Linear Systems Theory
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Module – 04
Lecture – 01
Part 02
The State Transition Matrix

So, let us come back to again our discussion on solutions of linear state space equations.

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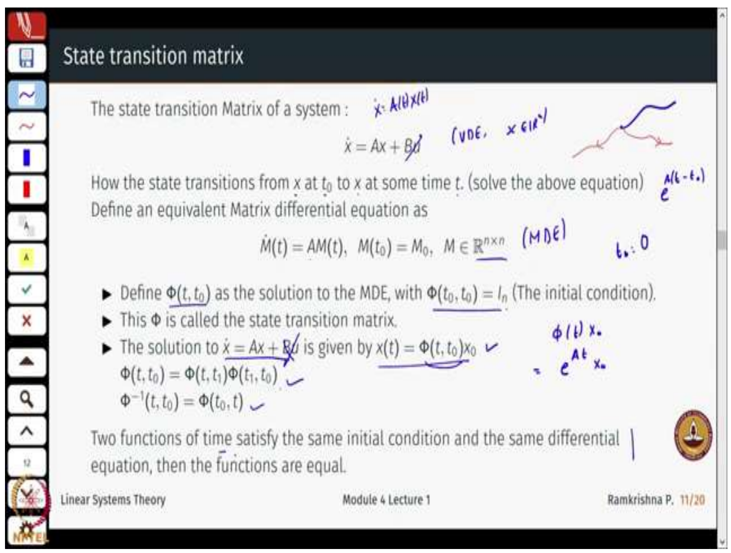

So, again we will start with $\dot{x}$ = A x for the moment we will leave out the input term and define slowly the concept of a state transition matrix. So, it essentially tells me how a state goes from a certain x at time t =0 to its any x at some time t. So, given an initial state how can I compute the state at some future time, time t, essentially like I am looking again at solutions of the differential equation.

So, let me first define an equivalent matrix so differential equation or this, this was my vector differential equation, where because x was in R^n. So, this is a matrix differential equation right, where $\dot{M}$ = AM(t); A is again n x n matrix and M is also now has a matrix representation ok. It slowly be clear why I am doing this, and we will derive this a little later, but for the moment we will just learn it a little not very constructively, but something which will help us understand or we understand the basics of state transition matrix ok.

So, let me start with this matrix differential equation and let me define as $\Phi(t, t_0)$, as the solution to this matrix differential equation with the initial condition being the identity ok. So, I am just using any arbitrary time as my initial condition even though t_0 can also be equal to 0. So, we will alternatively use this to make computations easy, where you know in many cases I will call or I will assume that t_0 is actually equal to 0; even though nothing changes in general generality ok.

So, this matrix Φ is called the state transition matrix ok. Now, if this is the state transition matrix, so what I claim here which I will prove is that the solution to $\dot{x}$ = A x ok, the B goes away here right. The solution to $\dot{x}$ = A x is simply given by this one, x(t) is $\Phi(t, t_0)x_0$ ok, now this is a matrix right. So, $\Phi(t, t_0)$ is a matrix, which is the solution to this matrix differential equation and just multiplied by the initial state ok.

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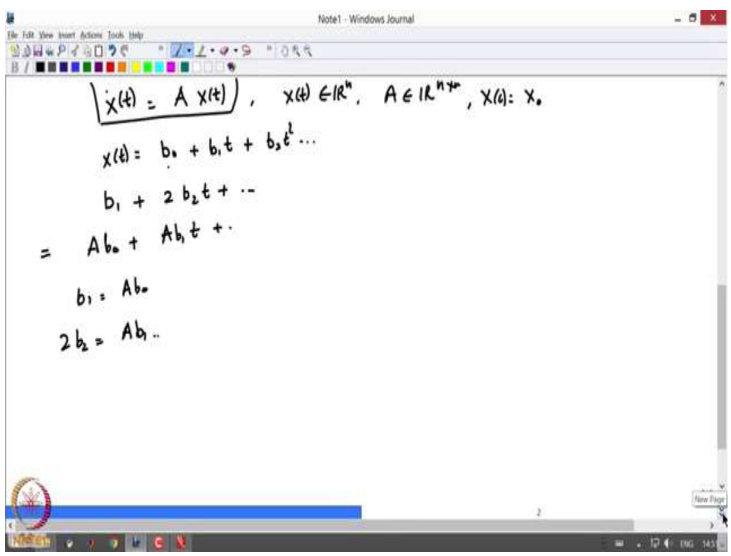

So, let us try and prove this one.

