Linear Systems Theory
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Module - 03
Lecture - 16
Tutorial
Linear Algebra

Hello everyone. In this video, we will have a tutorial session on the topics covered in module 3 of this Linear Systems Course. So, we will do one mathematical problem followed by 3 different proofs, which we have actually looked into during the lectures.

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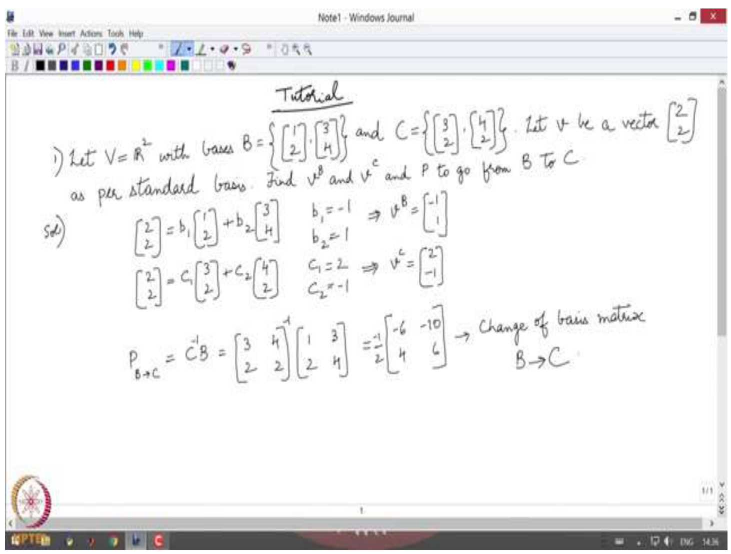

So, the first problem is as follows. Let V be a vector space R^2 and we are given two basis for R^2, B which is vectors $\begin{bmatrix}1\\2\end{bmatrix}$and $\begin{bmatrix}3\\4\end{bmatrix}$ and C which has vectors $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}4\\2\end{bmatrix}$ and let small v be a vector $\begin{bmatrix}2\\2\end{bmatrix}$ as per standard basis.

So, we are asked to find V with respect to basis B and similarly V with respect to basis C, and also the change of matrix, a change of basis matrix P to go from basis B to C, ok. So, now, how can we find out the coordinates of this vector $\begin{bmatrix}2\\2\end{bmatrix}$ with respect to the basis? So, when we say coordinates we can write them as a linear combination. So, I can write $\begin{bmatrix}2\\2\end{bmatrix}$ as a linear combination of the basis B. So, I will say $b_1\begin{bmatrix}1\\2\end{bmatrix} + b_2\begin{bmatrix}3\\4\end{bmatrix}$, ok, where b_1 and b_2 are the coordinates of V with respect to the basis B. Now, we can actually find out b_1 to be -1 and b_2 to be 1, you can verify this. So, which implies I will have V^B=$\begin{bmatrix}-1\\1\end{bmatrix}$. So, this is the representation of vector $\begin{bmatrix}2\\2\end{bmatrix}$ with respect to the basis B.

So, what about C? We will do the same thing with C. $c_1\begin{bmatrix}3\\2\end{bmatrix} + c_2\begin{bmatrix}4\\2\end{bmatrix}$and we will get c_1 to be 2 and c_2 to be -1, which implies $V^c = \begin{bmatrix}2\\-1\end{bmatrix}$. So, this is the vector V with respect to the basis C.

Now, we are asked to find out the change of matrix change of basis matrix P to go from B to C. So, this we have actually looked into a formula during the lectures which was given by C^{-1}B, where C is the matrix containing the basis vectors as columns and B is the matrix containing the basis vectors of B as columns. So, now I can get $\begin{bmatrix}3 & 4\\2 & 2\end{bmatrix}^{-1}\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ ok. And this will give me you can do the inverse by yourself. I will just give you the final answer -1 by 2 is actually the determinant 4 and 6, ok. So, you can verify this inverse and multiplication. So, this is my change of basis matrix to go from basis B to basis C, ok.

