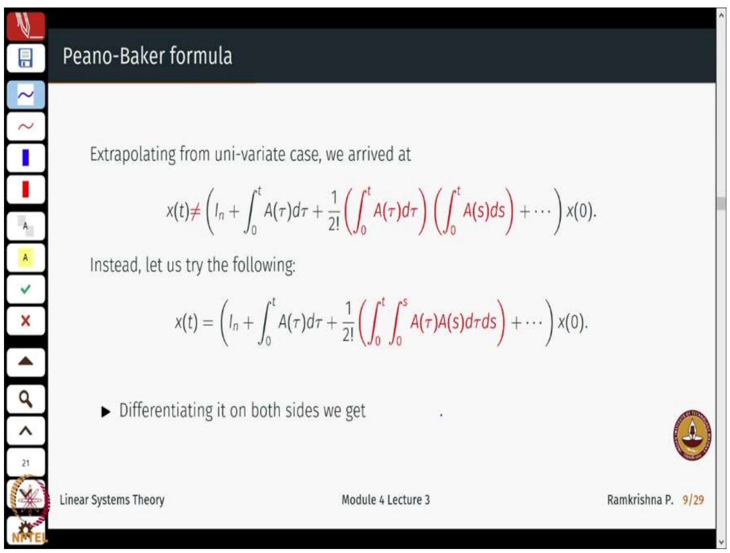

So, from the univariate case so, this is what we tried doing right in the in the univariate case right and this actually does not work right. So, instead so, let me try doing something like this right x(t) is identity this remains the same and I have some kind of I would double integral and then I can expand this to have triple integrals and so on. Let me see if I can verify this right.

So, first is this the solution? Well, this is satisfy the initial condition well. The answer is yes, because this integrated thing from 0 to 0 and you can retain the initial condition as true also in the previous case in this case we could retain the initial condition, but the solution did not satisfy the differential equation.

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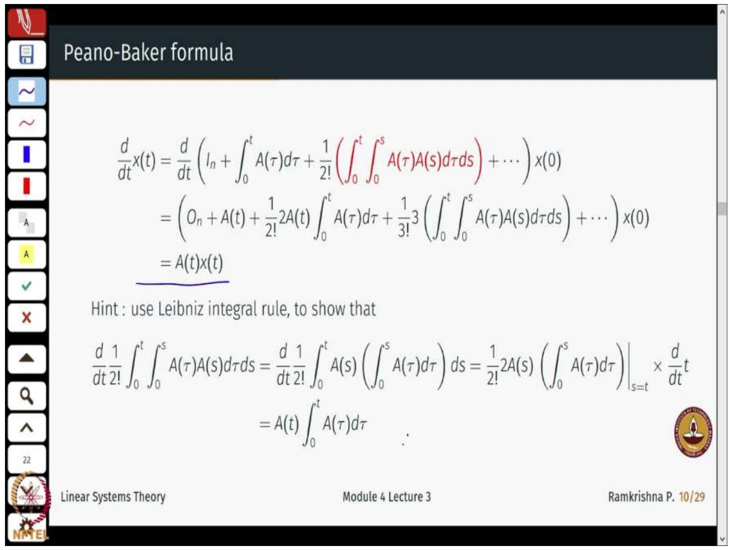

Now, I keep doing on all the keep on doing the differentiation and actually see that this satisfy my satisfies my original differential equation. Again, we use the Leibniz rule that we used yesterday to verify solutions of systems with inputs right. So, instead of doing a blind extrapolation we can do exactly different kind of extrapolation or possibly curve mixed with a bit of intuition to actually find that you know something like this as a solution can be proposed and can also be verified ok. I like it not do all the details of it I can just write down for yourself ok.

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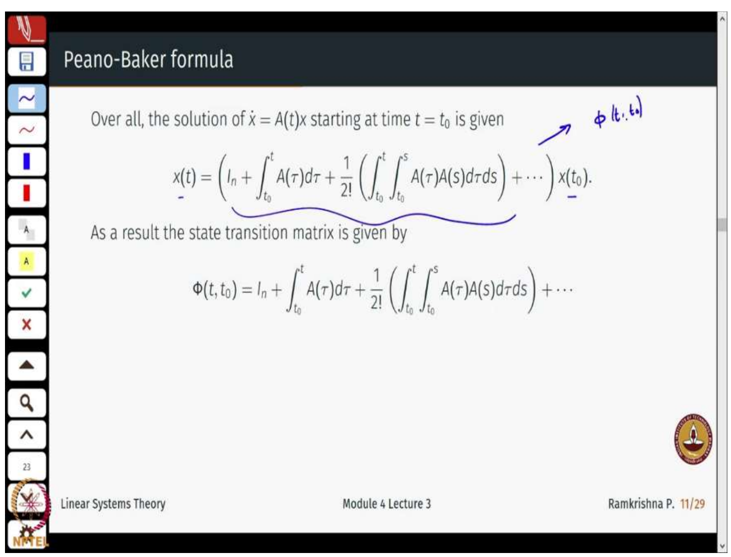

So, to conclude here about the solutions so, we will say that x(t) is given by this solution with a with lot of integral terms in the middle and this thing inside the bracket is what I call as the state transition matrix right Φ at t starting at any initial condition t_0 ok.