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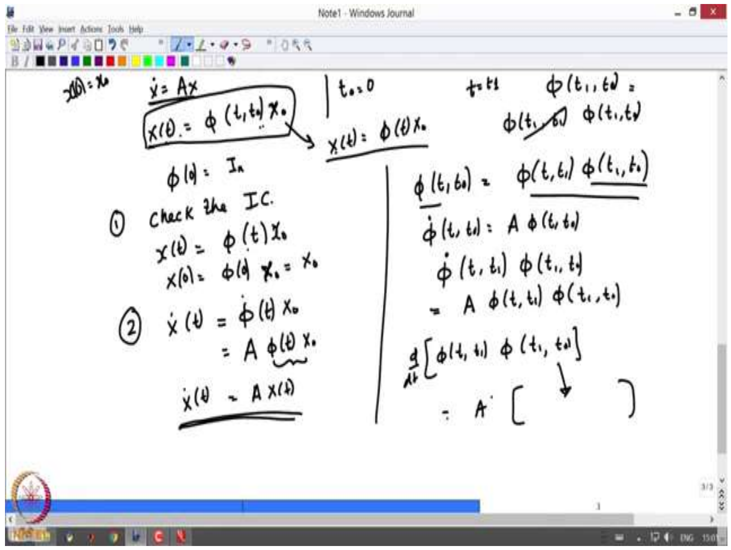

So, again, so I start with $\dot{x} =$ A x and what I claim is that this is further the solution to $\dot{x} =$ A x is given by x(t) is $\Phi(t,t_0)\, x_0$. So, let us for simplicity assume that $t_0 = 0$, even though nothing changes for this just my expressions just look a little easy ok. So, what is the property of the state transition matrix or how did I define this, well this was the first thing like $\Phi(t_0,t_0)$ was the identity or the initial condition of the matrix differential equation was the n x n and identity matrix, ok.

Now, what I claim is that the solution to $\dot{x} =$ A x is the solution to $\dot{x} =$ A x is just this expression here ok. So, how do I verify this ok, first is check the initial conditions ok, what is the initial condition for this for the original differential equation that x at 0 was x_0.

Now, I will just do this right. So, this is the solution which I proposed it x(t) is $\Phi(t,t_0)$ I will just say Φ, t_0 is 0 so I will just omit that times x(0) ok. Now, what is the let me verify the initial condition x(t) is Φ (0)x(0). So, what is Φ (0) is the identity, so this is x sorry, I should write this x(0) here sorry, this, x(0) this is to verify the initial condition is Φ (0) x_0. So, what Φ (0) is the identity and therefore, I retain the initial condition ok.

Now, second thing to verify is well if a certain x(t) is a solution to $\dot{x} =$ A x, then it must actually satisfy the differential equation right that is what we learn as a trick of backward substitution, when I may be writing a competitive exam; I do not really compute the solutions, but I actually checked if the solution satisfies the original differential equation right, ok.

So, let me do what is x dot t here, $\dot{x}(t) = \dot{\Phi}(t)x_0$, I will just again assume that $t_0 = 0$. Now, where does $\dot{\Phi}(t)$ come from well $\dot{\Phi}(t)$, so is the solution to this matrix differential equation which means $\dot{\Phi}$ is such that it satisfies, $A\Phi(t)\, x_0$. Now, what is $\Phi(t)\, x_0$? $\Phi(t)\, x_0$ is exactly this, this is A x(t). So, I recover starting from this solution which I rewrite for $t_0 = 0$ as $\Phi(t)\, x_0$. I start from this a differential equation I sorry, this solution which I proposed for $\dot{x}$ =A x, I substitute for x equal to here and I recover again the original differential equation.

So, the initial conditions agree and also the differential equation agrees, it satisfies so this function which I proposed as a solution, satisfies the initial condition and it also satisfies the differential equation right. And therefore, well this is the so this statement is true right, then the solution to $\dot{x}$ =A x is given by $x(t) = \Phi(t, t_0)\, x_0$, right.

So, they also a little goes on to say or what we use here that if two functions of time, they satisfy the same initial conditions and the same differential equations and these functions are equal right, so that is what I make use of our which kind of in a way obvious. And what is a little key here which is not explicitly, discussed when we do our basic state space representation in control one is that solutions to a given differential equation exist and they are unique.

