Linear Systems Theory
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Module – 4
Lecture – 1
The State Transition Matrix Part 01

Hello. Welcome to this week 4 of lectures on the course on Linear Systems Theory. And if you have come this far you are actually doing pretty well I presume in the course. So, we started off in week 1, we just some general examples of building mathematical models of systems and of course, we did also investigate certain properties of systems and what can be inferred from the models and so on. Week 2 and week 3 were just dedicated to just equipping ourselves with tools from linear algebra, right. So, and this tools would be very useful for us to the to the remaining 8 or 9 weeks of the course.

So, today we will start with some basic state space models and which are essentially in form of differential equations and what does the solution of those equations mean. Can we use some properties that we learnt earlier; some things might be obvious to us, something might be a little new, but they can be yeah you know kind of derived from the from the from the background we already have.

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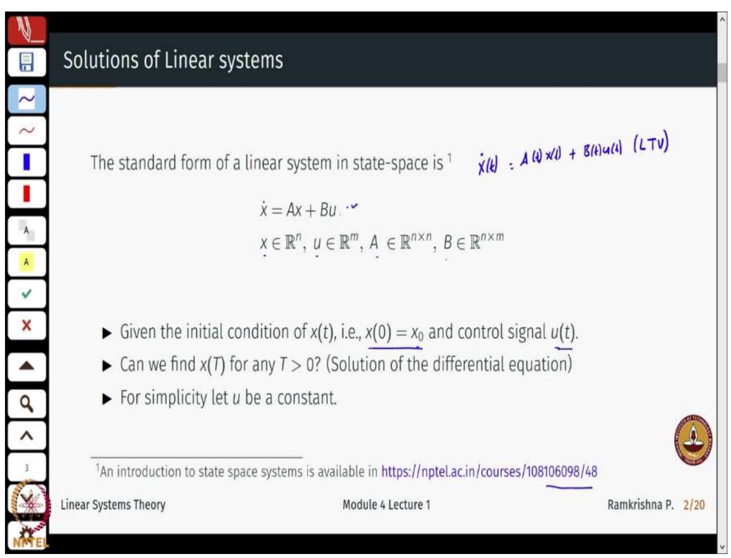

So, so that will. So, the that concept is called the a state transition matrix that we will spend a bit of time on today. So, we start with a standard form of the of a linear state space systems. I am not at the moment talking of outputs just look at the state equation $\dot{x}$ = Ax + Bu. I use (Refer Time: 01:52), ok.

So, I can write this more generally as $\dot{x}$(t)= A(t)x(t) + B(t) u(t). So, this is like linear time varying system whereas, this is just a time invariant system. So, we will deal with both of them and start with our little simpler case of time invariant case and then slowly build up our case to solve to look at solutions of a time varying systems, ok.

Now, how to write a state space model for a given system? So, we have done a bit of that in our first course on control engineering. And I think much of you who are listening to this course would have had some course in basic control with a little bit of exposure to state space analysis of systems. So, if not then you can just revisit those lectures over here, so I would not want to redo that because it is essentially the very similar kind of stuff that we did over there. So, we will assume that we know how to construct state space models $\dot{x}$ = Ax + Bu, where x is a state vector, u is a input, A is a system matrix and B is the input matrix of appropriate dimensions.

So, the question that we will ask here is well given a initial condition of the state x(t) at t = 0,let me call it x_0 and some control input u(t), can we find. What is given? Given is the initial state of the system and a certain control input. Can we find the state the value of state at any time T > 0? Could be 10 seconds, 20 seconds, a million seconds and so on, ok.

So, essentially in terms of mathematics, in mathematical terms I am looking at can I solve this for this equation, right, there is a differential equation, can I find a solution of it. So, let us begin with a little example to motivate our case and also further simplify it by letting u to be a constant input. Just like, and if I take and a circuit like this we will just be 1 volt or 2 volts or whatever, ok.