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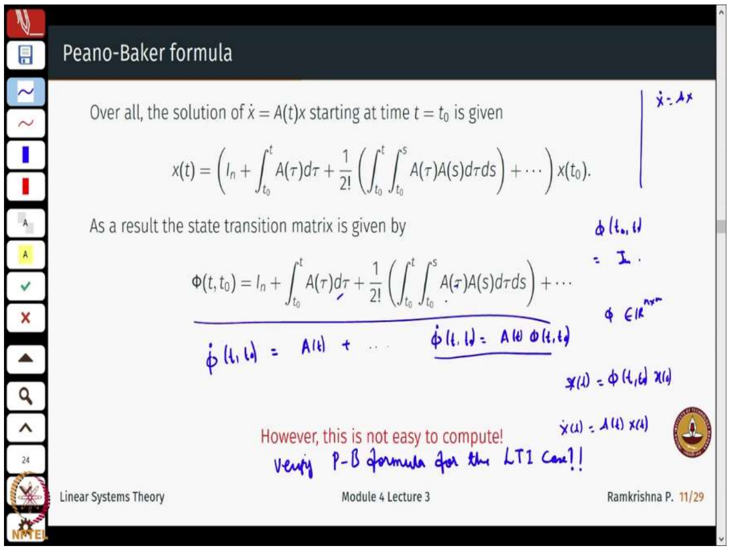

So, one thing about this what is this Φ and yesterday or in the previous lectures when we were talking of solutions to even LTI systems we started also comparing it with the solution to a matrix differential equation and that is what really obvious of where it actually you know comes from. So, if I look at this phi let me check right. So, what does this Φ satisfy Φ? $\Phi(t_0)$ the initial condition would just be the identity right because all the integrals would just vanish ok.

Now, I do what is $\dot{\Phi}$; $\dot{\Phi}(t, t_0)$ I differentiate the first term I get A(t) plus all this terms right. So, what I can compute again I skip the computations and this should be easy to follow that Φ(t, t_0) can be shown to be A(t)Φ(t, t_0) ok. So, this is how we actually derive the matrix differential equation or how we actually infer that the solutions actually can be interpreted as coming from a matrix differential equation. This Φ here is this is now when n x n matrix and that is why we could actually do it that way.

So, what does it satisfy? Well, this initial condition is identity that is what we said that the Φ actually has an initial condition identity and therefore, x(t) is Φ(t, t_0) with x(0) is actually the solution of $\dot{x}$(t) = A(t)x(t) right. So, this is where the matrix a differential equation comes from and that is also the reason why I am actually doing this ugly looking series very explicitly I just to show you the relation with the matrix differential equation.

Now, you can write this entirely for the LTI case $\dot{x}$ = Ax right and just do the substitutions I will not I will possibly skip those and you can just verify of how this Peano-Baker formula holds for the LTI case. I will just as a exercise just verify the Peano-Baker formula for the LTI case results would not be surprising, but yes it would be easy or it would be helpful just if you write down the details for yourself just to your kind of get yet concepts engrained into your mind ok. But, I do not want to do this kind of integrals all the time right now are there easier ways to find this.

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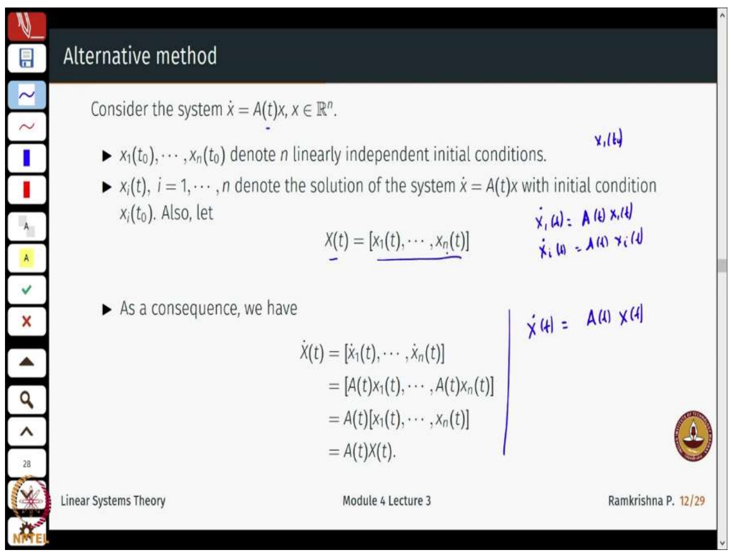

Well, it turns out that we can we can do something nice here right. So, again let us start with the LTV the Linear Time Varying equation $\dot{x}$ is Ax with a being time varying and let me start with n linearly independent initial conditions ok. There is a reason why I am actually doing this ok. So, with each initial condition I will have a certain solution right.

So, let me denote those solutions starting from initial condition say $x_1(t_0)$ as $x_1(t)$. The solution starting from $x_n t_0$ as x_n(t) right, let me have n different solutions and as a consequence if I differentiate this, what I have is $\dot{X}$ is of course, differentiation of each of this. Now, what does $\dot{x_1}$ dot satisfy? $\dot{x_1}$ with an initial condition x_1 at t_0 is also a solution of this equation. So, I will have A(t) x_1(t); similarly, for any i $x_i\dot{(t)}$ is A(t) $x_i(t)$ and therefore, I can write this as again some kind of a matrix differential equation right. So,

$\dot{X}$(t) which is X, X is an have n x n matrix is A(t)X(t). So, I am just combining this solutions right.