Essentially means given an initial condition, I can only travel in one particular direction. There is there will be only one solution or that I stand here and I can go this way, this way, up, down all those are not allowed right. So, what so there are some conditions which guarantee existence and uniqueness of solutions to differential equations. We will not do the details of that, but for our course we will assume is not an assumption that it is actually true that solutions will exist and they will be unique.

So, my uniqueness it means given an initial condition there will be only one solution that will emerge out of that initial conditions. So, if I were just to draw a trajectory, say this is x_0 and the solution goes this way, this is the only possible one there will be nothing which goes maybe this way or even this way ok. So, existence and uniqueness is this is guaranteed, but we will not really go into the details of that ok.

The second thing which we can prove is the following right. So, x sorry, we are write this term is $\Phi(t, t_0)$ is $\Phi(t, t_1)\, \Phi(t_1, t_0)$ ok. So, again Φ is a solution so let us just check if the left hand side and right hand side, are do they satisfy the same initial conditions and they satisfy the same differential equations. And then I exploit this statement right. So, if two functions satisfy the same initial condition and the same differential equations and essentially the functions are equal ok. So, in the left hand side I have the following, so I know this already right $\dot{\Phi}$(t,t_0) is A $\Phi(t, t_0)$ ok. So, first check for any initial condition right or just check for any value of time t if they agree.

So, let me check at Φ say at t = t_1, on the left hand side I have $\Phi(t_{1,}t_0)$; on the right hand side what I have, I have $\Phi(t_{1,}t_1)$ $\Phi(t_{1,}t_0)$. Now, what is phi t 1, t 1 this is the identity and therefore, well they both agree at point t 1. So, let me assume that as my you know as a condition where I am checking if they are they are equal or not, ok.

Now, do they satisfy the same differential equation well, I know already they said that $\dot{\Phi}$ is A Φ. Now, let us look at the on the right hand side, on the right hand side well if I differentiate this well this is all these are constants, because I can compute them at t_1 and t_0. So, this is the time varying terms I have $\dot{\Phi}$(t,t_1) $\Phi(t_{1,}t_0)$ ok.

Now, what is $\dot{\Phi}$? $\dot{\Phi}$ is A Φ(t,t_1) Φ(t,t_0). So, they so these two functions satisfy the same differential equation, so such that so I have on the left hand side, I just differentiate this function. So, I just write this side, so d by d t of this entire thing here write Φ(t,t_1) Φ(t,t_0) will give me A times the entire function rewritten here itself ok. So, the second thing is also true and I think well, this can be a little easy to prove. So, I will just leave that as a little exercise right, ok.

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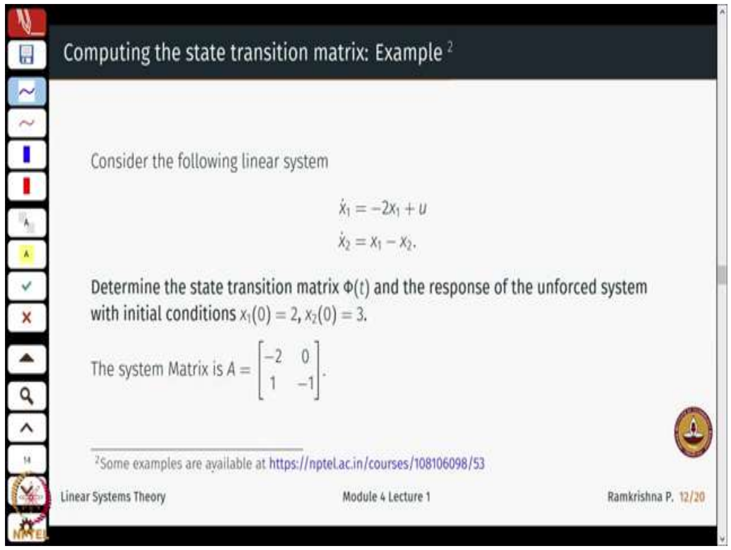

So, let us spend some time just solving particular thing right. So, again unforced system, so u goes to 0 that I have that the system matrix is of this form. So, just before this way. So, why am I writing this in a little more generality is because I also I like this notations can also be used, when I am using a time varying system for example.