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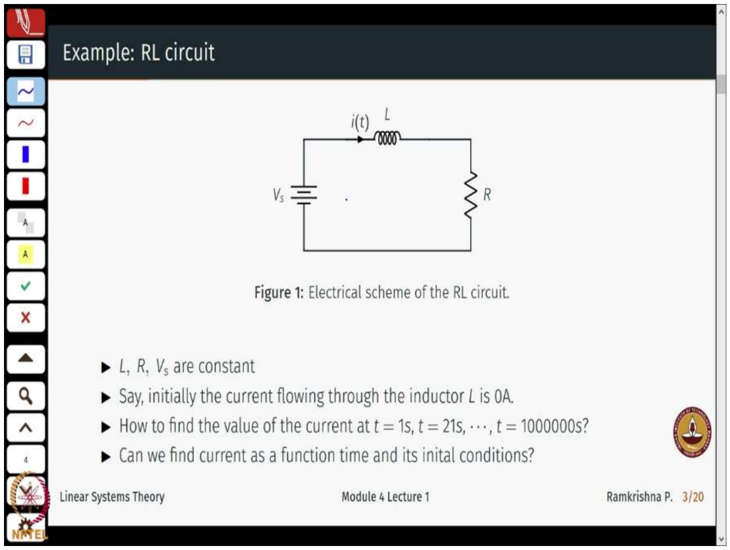

So, in this system, so I will have R, L circuit with a constant voltage input. And so, what is given to me are L and R which are the constant parameters of the system the input voltage. So, what I would like to see is the evolution of current. So, as soon as the voltage is applied at say t equal to 0. Let me also assume that the initial current through the inductor or and therefore, also so the resistor is 0.

So, the question is how do I find the value of current at t equal to say 1 second, 20 seconds and so on, right. And what will one expect it to be a function of when the current will definitely be a function of course, the system parameters it will also be a function or it will also depend on its own initial condition, right and of course, it will have it will depend on what is the input also, right. So, can we find the current as a function of time and its initial condition? Is our question.

And, we would have solved this equations several times in maybe first course on networks, we have possibly even a first course on control engineering, more immediately in a you know more obviously, in a in a first course on differential equations and so on. So, we will just revisit that you know for the for our own benefit or to make a bit of continuity or self-containment in the course, ok.

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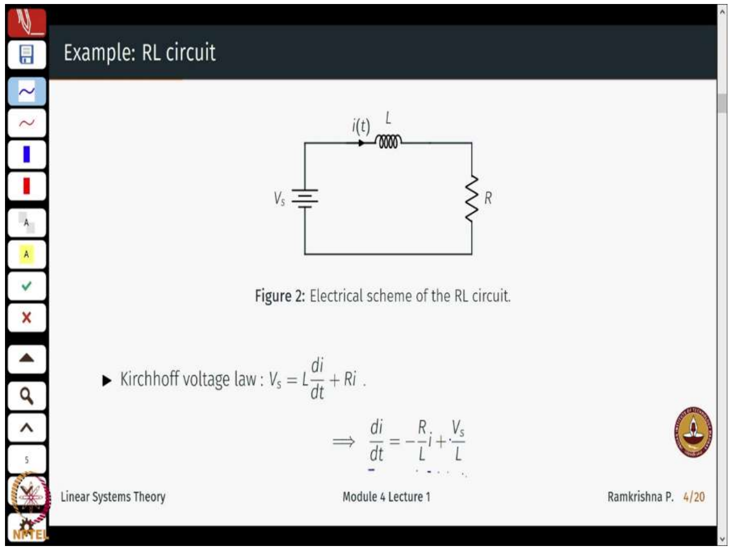

So, where do I start with I know that I can write down the differential equation which comes from the Kirchhoff's voltage law and I can just write a differential equation $\frac{di}{dt} = \frac{-R}{L}i + \frac{V_s}{L}$, ok. So, this things we already know.

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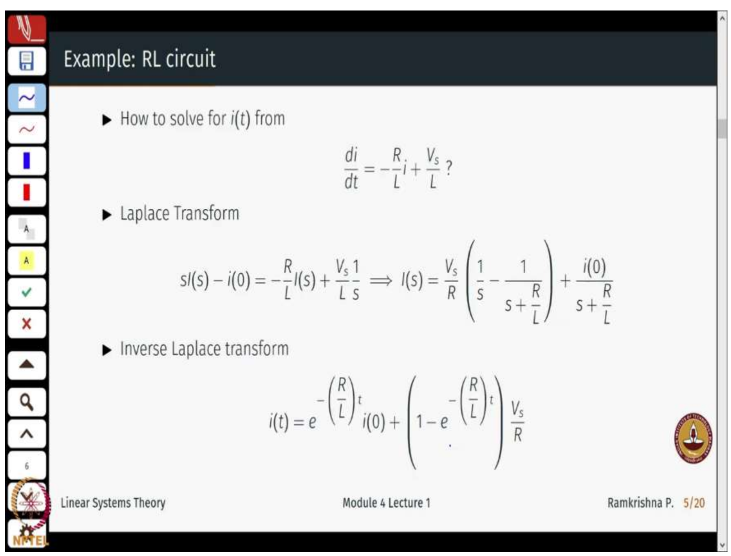