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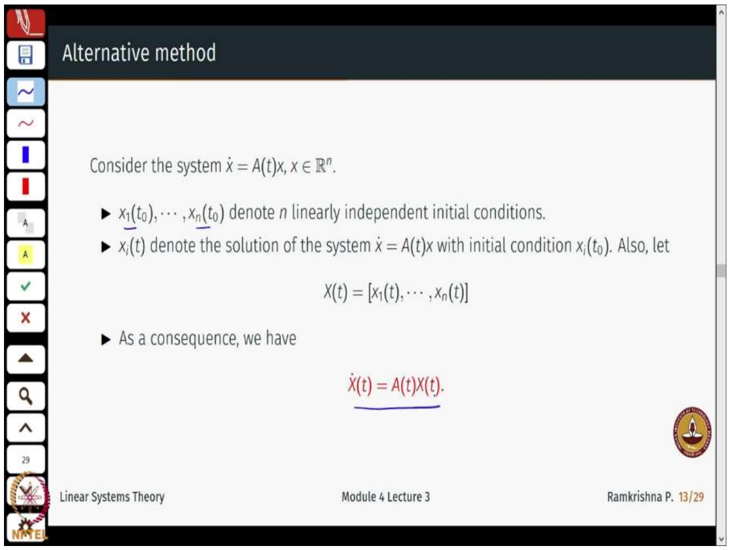

Now, well key here is that this $x_1(t_0)$ till $x_n(t_0)$, the n initial conditions which I choose must be linearly independent I will shortly tell you why that is important. So, because of that we have now a matrix differential equation and ok.

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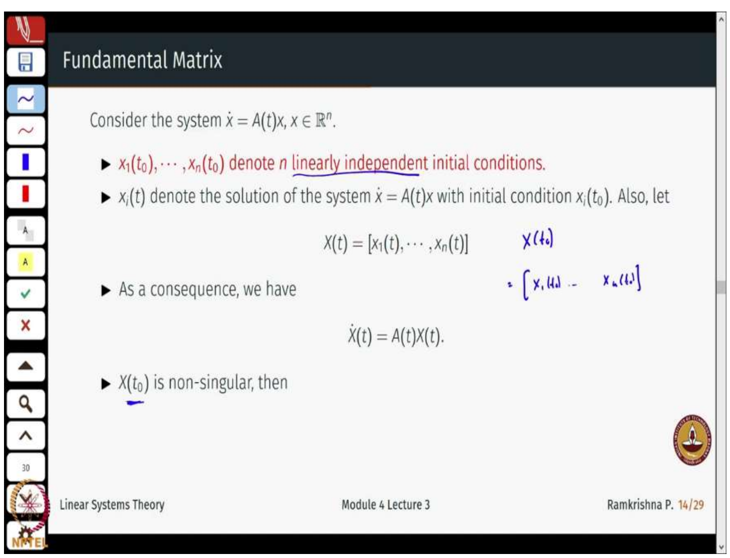

If X(t_0) because I assume that these are n linearly independent conditions. So, this will turn out that X at t_0 which is essentially this vector right if I just write down X(t_0) is just this. In fact, this matrix collection of all these vectors $x_1(t_0)$ till $x_n(t_{0)}$ ok.

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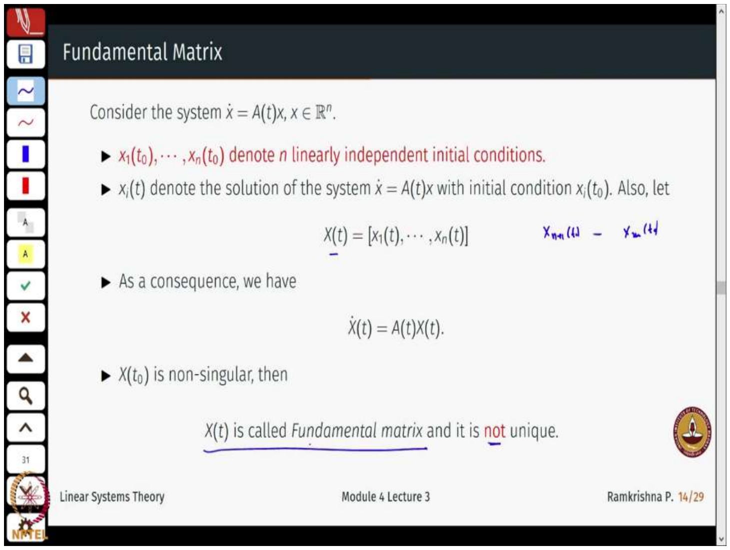

If this matrix is non-singular, then well, what I call X(t) I call it as the fundamental matrix ok. So, the reason I also said it is actually not unique because I can choose any different set of n conditions and that will give me just say I choose say $X_{n+1}(t_0)$ till X_{2n} (t_0) which is whatever initial condition in this states that will give me different set of capital X and that will lead me to a different set of a fundamental matrix. So, I call this a fundamental matrix, but this is not always or this is not unique it need not be unique ok.

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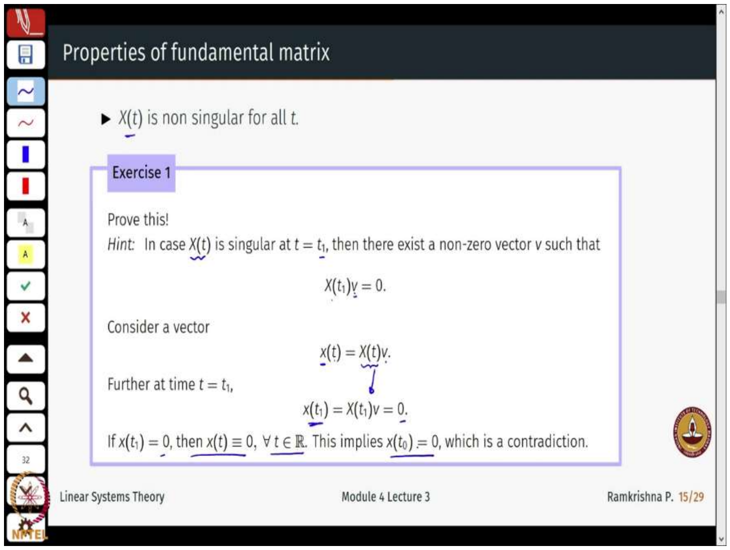

So, the first thing let us see of why so, if I start with linearly independent initial conditions. So, my claim is that X(t) is non-singular for all t right the fundamental matrix if I start with linearly independent conditions then this matrix is non-singular which means it is it can it is invertible, its determinant is non-zero and so on ok. Let us see how we will do this even though this is say prove this I think I actually proved it here.