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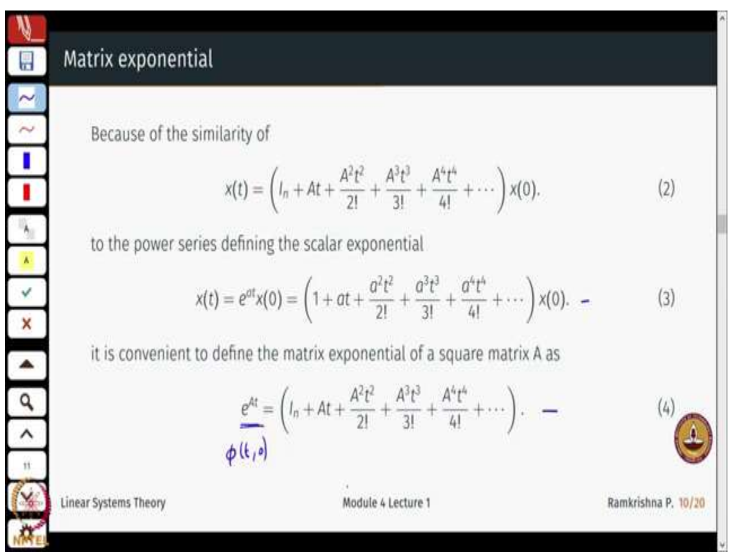

And for now it must be little easier to guess the following that in the case of LTI case, this is my state transition matrix starting from time t equal to 0 and at any time t and because

I get the solution if you look at it, so what I call it, so the solution to $\dot{x}$ = A x + B u or just or without the input is given by Φ(t), let us assume t_0 = 0, x_0.

So, what is Φ(t), Φ(t) is e^{At}. It also satisfies the property that this initial condition is the identity, just put t equal to 0 here and you get the identity. For if you if it just for any arbitrary t, (t,t_0), you will have things like this; $e^{A(t-t_0)}$ that satisfies that a t = 0, this initial condition is the identity. So, a little so this is an easy to guess starting from here right that e^{At} is essentially the state transition matrix for a linear time invariant system of the form $\dot{x}$= A x + B u ok.

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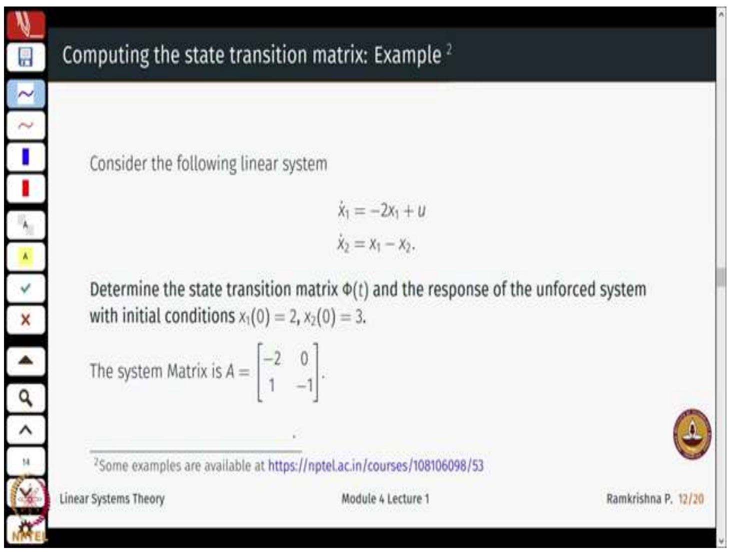

So, let us see how to compute this well, I have given two differential equations with certain initial conditions, system matrixes is just like this and I just do this way right.

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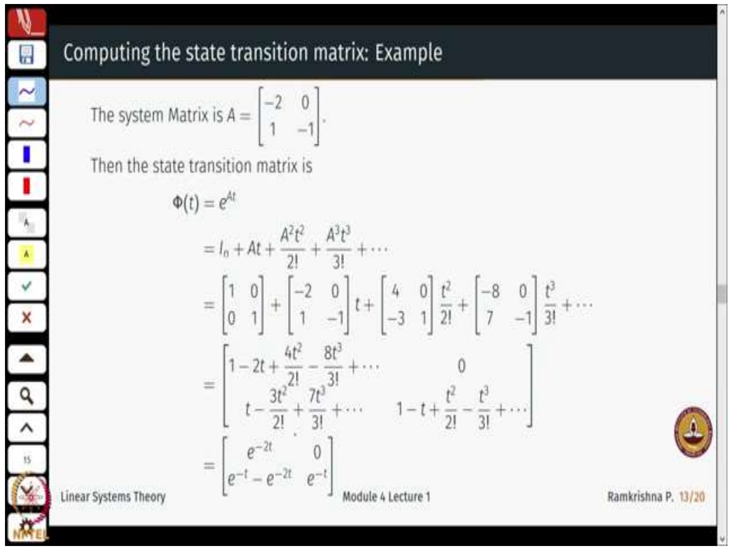

So, I just expand it in powers of A and I do all the complex math and I can conclude that this is the solution e^{-2t}, because this has the form of e^{-2t}, 0 here and so on and so forth ok. Now, do I really do this all the time right I just give me any matrix I should I expand it at in powers of t and A and then try to get a solution right.

