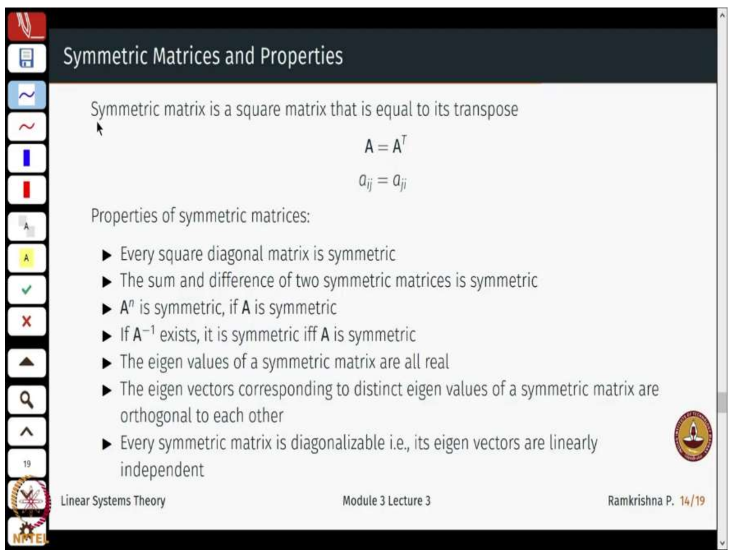
Linear Systems Theory
Prof. Ramkrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module - 03
Lecture - 03
Part 02
Math Preliminaries: Linear Algebra

So, before we deal with rectangular matrices, let us just talk a bit about some one other special matrix; we introduced them a little earlier just as definition of what are called as symmetric matrices.

(Refer Slide Time: 00:33)



So, these are matrix which are equal to their transpose $A = A^T$. And this comes with some nice properties, something which are easy to check is every square diagonal matrix is symmetric. If I add two symmetric matrices I will again get a symmetric matrix. If A is symmetric A^2, A^3 and all higher powers of A are also symmetric. And similarly if A^{-1} exists it is symmetric if and only if A is symmetric ok.

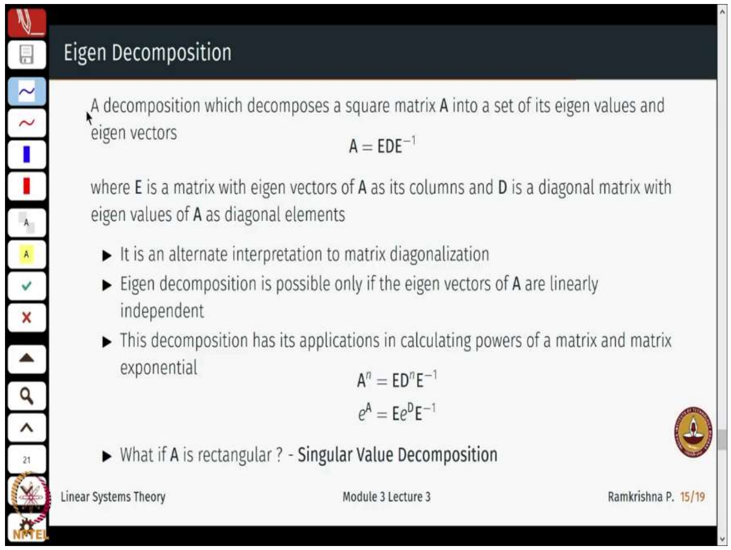
So, take a invertible A, if A is symmetric A^{-1} if at all it exists will be symmetric. Useful property which I not prove but, this is interesting thing to note that all eigen values of a symmetric matrix are real. And this is one of the properties which will also explain later in the course. But I will leave this for you to do this proof as a little exercise; if you cannot do it just let me know whether this is an interesting proof. So, eigen values of all symmetric matrices are real and the eigen vectors are orthogonal to each other right. So, if so we know the definitions right.

So, there you take the dot product and then get you 0. And lastly every symmetric matrix is diagonalizable and of course, its eigen vectors this is a (Refer Time: 02:13) thing we will think that and its eigen vectors are linearly independent. This is nothing really special to say about here ok.

(Refer Slide Time: 02:22)

We now introduce the concepts of eigen decomposition and singular value decomposition. Singular value decomposition is essentially when you have a rectangular matrix.