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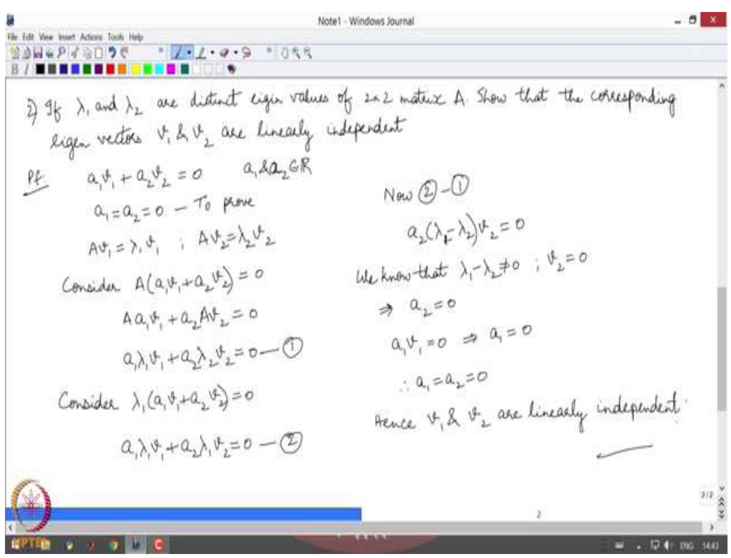

So, this is the first question, going to the second question. It is actually a proof given two Eigen values λ_1 and λ_2, which are said to be distinct of a 2 x 2 matrix A. So, now, we have to show that the corresponding Eigen vectors V_1 and V_2 are linearly independent, ok. So, we are asked to show this. So, you will go through the proof.

So, to show linearly in the linearly independent vectors we will start with a linear combination as follows $a_1v_1 + a_2v_2$= 0, where a_1 and a_2 belong to R. So, I am taking two scalars and writing the linear combination and saying it to be 0. Now, if v_1 and v_2 are linearly independent then $a_1 = a_2 = 0$. So, this is what we need to prove to show that v_1 and v_2 are linearly independent.

Now, we know that v_1 and v_2 are Eigen vectors. So, I can write of A. So, I can write A$v_1 = \lambda_1 v_1$, similarly Av_2=$\lambda_2 v_2$. So, consider this. A times the linear combination $a_1v_1 + a_2v_2$ will obviously be 0 because we assumed that the linear combination gives us 0. So, now, we expand it, and then I substitute this into this equation which will give me $a_1\lambda_1v_1 + a_2\lambda_2v_2 = 0$. So, we will say this is equation 1.

Now, I will consider another equation and just multiply λ_1 with the linear combination which will again will be 0 and will be $a_1\lambda_1v_1 + a_2\lambda_1v_2$ will be 0. So, I will call this equation 2. Now, what if I subtract 1 from 2? I will get the first terms will get cancelled out, so I am writing only the second terms in both the equations, which will give me $a_2(\lambda_1 - \lambda_2)v_2 =$ 0, ok. So, this is the equation we get.

Now, we know that $\lambda_1 - \lambda_2 \neq 0$ because that is the assumption that we started that they are distinct and then also $v_2 \neq 0$ because it is a Eigen vector and it cannot be 0. So, the only way that this equation can become 0 is a_2 being 0. So, now if a_2 is 0 and I take this and substitute in the linear combination then I get a_1v_1 to be 0 and v_1 obviously, cannot be 0, so it again implies that a_1 should be 0. So, now, this proves that a_1 and a_2 are equal to 0, and hence v_1 and v_2 are linearly independent. So, this is the end of the proof. Now, you can actually prove this for any number of n Eigen distinct Eigen values and corresponding Eigen vectors. So, this can be extended to any number.

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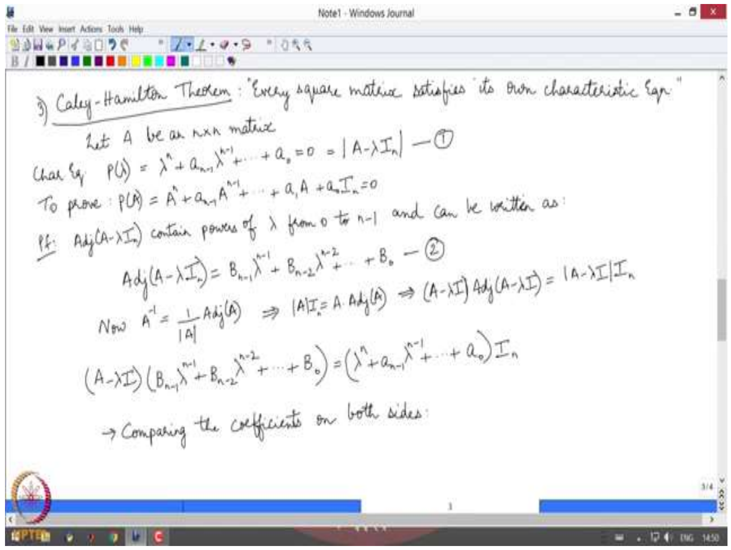