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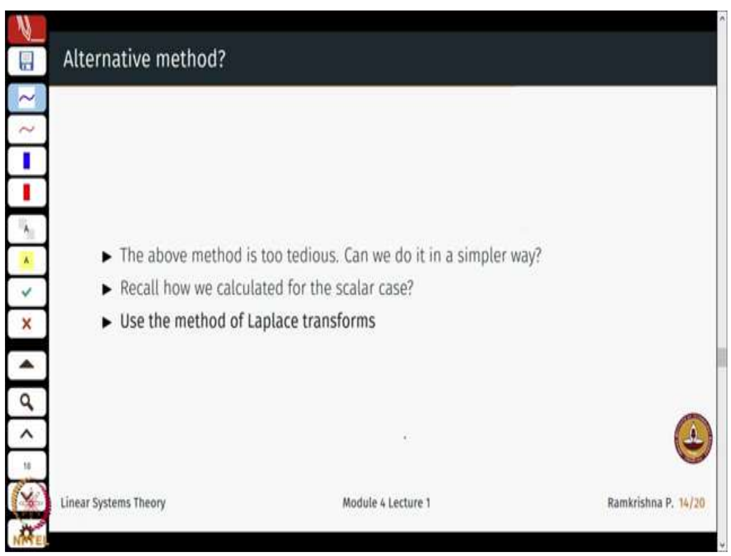

So that might be a little tedious, but what we know which we already started motivating with is the use of Laplace transforms, now that makes life much easier for me right.

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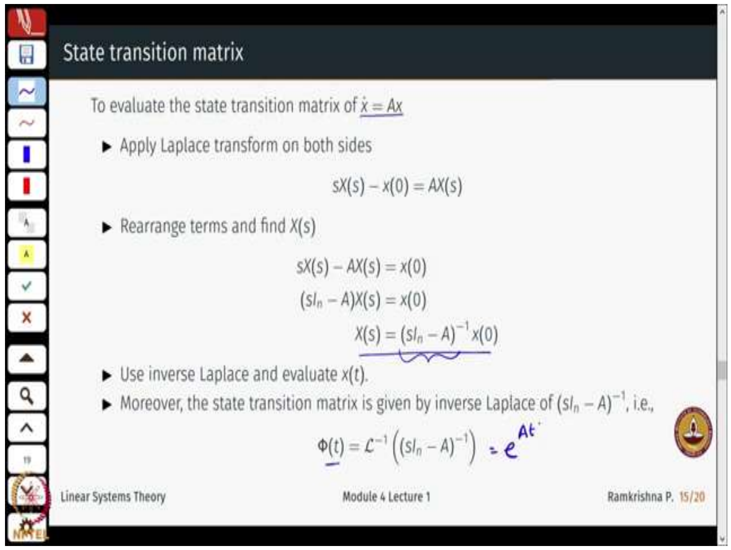

So, if I have a general state space representation $\dot{x}$= A x, this the state transition matrix ok. So, let us see how it goes right, so I have X(s) as $(sI - A)^{-1}\, x_0$, this (sI-A) is always invertible and I think we should remember this from our basic control course if not, just let me know I can do maybe proof of that in the forum.

But a very important thing to understand here reset (s I- A) is always invertible ok. So, once I know this if I write this in the time domain well, this essentially the inverse Laplace of $(sI - A)^{-1}$ is essentially my state transition matrix, which also takes to this form e^{At}, right ok.