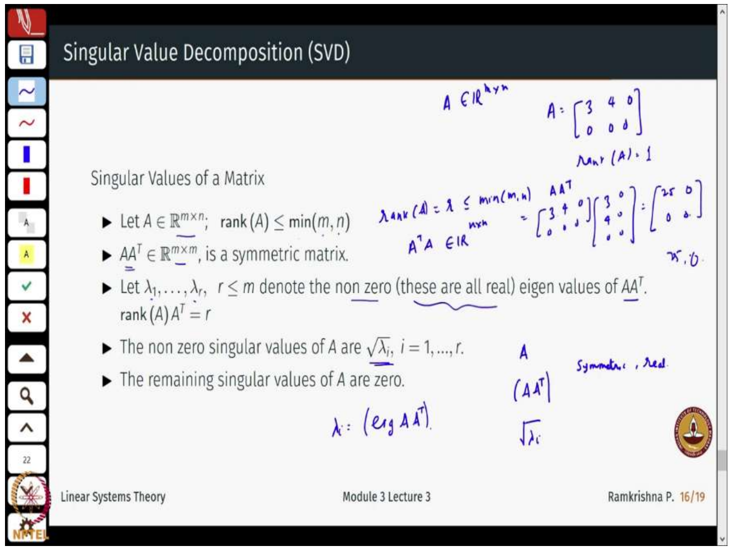
(Refer Slide Time: 02:35)



So, the eigen decomposition is one which decomposes a square matrix A into its set of eigen values and vectors. So, what does it mean? That if I can write a matrix A as EDE^{-1}, E is a matrix with eigenvectors of A, as its column D is a diagonal matrix with eigen values of a as its diagonal elements. Its a it looks very singular to what to what we studied in diagonalization its just a little alternate representation. And this eigen decomposition of course, is possible if and only if the eigen vectors of A are a linearly independent.

So, how does this help us? Well, as usual we were talking of computing higher powers of A. So, A^n would simply be E $D^n E^{-1}$. So, interesting question is what happens when A is a rectangular matrix? So, the answer to that will be by introducing the concept of single singular value decomposition ok.

(Refer Slide Time: 03:46)



Before we do a singular value decomposition, let us first define what are singular values of a matrix. And I will, usually associate this with a rectangular matrix, A belonging in to A^{mxn} right. So, if I have a rectangular matrix, its maximum rank can be the minimum of m and n. So, let us say I just call this rank of A; let me call this as some number r, which can add best be the minimum of m or n. What is easy to check is that AA^T is a square and a symmetric matrix. Similarly AA^T will be in A^{nxn}. And this will also be a symmetric matrix right this is easy to check ok.

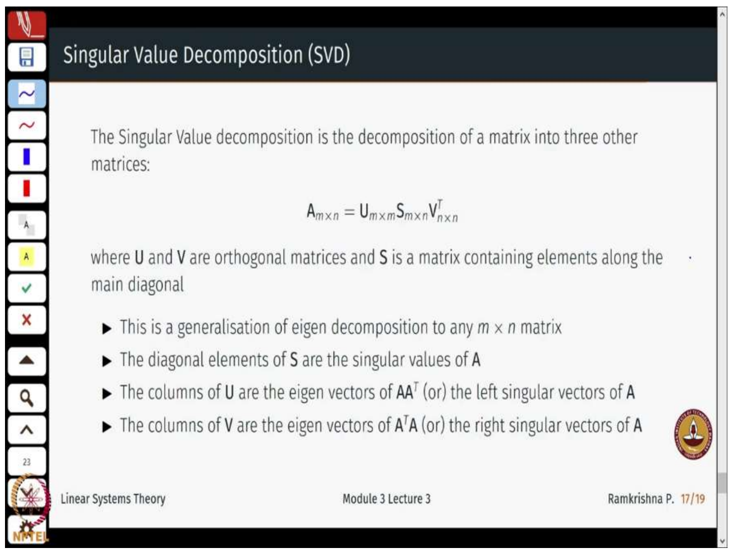
So, let us start with this and say that, I have λ_1 to λ_r let them denote the nonzero eigen values of AA^T. And from the property of symmetric matrix, I know that this all will be real eigen values ok. So, I start with a matrix A right, which is a rectangular matrix. Then I compute this matrix AA^T which is symmetric and of course, square which goes without saying all its eigen values are real ok. Now the singular values by definition are the square roots of eigen values of AA^T, so λ_i; here are the eigen values of AA^T ok. And the remaining singular values will be zero, let us see if we can work out pretty simple example.

So, I take a matrix A; which is $\begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank of A and I can trivially check it to be 1 ok. Now what is AA^T is simply this multiplication, $\begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. I have 3 4 0 and all other 0s. So, this will be 9 plus 16 is $\begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$. So, singular values here would just be the square root of this of eigen values of AA^T would just be the diagonal matrices, 25 and 0 right. And then you can just compute the singular values by this these numbers ok. So, there will be one nonzero and the other will be 0 singular values are positive the square root of 25 will be plus and minus 5 right.