We also know that, well I can solve I will use the method of Laplace transforms at which was one of our earlier tools that we learned in control engineering or even in mathematics of how to use a certain Laplace transform to solve ordinary differential equations. The advantage comes that it converts a differential equation into a linear equation, multiplication, so convolutions become a multiplication. So, it is a little easier for me to handle, right.

So, I just use the Laplace methods and then compute the solution to be of this form. Like something depending on the initial condition, something depending on the voltage and of course, the system parameter R and L. And the of course, the variable which is the current here it evolves over time and the t is captured over here, ok.

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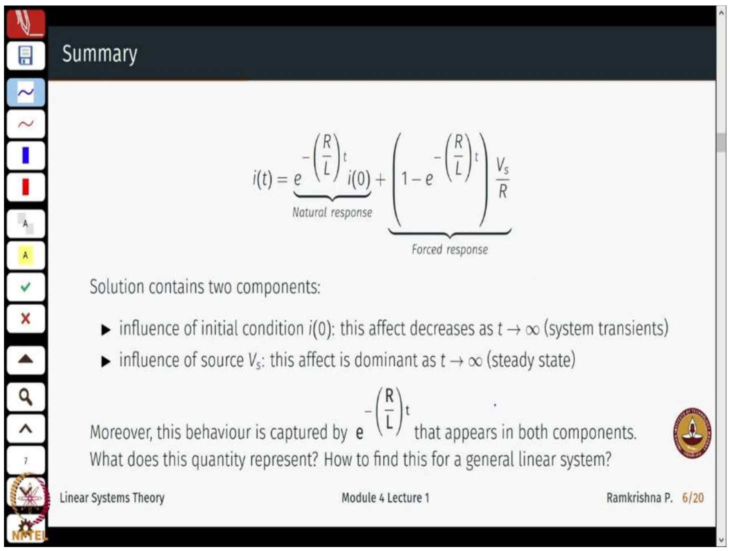

So, the system will essentially have two components or the profile of the current which we are interested in observing will have essentially two components, the natural response and the forced response, ok. So, the natural response or this takes care of the system transients and also with the influence of initial condition. If the initial condition is 0 as we assumed, so this term will just go away. So, everything will just be based on this, ok.

So, the two components are the natural response, and then the effect of the external input the voltage in this case, right. And if you look at it say assume that I(0) $\neq$0, say it is equal to 1 and as time progresses what we see is that the effect of this term it goes no it asymptotically goes to 0. As t becomes very large this natural response the term it goes closer and closer to 0, right.

So, we can say that this effect of the natural response of course, I am talking of stable systems here, so that it is so this, so that is effect is a little natural that the effect decreases as time progresses, right. So, this is also like the system, a part of the system transients. Then you have the influence of the voltage and this you gets dominant as time increases. So, the effect of the natural response decreases and the forced response increases, and then it will, so what will remain at the steady state is only the terms corresponding to the forced response, ok.

So, what is interesting to observe is this term $e^{\frac{-R}{L}t}$ appears in both components, ok. Now, is this general or is there any interpretation to this kind of quantity here? Right. An exponential term which of course, which has the time component and also to do with the system parameters. So, we will try to interpret this in a in our little general setting now, ok.

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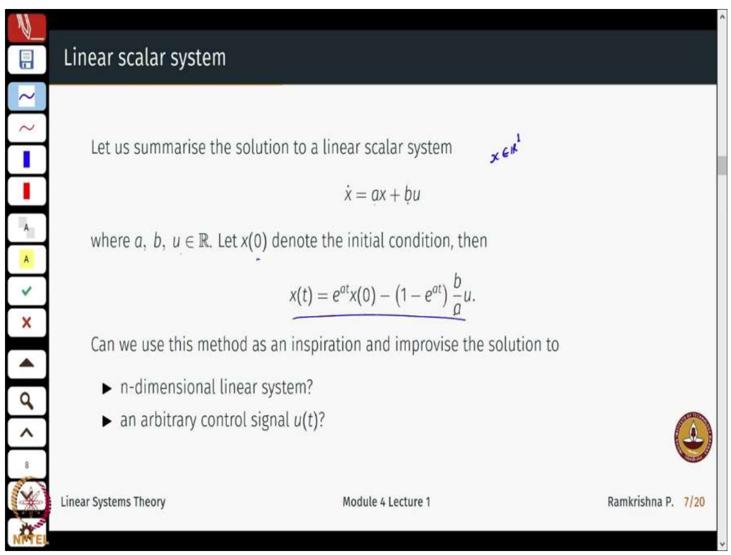