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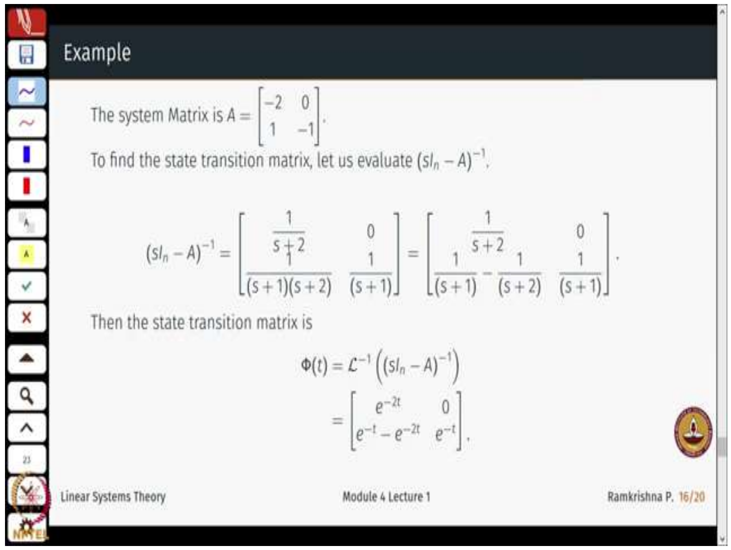

And then I can do the entire stuff again and it actually becomes a little easier right. So, I can so given transfer function kind of thing here I can always rewrite it in this time domain, why are the inverse Laplace transform. I do not need to teach you this, this has been done in a control one also some basic courses on differential equations.

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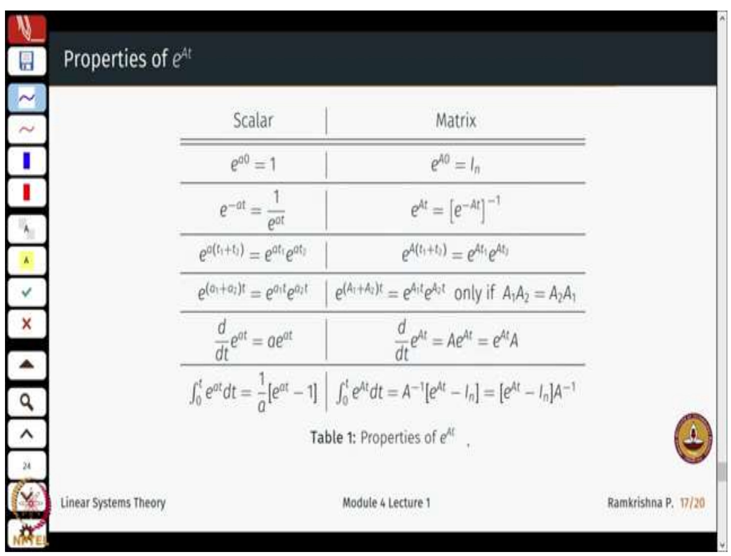

Some little properties well, at t = 0 is of course, identity this well all this I think these are these are easy to prove right; $e^{At} = (e^{-At})^{-1}$, this is also easy to prove $e^{A(t_1+t_2)} = e^{A(t_1)}e^{A(t_2)}$, this look like those lots of indices which we study in high school ok. This I will leave for you to derive by yourself that $e^{(A_1+A_2)t}$ is $e^{(A_1)t}e^{(A_2)t}$ if and only if the matrices commute that $A_1A_2 = A_2A_1$.

And lastly you could $\frac{d}{dt}(e^{At})$ is this one and this is also equal to $e^{At}A$ and lastly with the integral version of A. Now, this scalar case is actually obvious to like relate.

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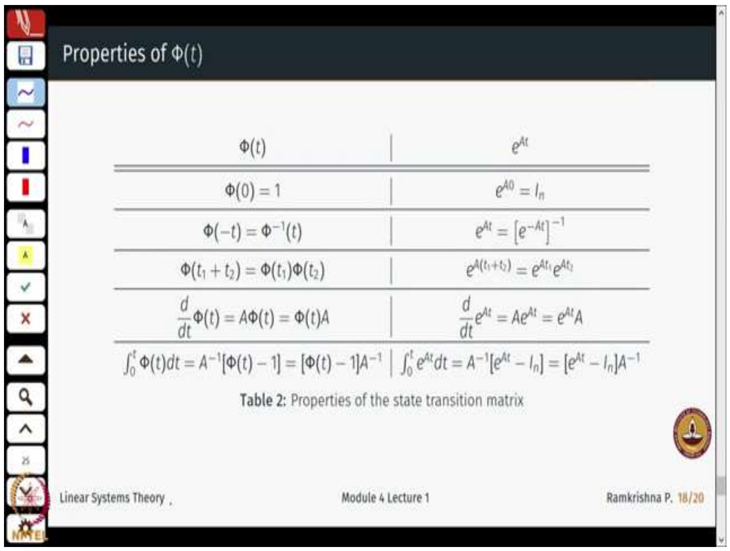

So, in general well if I am also talking of a time varying case or whatever I will just have instead of e^{At}, I just have some straight transition matrix which is called Φ(t) which has its own set of properties. Now, couple of things we already derived over here right like sorry, I go over here. So, in general what will the state transition matrix is satisfy ok. So, just these are the properties, but and you can relate them to each time to the to this special case of state transition matrix which is e^{At} ok.