So, let us start with this matrix X(t) and let us do the proof by my contradiction. Let us assume that it is actually singular at some time value t_1 ok which also means that there will be a vector a non-zero vector such that X(t_1)v = 0 ok. Now, given this X at some other time t ≠ t_1, I can always construct a vector small x(t) as this capital X(t) v this v and this v can be the same they are like non-zero ok.

Now, if I compute the value of this small x at t_1 what is small x at t_1 is capital X(t_1)v = 0 ok. If x(t_1) = 0, then x(t) = 0 for all time t in R right. What does it imply that x(t_0) = 0 which is a contradiction right. And therefore, we can prove that or we can show that X(t) is non-singular for all t which essentially means that if x(t_0) = 0, I start with 0 initial conditions I just end up with the solution being also staying at 0 and that is not very interesting to me.

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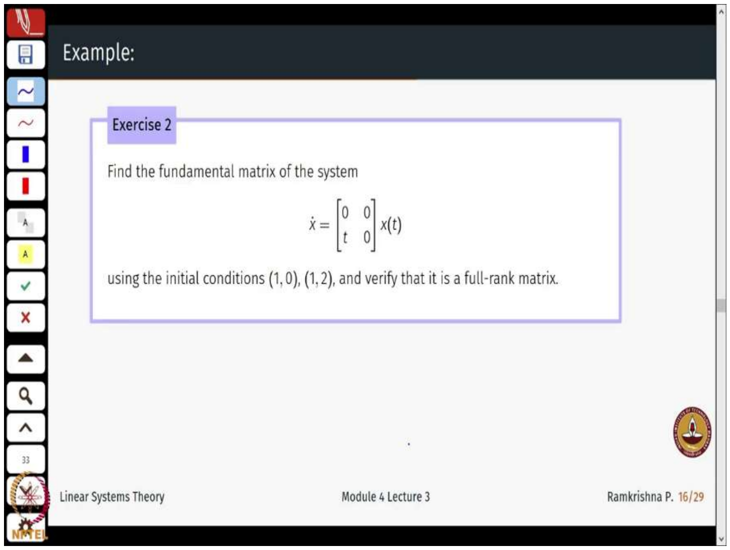

So, this is example which we started with. So, let us see if we could find the fundamental matrix of the system using two initial conditions so, (1,0) and (1,2) ok. Do not really confuse it with you know that I am having a zero initial condition. Well, this is actually a non-zero initial condition because both vectors are not 0 both the initial conditions are not are not zero ok.

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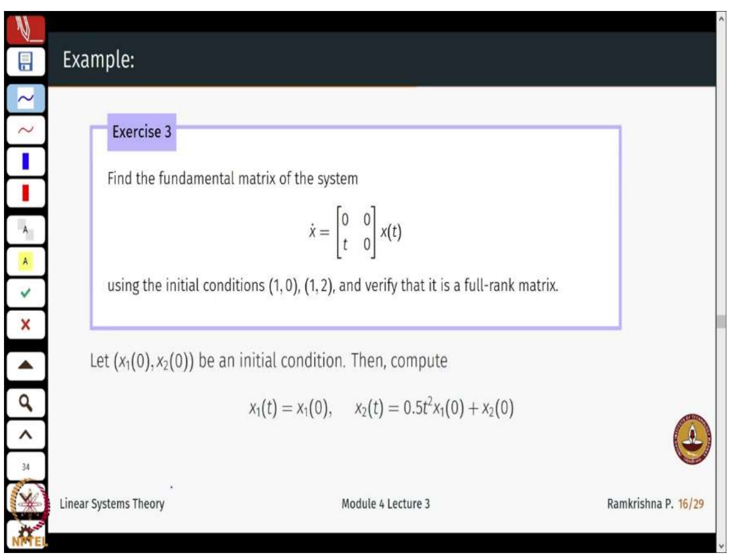

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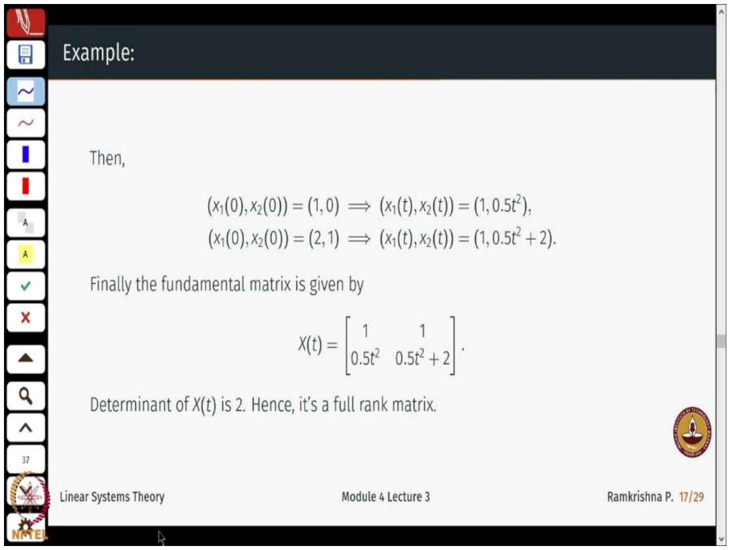

So, the first one we could we could compute. And, again all the steps are exactly the same that we did earlier and therefore, I can say that the fundamental matrix is given by an expression like this ok. This is straightforward a consequence of what we started off the lecture with by defining an example, we can check that this is of a full rank because the determinant is 2 for all times t ok.