Student: We taken only one.

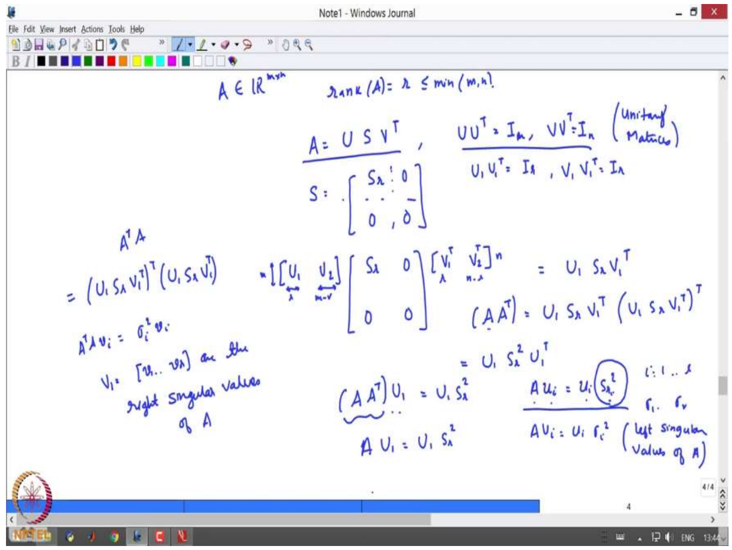
The positive one ok.

(Refer Slide Time: 07:13)



So, given a matrix how would I go about finding its singular value decomposition or what does it even mean by singular value decomposition? In the diagonalization I had a transformation E D or E^{-1} where that the matrix D was just a collection of the eigen values on the diagonal and everything else was 0 ok. So, what does what does this just mean in terms of a rectangular matrix? So, let us do a little derivation here, (Refer Time: 07:57) I have written ok.

(Refer Slide Time: 08:05)



So, I start with A from R m cross n my question is there is its a m by n matrix, again the rank of A is equal to r, which is less than or equal to the minimum of these numbers m , n ok. Now the definition here says that A can be written as U and some S and V^T ok; where this U and V are such that UU^T is the identity and VV^T is the identity.

So, this UU^T would be the m dimensional identity and this would be the n dimensional identity equality here these are also called as unitary matrices ok. Now suppose that I can write or S as $\begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix}$ (Refer Time: 09:31) like. So, this will correspond to the r non zero singular values and then the remaining would be would be 0 ok.

So, this thing I can write down now as, so I have $[U_1 \quad U_2]$ I have $\begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix}$. So, now, this has dimension m this is r and this is m - r. And similarly here so, I will have V_1 and say some V_2 with its transpose. So, V of n this will be r and this will be n - r. So, again I can rewrite this as $U_1 S_r V_1^T$ and then there is a reason why I am doing this why I am I assuming that there are r nonzero singular values and in the remaining are are 0. So I know these two things, so an immature thing to also check is $U_1 U_1^T$ would be the are dimension identity similarly V 1 with V_1^T would be also the r dimensional identity ok.

Now coming back to this matrix A; let me take this AA^T right, this is what how I defined you know the singular values. That these are the singular values are the square roots of eigen values of A transpose A transpose is A square matrix. So, this will be U_1, call this as $S_r V_1^T$ ok, multiplied by the same write $U_1 S_r V_1^T$ and the entire transpose of this. So, this will be ok, I just do all the math; this will be $U_1 S_r^2$ and V_1 we can transpose. So, this will be U_1^T ok.

Now, if I multiply this by say I have $AA^T U_1$ is $u_1 S_r^2$ ok. So, now, this is let me call this some matrix A; this is A set of vectors right A, $u_1$is some u_1 again the same, S_r^2 ok. Now if I write down each element it will look as A; so, you call this small u_i u_i is $u_i S_{ri}^2$ ok.