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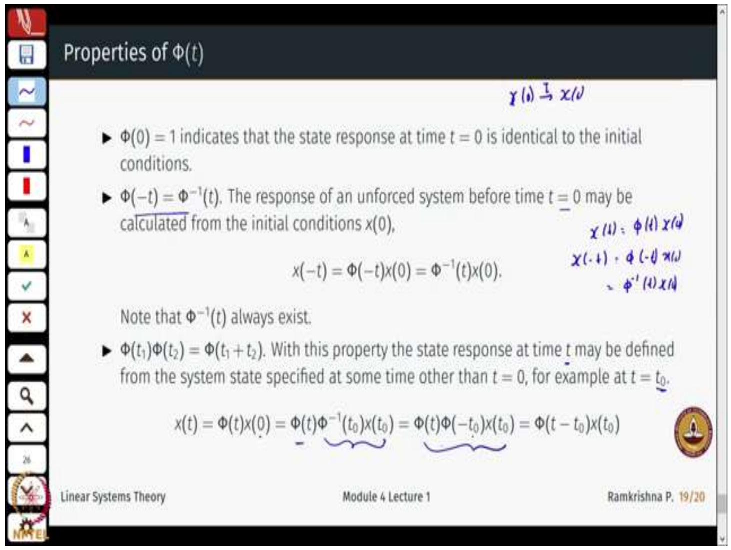

So, $\Phi(0)$ = 1 what does it mean that the state response at time t = 0 is equal to the initial condition. So, if I start at t = 0 and I measure at t = 0, I always be at that same point. So, the transition is just the identity like what takes me from x at t = 0, to x at t = 0 that is identity ok.

Now, $\Phi(-t)$ is simply the $\Phi^{-1}(t)$, physically what does it mean this is the response of an unforced system before time t equal to 0 maybe calculated and this can actually be calculated from the initial conditions, because what is my solution; solution I always wrote in terms of $\Phi(t)$ sorry, x(t) is $\Phi(t)$ assuming that the initial time is 0 times x at 0 ok.

Now, I just look at what is x at -t this is simply $\Phi(-t)\ x_0$ and this is $\Phi^{-1}(t)\ x_0$ right, assuming that we could actually derive this thing ok, it also in some way resembles this one. So, I just leave that for you to derive for its this is again in the same steps as what we did in the first two; in the first two lines of or the first proving the first two statements ok.

And Φ^{-1} always exists, again I do not need to prove why that is true, but it is kind of obvious on the statements that we have made the since the beginning of this lecture. And similarly $\Phi(t_1)\ \Phi(t_2)$ = $\Phi(t_1 + t_2)$ ok, this also you can prove fairly easily ok.

What does this property tell me, with this property the state response at any time t may be defined from the system state specified at some other time then t_0. For example, if I do not know what is the at say that at time t = 0, but I had no at some t =t_0 ok. So, how do I find that; so I do not know this information, I do not know what is x at t_0, but I would want to, but I know what is x at some arbitrary time t_0 ok.

So, I just use the properties which I know earlier. So, Φ(t) I can write it this way ok, and then I can use this property to rewrite ok. So, I can use the second property to rewrite it in this way and then the third property we just plug in here as $\Phi(t-t_0)x(t_0)$ok.

So, it is just rewriting the first two properties and then making use of this. So, with this property again I repeat that if I do not know what is x at t = 0, but I know something at x equal x the value of x at some time t naught. Can x will compute x(t), well the answer is yes and I just make use of a property which is kind of beautifully looks like this.