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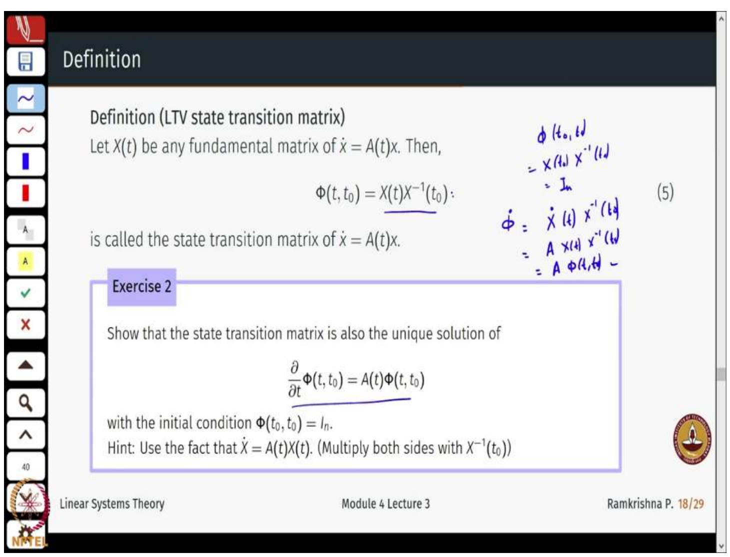

Now, how do we construct a state transition matrix of LTV system starting from this definition of the fundamental matrix. So, the claim here is that start with any fundamental matrix of $\dot{x}$ = A(t)x, then the state transition matrix is simply X(t)$X^{-1}(t_0)$ we also eliminates the thing that I am not really looking at any zero initial condition that not does not really exciting to me. Now, this $\Phi(t, t_0)$ is called the state transition matrix of A (t) x ok.

As usual we will have to verify this claim right. So, can I show that well that the state transition matrix is also a unique solution of this one ok. In the previous lecture we try to verify how the state transition matrix is actually a solution to the matrix differential equation and also to the to the vector differential equation again use one is by checking the initial condition and second also that the solution must satisfy the actual differential equation ok.

So, let us first say that is this solution to my differential equation. So, first just verify what happens $\Phi(t, t_0)$, I have X at t_0 and $X^{-1}(t_0)$. So, I have the identity ok. Second thing is let me check what happens with $\dot{\Phi}$; $\dot{\Phi}$ = $X(\dot{t})\, X^{-1}(t_0)$; what is $X(\dot{t})$? $X(\dot{t})$ was A X(t)$X^{-1}(t_0)$ this is nothing, but A. So, X(t)$X^{-1}(t_0)$ is again $\Phi(t, t_0)$ right. So, this actually is solution.

The uniqueness is guaranteed. Well, we assumed that we know that uniqueness is guaranteed I will not verify those so called Lipschitz kind of conditions we will not complicate out our lives. So, we just assume that god has given us all the equations for which there is a unique solution right. So, let us make our life a simple for the time being, we can dig deeper a little later in the course ok. So, this is nice here right. So, this actually is the solution to the differential equation ok.

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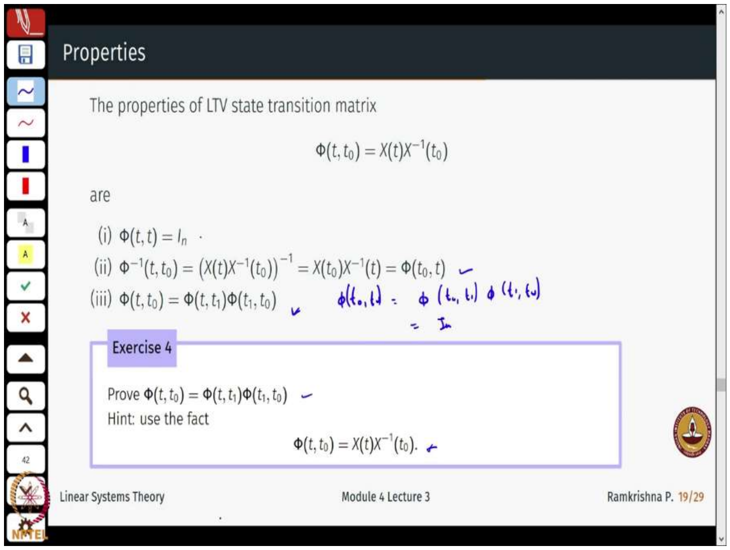

So, few properties which well we can I should take you from yesterday's thing right. So, show that this is easy to show we prove these two properties yesterday in the LTI case, but the proof will kind of be the same or just follow the similar patterns right. So, the first one is freewheel, the second one is also easy to show, the third one we just say that well these are two equal functions. So, just check what happens to them at some initial conditions at say t_0, t_0 and then just satisfies I will just check if three satisfies the same differential equation then the two functions are the same ok.

So, let us just at least satisfy just at least check the initial condition at t_0, t_0 is $\Phi(t_0, t_1)\ \Phi(t_1, t_0)$ ok. So, if you use; so just the previous property this is the inverse, this is phi inverse. So, I just have the identity and I can still I just look at the differentiation of this and claim or easily verify that they satisfy the same differential equation. We have done enough of those steps. So, I will leave that as an exercise for you ok. You can just also use this specific form of solution just to make the proof a little easier ok.