So, let us start building our case with a scalar system, right, x is just in R as opposed to say R^n, as opposed to an n-dimensional state vector. In the for a and b will just be will also be in R, u will be in R and so on. So, if x(0) is the initial condition then the solution can be written in this way. So, this is in some sense also you could go back to this equation that we had here and then just compare what we are trying to do here, right. So, this is also a first order equation with certain initial conditions, and then I can if I generally have $\dot{x} =$ ax + bu I can write a solution which looks like this, ok.

Now, can I do something more? Right. Can I write this for an n-dimensional system that is what we will be interested in and of course, any arbitrarily time varying control signal u(t). So, that is what we will spend time doing that, ok.

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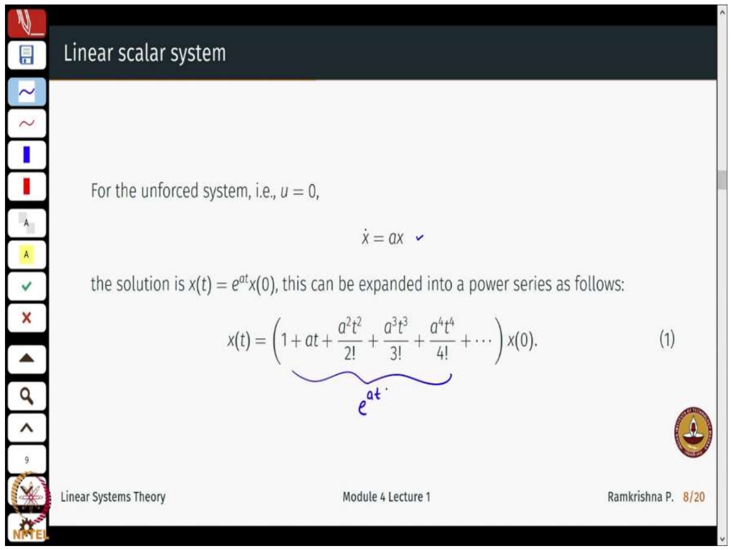

So, let us first start with the unforced system. This, it is equally interesting to look at the solutions of the of the unforced system. So, essentially let us start with a scalar case $\dot{x} = ax$ and let me assume that, ok. Let us just write down few things here.

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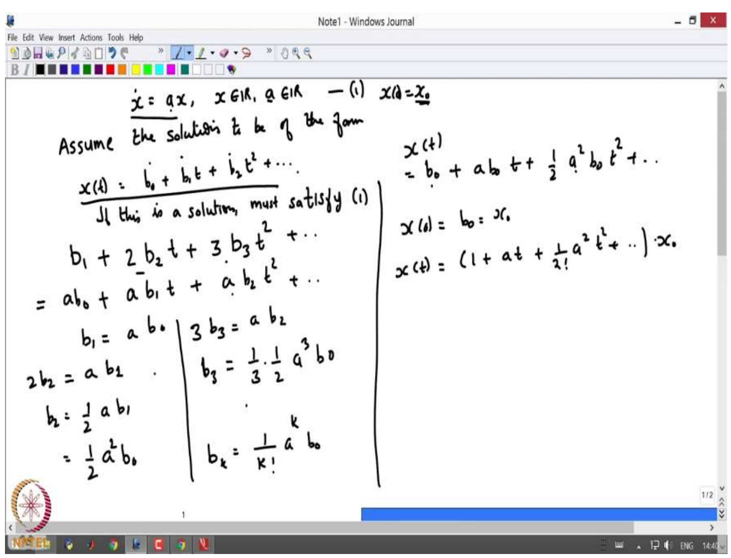

So, I have $\dot{x}$ = ax at everything is in R, x is in R, a is also in R, ok. So, let me assume the solution to be of the form say it is it is like a series in time, right. So, this has b_0 + b_1t + b_2t^2 and so on like a like a power series in time, ok. Now, well, is this a solution, right. And if take assume that this is a solution and if this is a solution then it must satisfy. Let me call this equation 1, and it actually satisfy the equation 1, the ordinary differential equation 1, ok.