So, now we will prove the Caley Hamilton theorem which was introduced during one of the lectures of this week. So, Caley Hamilton theorem says every square matrix satisfies its own characteristic equation, ok. So, this is what the Caley Hamilton theorem says. So, let us assume A an n x n matrix which has the following characteristic equation. I will call it P(λ). It is an nth degree polynomial. So, I say its λ^n+$a_{n-1}\lambda^{n-1}$ 1 + so on, $a_0 = 0$. If it is obviously, the |A-λI_n|, ok. So, I will just put this as equation 1.

Now, what we need to prove is that P(A) which is A^n + $a_{n-1}A^{n-1}$+ so on, a_1 A + $a_0 I_n$ = 0. So, this is what the Caley Hamilton theorem says and we will prove this. So, so the proofs will actually take the adjoint of the matrix A -λI_n. So, this actually contains powers of λ from 0 to n - 1, and can be written as follows. Highlight, adj(A-λI_n) is equals to, I will take some coefficients $B_{n-1}\lambda^{n-1}$ + $B_{n-2}\lambda^{n-2}$ + so on up to B_0, where B_{n-1}, B_{n-2}, B_0 are all n x n matrices. So, this I will assume denoted by as equation 2.

Now, if we take the formula for A^{-1}. $A^{-1} = \frac{1}{|A|}$adj(A) and I can send |A| to the other side and write it like this. So, I took A^{-1}, I put A to the other side now because it is a matrix there will be I_n here. So, the same thing I can write it in terms of A-λI. So, I will say A-λI, this is the matrix times adj(A – λI) is the |A-λI|I_n. So, now, in this equation I will substitute 1 and 2, in adj(A)and |A-λI|. So, then I will get something like this. This is equals to λ^n + $a_{n-1}\lambda^{n-1}$ + $a_0 I_n$, ok. So, what we now do is we compare the coefficients.

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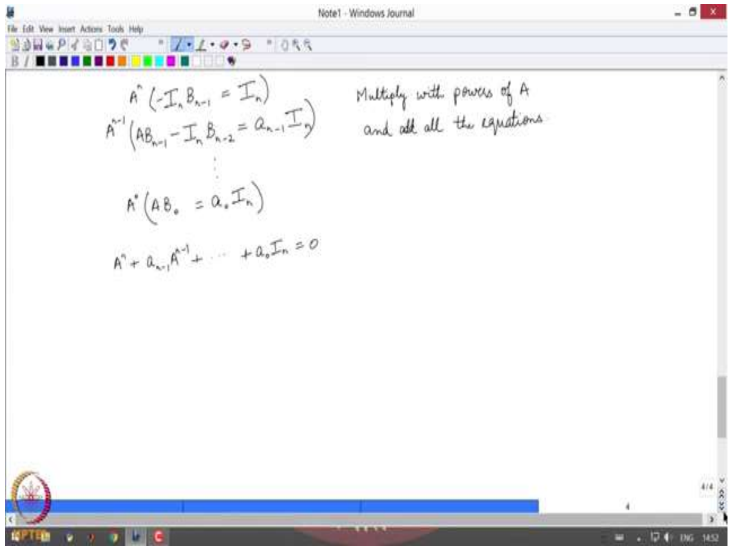

So, when we compare the coefficients I can write. So, you can actually check for yourself that when I do this multiplication this A gets multiplied with this whole thing and then λ I gets multiplied with this whole thing, and then all there are all powers of λ on both sides going from λ^n to λ^0 I can say. And then I just equating all the coefficients on both sides and this is the set of equations that I get.