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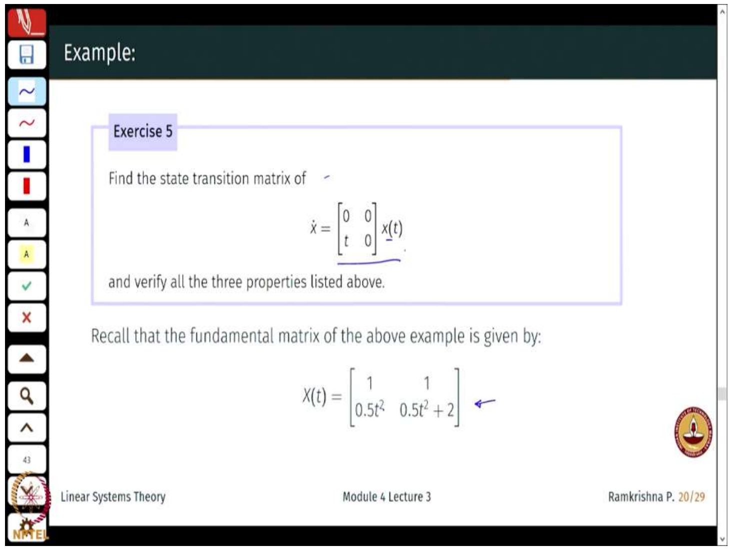

As in as an illustration let us recollect the example that we started with that $\dot{x}$ was $\begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$x(t) and we wish to find the state transition matrix of the time varying system given by this a matrix. So, we earlier computed the fundamental matrix to be of this form a couple of slides ago.

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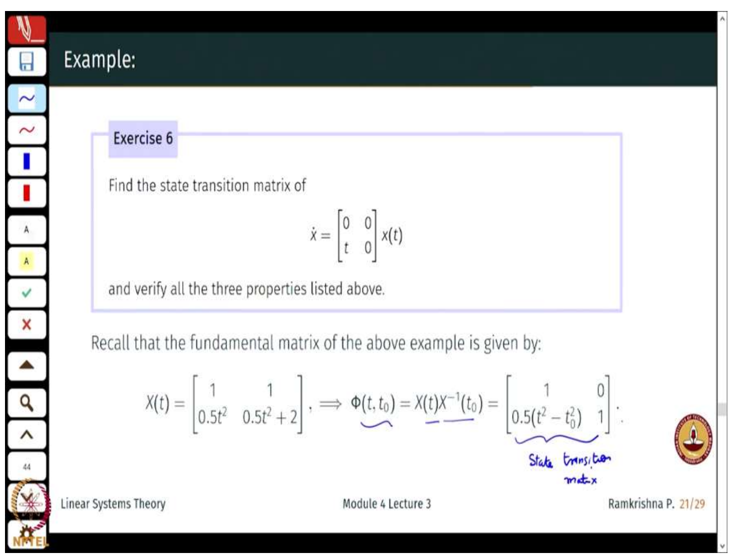

So, to compute the state transition matrix would just be application of this formula right Φ(t,t_0) is X(t)$X^{-1}(t_0)$ and if I just used the definitions it just turns out to be that this matrix over here is my is my state transition matrix for the given system ok. So, that is a straightforward procedure to compute the state transition matrix of a of a linear time varying system ok.

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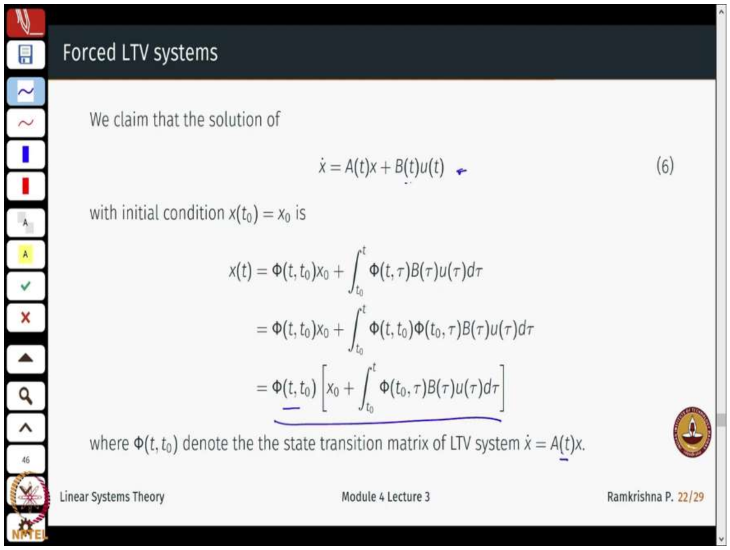

So, the obvious way to look at is again it is it is a simple extension and much of the steps here will be similar to the previous lecture right when we were looking at forced LTI systems ok.

So, much of most of the things remain the same right. So, we claim that the solution to this forced LTV equation where now B is also varying with time is just given by this expression you can just do the steps and arrive at this right and again $\Phi(t, t_0)$ denotes the state transition matrix of the system $\dot{x}$ = A(t) x ok.

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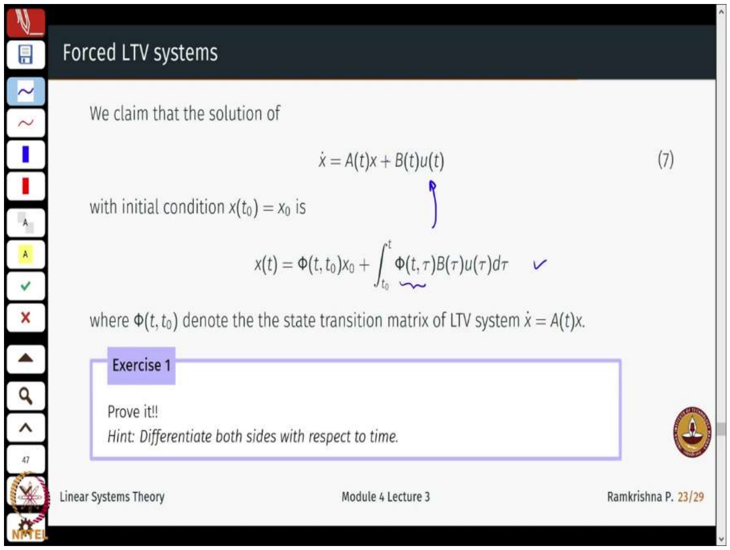

So, again the way we verified yesterday making use of the Leibniz rule, we can still verify using exactly the same procedure. Just that in that case this Φ was $e^{A(t-\tau)}$ now I have a little more general Φ, but nothing really changes in those steps right. So, you just verify just by differentiating both sides and you can say that even actually you realize these two that x of t actually satisfies this differential equation. Again, the steps are skipped for obvious reasons and I would really want you to work out few of these things by yourself to get this concepts and grain into your mind.