So, let me do, let me differentiate. So, I just want to write it in this right $\dot{x}$ = ax. So, on the left hand side $\dot{x}$, so I just differentiate this I will have b_1 + 2b_2t + 3b_3t^2. . . . So, ax, this is my x. So, this will be ab_0 + ab_1t + ab_2t^2 and so on, right, and all the all the higher powers, ok.

Now, if this is the solution, let us also assume that x at 0 is some x_0, ok. If this is a solution, what are the unknowns? Unknowns are b_0, b_1 and b_2 and so on, right until either infinity. What is given to me? Given to me is the initial condition and this number a, ok.

Now, can I write these bs which are unknowns in terms of what do I know? I know a and I know x_0, ok. So, let look let us look at the first term, right. So, and I just equate or the left and right side and then the coefficient corresponding to t^2 should be equal, look similarly the coefficient corresponding to t should be equal and so on. So, first I think would give me that b_1 is ab_0, ok. And the second expression will give me, so this, this and this are 2b_2 is ab_1 or b_2 is $\frac{1}{2}$ab_1. And what is b_1? b_1 from here is bs ab_0. Similarly, I looked for the third one, 3b_3 is ab_2 and which will give me b_3 as $\frac{1}{3}\frac{1}{2}a^3b_0$. And similarly, b_k would be $\frac{1}{k!}a^kb_0$, ok.

So, I have the solution now in the following terms. So, x(t) now is, ok. What is b_0? Ok. So, I have everything is in terms of b_0 plus. What is b_1? b_1 is ab_0 + b_2t^2. What is b_2? $\frac{1}{2}a^2b_0t^2$ and so on, ok. Now, I have everything now the solution written down in terms of a will be (Refer Time: 15:14) is given to me and b_0, ok. How do I find this b_0? Well, b_0 is simply the initial condition x at 0 would simply be b_0 and this will be x_0, ok, that is that is that is easy.

And therefore, now I can write down x(t) as 1 + a + and so on with x_0, ok. So, that is what is happening here, right. So, the solution can be expanded into a power series of this way in this form and I know that this is this is actually is e^{at}, right. So, this we know. So, I am just constructing the solution in a slightly better or slightly in a in a more natural way that one would do before we sophisticated in to close from expressions for all these differential equations, ok.

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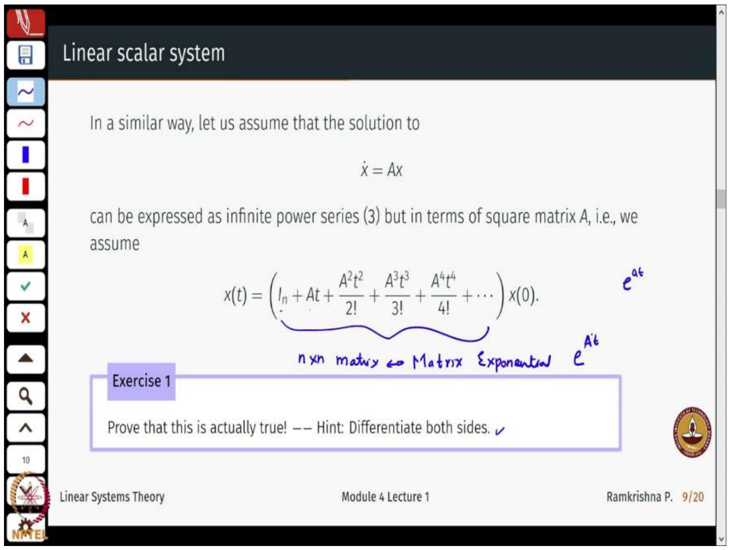

Now, I am looking now at the solution to a matrix you know when, so to a vector differential equation. So, this is usually called the scalar differential equation this is D E, where the states is just in R, ok.