Now, what I will do is I will multiply these with powers of A. The first equation I will multiply it with A^n, the second I will multiply it with A^{n-1} and last I will multiply it with A^0 which is just identity, ok. So, now when I do this, I will and add up all the equation. So, multiply with powers of A and add all the equations. So, when you add all the equations actually all the terms on the left hand side get cancelled and will be left only with the terms on the right hand side. And what are those terms on the right hand side? The first term will be A^n because A^n is getting multiplied with I_n and the second term will be $a_{n-1}A^{n-1}$ and so on $a_0 I_n$ because A^0 is just I_n and because all the terms on the right hand side got cancelled, I can just assume it to be 0. So, now, this is what the Caley Hamilton theorem says and we were able to prove it.

So, going to the next proof it is another theorem which says two square matrices are similar if and only if they have the same Jordan form, ok.

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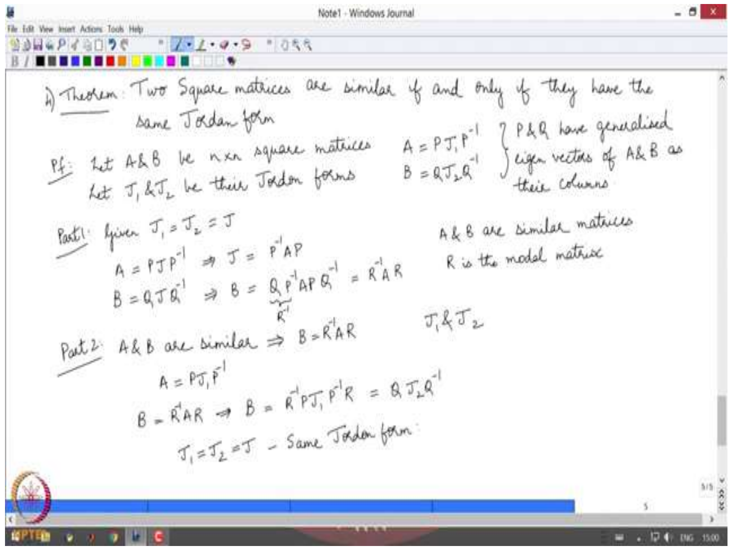

So, we will try to prove this. So, in the first part we will take the 'If' part, that is we assume that they have the same Jordan form. So, if I write it in terms of matrices. If I take J_1, ok. First I will write down some notation and then go to the proof. So, let A and B be n x n square matrices and let J_1 and J_2 be their Jordan forms. So, which means that I can write A as some PJ_1P^{-1} and similarly B as some QJ_2Q^{-1} and you can actually see that P and Q have the generalized Eigen vectors of A and B as their columns. So, this is what was discussed during one of the lectures when the Jordan forms were introduced and how can we convert a matrix to its Jordan form. So, that is what P and Q will be.

So, now, we will go to the proof. So, the if part we assume that that is given that the Jordan forms are same which means J_1=J_2=J, and then I can rewrite the equations A = PJP^{-1} and B = QJQ^{-1}. Now, I can write J to be P^{-1}AP and then substitute this in B, I will get QP^{-1}APQ^{-1}. So, if I take this to be R or say I will say R inverse, I can write it as R^{-1}AR. Now, this implies that if I am able to write B as R^{-1}AR, then A and B are similar matrices with this R is the model matrix.

Now, we are done with one side of the proof by assuming that we have the same Jordan form and we are able to prove that they are similar matrix. What about the other side? So, if we take the square matrices to be similar, so this is say part 1 of the proof. Part 2 of the proof says that A and B are similar which implies I can write B is some R^{-1}AR, ok. So, now, so if they have separate Jordan forms I will assume them to be J_1 and J_2 to begin with. So, now, I can again say that A = P$J_1 P^{-1}$. I already have that B=R^{-1}AR and I will substitute A in this, which implies B=R^{-1}P$J_1 P^{-1}$R which is equal to Q$J_2 Q^{-1}$ because that is coming from the Jordan form of B.

Now, if I compare these two J_1 and J_1 are Jordan forms, so obviously, R^{-1}P = Q which implies that J_1 =J_2 and I can again call it J. So, then they have the same Jordan form. So, this brings us to the end of this proof. So, we have covered 1 problem and then 3 proofs in this lecture. So, there is proof some more like these which you can work them out and we will try to put them out as an exercise during the course of this week.

Thank you.