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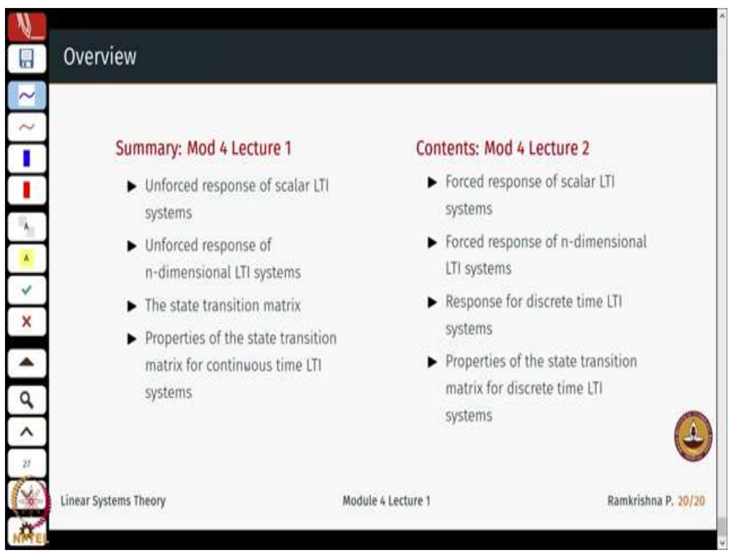

So that concludes the lecture – 1, where we started off with scalar differential equations unforced and saw the unforced response and generalized that to an n-dimensional state space system and defined the concept of state transition matrix and listed out few properties, which will be useful for us ok. One thing before I conclude is the following right.

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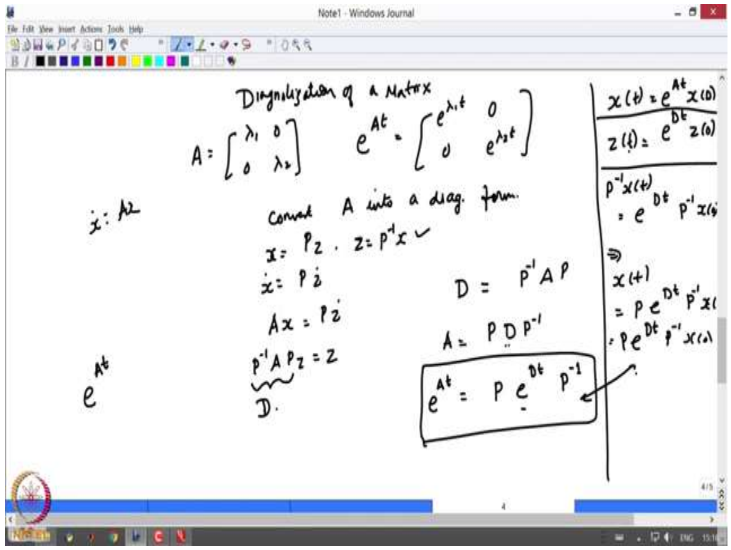

So, what we did in some of our previous lectures was about the diagonalization right of a matrix ok. So, let us see I am given a matrix A, which looks like this $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ ok. It is easy to verify that e^{At}, in this case is simply a matrix which looks like this $\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{bmatrix}$.

Now, I know that if A is not in a diagonal form, I can convert it into a diagonal form; A into a diagonal form ok. Now, let us say I can convert say x =Pz. So, $\dot{x}$ = A x =,P$\dot{z}$, I just do P^{-1}A Px = z. Now, I know that I can convert this into a diagonal form ok.

Now, let me call this D ok, this D is P^{-1}A P and I wish to find what is e^{At} or the in other words I can also write A as P DP^{-1}, where D is a diagonal matrix ok. Now, it turns out this is a little proof actually a small proof e^{At} = P$e^{Dt}P^{-1}$, which is easy for me to prove or easy for me to compute. So, this is actually a more computationally efficient way of computing e^{At}, where the diagonalization ok, this actually will be z here ok.

So, let us see if we can actually prove this. So, I know that x(t) = e^{At} x(0) ok. So, what is z(t) what is the relation between z(t) and x(t) I know that z(t) can be written as in terms of x as so this is z = P^{-1}x. So, I have P^{-1}x(t) is on the right hand side I have e^{Dt}; what is z_0, z_0=$P^{-1}x_0$. Coming again from here, so this implies that x(t) = P$e^{Dt}e^{-1}x_0$, sorry they said. So, this is P$e^{Dt}P^{-1}x_0$ ok, so this is a little proof of how I can compute.

Now, e^{At} when I know that I can diagonalize the matrix A. Similar expressions also exist, when I have when I cannot diagonalize A matrix, but I have the equivalent Jordan form of the a matrix, so that was a little illustration of why the kind of tools we learnt in the previous 2 weeks of lectures were actually useful.

So, we will in the next lecture do solutions to equations which have input in them or we will look at in other words, force response of scalar LTI systems and then we go to n-dimensional systems and also have a little bit of introduction to discrete time systems, so that just coming up shortly.

Thank you.