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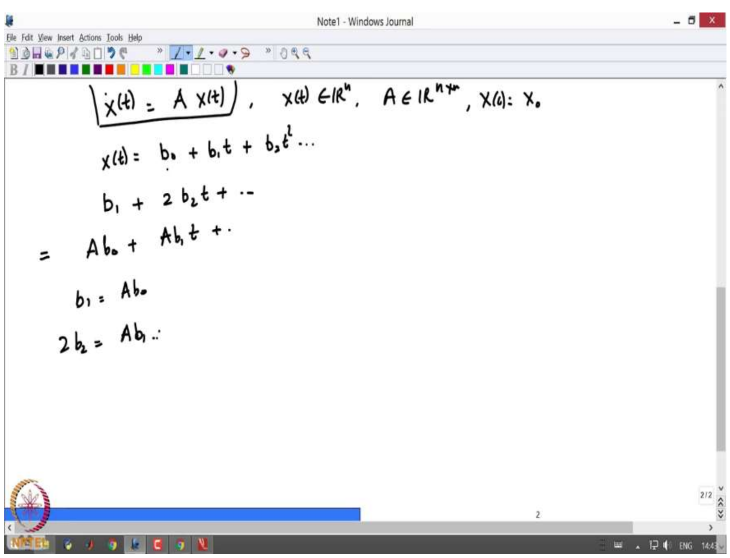

Now, I will go to say vector differential equations when I have $\dot{x}$ (t) = Ax(t), where now x is n-dimensional vector A is n x n matrix and then you will have certain initial conditions the initial state director would be x_0, ok.

So, let me similarly you know write as I did in the in the previous case like this. So, let me assume that there is a solution which looks like this x of t is now these are all vector right, b_0 + b_1t + $b_2 t^2$ and so on, ok. Now, how do I verify of assuming this is a solution what are the bs is again this it will be the same problem for me, right. So, I just write it again, so what is $\dot{x}$ would be b_1t plus sorry, I differentiate this I just I just substitute the value of x into this given differential equation. So, the first term will be b_1 +2b_2t and so on. On the right hand side, I will have Ab_0 + Ab_1t and so on, ok.

So, again I can just do the same trick to find the coefficients. And so, again, so b_1 is Ab_0 similarly twice b_2 is Ab_1 and so on, right. So, this will give me a solution which looks like this, right. So, I have the inclusion identity matrix At and all these higher powers of time, ok. So, of course, this is true, I can easily check by differentiating both side. That is one of the way to check if a certain x(t) is the solution to a particular initial condition or not. So, in the similar way, so this is, this is an n x n matrix, ok.

So, similar to what was happening previously here this and this is loosely speaking, this is just a substituted by the by the matrix A, right. So, we can call this as the matrix exponential, or also I can write this as e^{At}, in the similar way as I had e^{At}, where A was a was a scalar, right and I have e with a matrix on top e^{At}, and then of course, the initial conditions here, ok.

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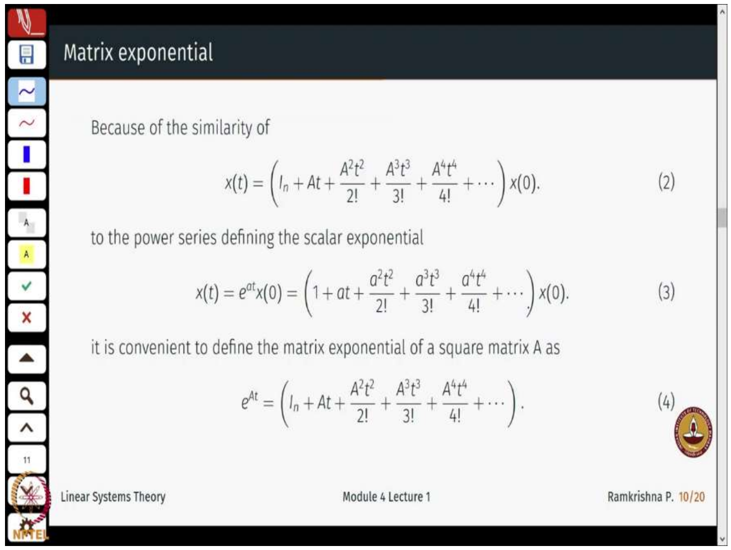

So, they are just rewriting this again. It is starting from a scalar differential equation to a to a vector or differential equation, right. So, that is the first step towards writing solutions of state space equations, assuming that u is 0. We are not we are not at the moment worrying much about what is the nature of input. So, we just look for autonomous systems $\dot{x}$=Ax, ok. So, it, will pause here for a while, and then continue this with this lecture.