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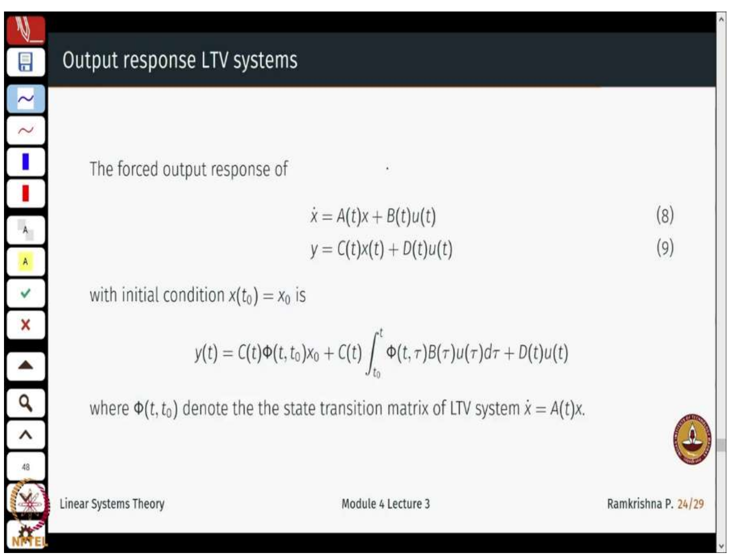

So, once I know x of so, when I once I know solution I can always compute what is the output that is again a very straightforward extension of things that we that we have been talking so far.

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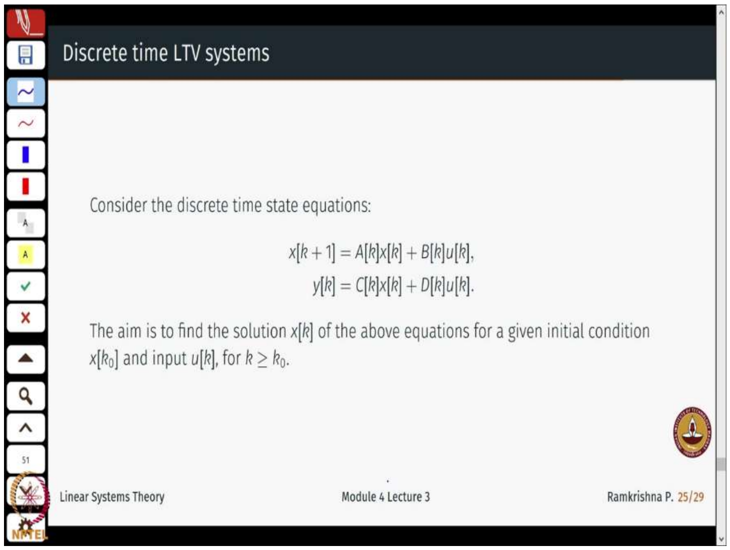

Now, so, the last topic of today would be how can how do I handle systems which are discrete time and also varying discrete time LTV systems. So, the model would be the same as just as that A would now be depending on k. So, the aim is to find solution x(k) given certain initial condition and some input. So, the steps are again similar.

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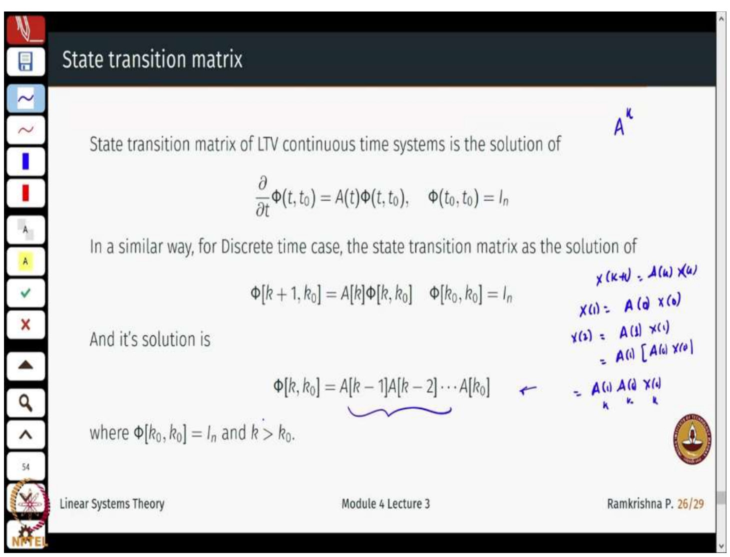

So, we define an equivalent notion of a state transition matrix for the discrete time case. So, the steps all remain the same right. So, I have discrete so the state transition matrix is a solution of this matrix a differential equation and with solution being of all this form. I guess it will be little easier for me to compute the closed form expression in the discrete time as was also little easier in the continuous time what where we had was it was all higher powers of k.

You can also check realize that condition here right. So, if all these A's are constant, you just have A^k ok. So, I did not really write down things to explicitly you know. So, or maybe just for see if I have x(k+1) = A(k)x(k) ok. So, what happens when k = 1? X(1) = A(0)x(0), similarly x(2) = A(1)x(1) this is A at 1. What is X of 1? Is A(0)x(0). So, this will be A(1)A(0)x(0) or here I just call it as k_1, k_0, and so on right. So, that is you know little easier to verify than in the in the continuous time case ok.

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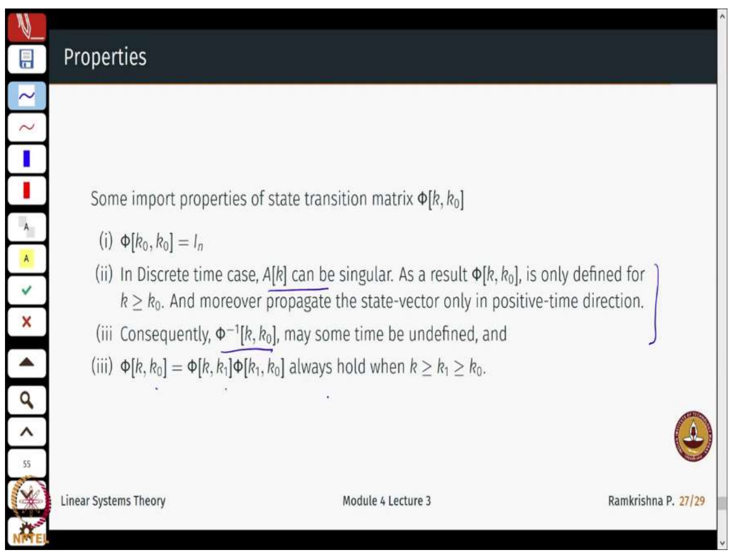

So, the properties again we will translate very nicely from the continuous time case to the discrete time case of the of what is the identity and so on. The only thing we could encounter and we will talk about this explicitly while we are talking of controllability of our discrete time systems is that this matrix A; A times A of k can sometimes be singular and therefore, the Φ^{-1} may not be necessarily defined all the time ok.

We will postpone this discussion when we actually encounter a situation like this whereas all the other cases actually remain the same. All the other properties of the state transition matrix remain actually the same ok.

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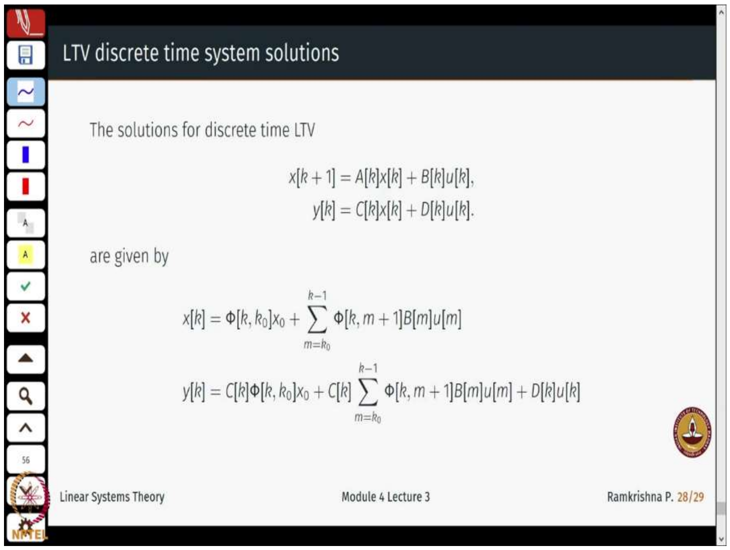

So, similarly if I have input so, I just augment to my solution with the with the input terms right. So, and similarly I can compute the output. So, just straightforward exercise here once you have understood the continuous time case.

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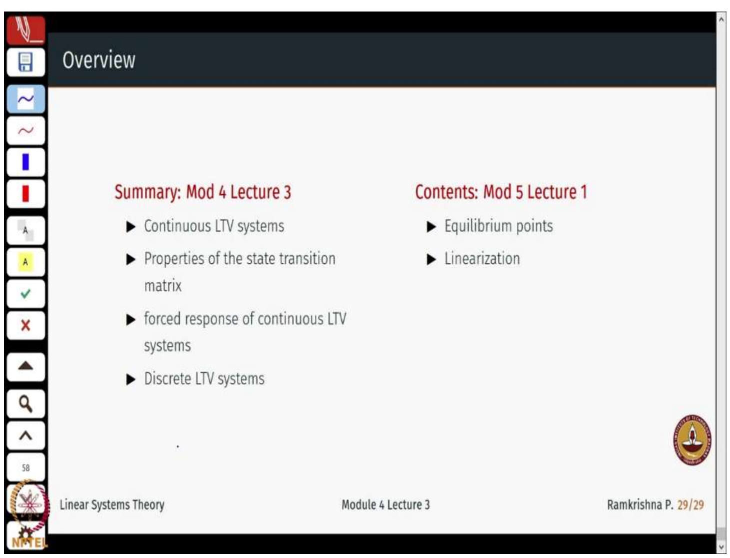

So, to conclude so, we started this lecture by looking at solutions to LTV systems and we actually made a nice justification to why we use the matrix differential equation as a basic building block to compute what is called the state transition matrix and we also looked at towards the n discrete time LTV systems.

So, this concludes week 4, starting week 5 we will look at different kinds of equilibrium points of systems both from the linear and the non-linear case and then motivate our case towards linearization of non-linear systems and what are the properties that are retained and what are the properties that may not necessarily be retained while we do the linearization procedure is there only one way of doing linearization or there are multiple ways. So, all this will be covered in the next weeks lecture.

Thank you.