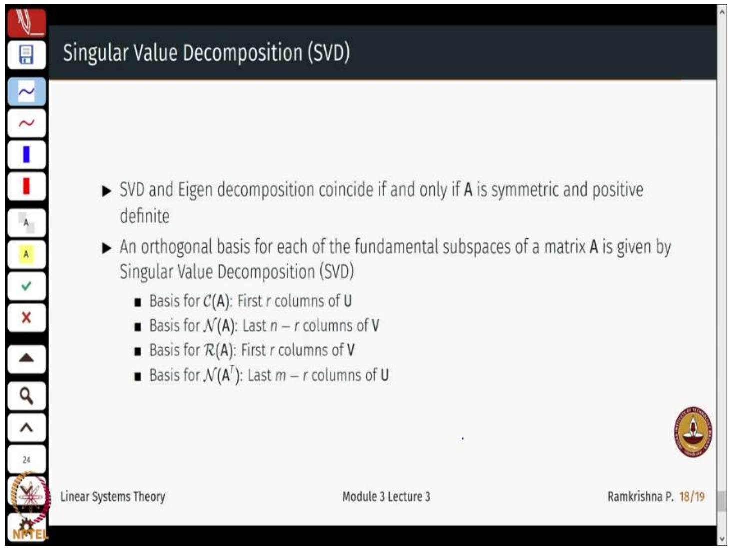
So, see whatever is the ith diagonal entries I; so, i going from 1 to r. Now this has a very nice interpretation here right. So, I have a matrix A multiplied by A vector will give me again that vector multiplied by the square of the ith diagonal element, which are essentially the singular value. So, let me call this the this singular values to be σ_1 till σ_r.

So, I have AU_i is $U_i \sigma_i^2$ ok. Now this use or this eigen vectors are called the left singular values of A ok. Similarly, I can do with the other thing also right. So, I just take and the other so, I have take AA^T so, what is AA^T, this would be sorry not A^T; I take the another one $A^T A$ is. So, sorry, I get $U_1 S_r U_1$: $U_1 S_r V_1^T$ the whole transpose of this times $U_1 S_r V_1$ with the transpose.

And I do all this stuff and I get that AA^T with A may be a small v_i would be again this list number $\sigma_i^2 v_i$. And this V's which are now elements of v_1 to v_r are the right; singular values of the matrix A. So, this is a little proof of how you know these things work and what is the relation why the what is the relation between the eigen values of AA^T and the single singular value.

So, why do we call them as the singular value? So, this is a little illustration of that ok. So, let us go back here and then read out what this what the entire steps that we did so far mean. So, I take a matrix A I can write it as a product of 3 matrices U S and V^T; where S we will consider will consist of all the same singular values along the diagonal which is a generalization of eigen decomposition. The diagonal element of S are the singular values of A; the columns of U were called the left eigenvectors the column for V are were called the right singular eigen vectors of A. Yes, it is again you can just go through this sense and relate each statement to the steps which we are followed over there ok.

(Refer Slide Time: 16:45)

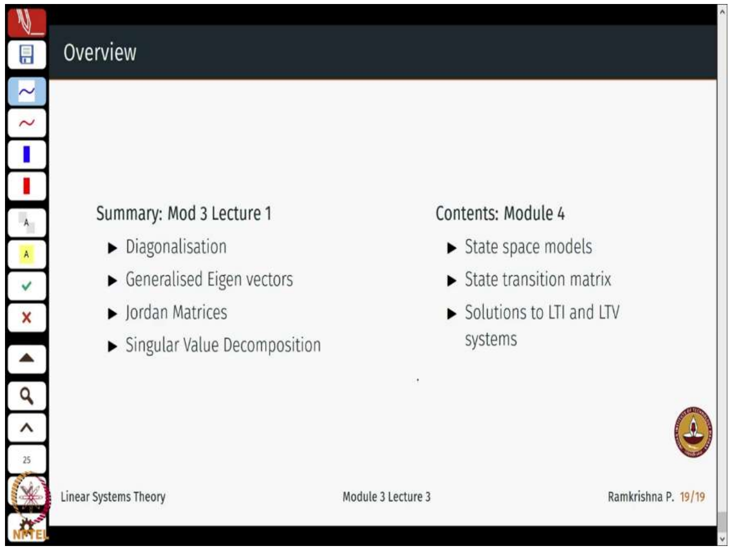


So, when do this see when is the singular value decomposition and the eigen decomposition, when are they same well its again easy to check, if the coincide if and only if a is symmetric and positive definite you can just you know start from here. And say when A is defines symmetric and positive definite then a positive definite would mean that all eigen values are greater than 0; that was its the property of the symmetric matrix or sign definite symmetric matrix.

So, A^TA would be A^2 and then you can just re rewrite all these steps to validate this statement ok. So, the reason I also did this that I just assumed that r, there are only r; singular values which are less than the minimum of this number. In the remaining go to 0, is we can get now a nice interpretation of what we call as the row space and so, the column space and the null space of A and relate them directly to the singular values or the singular value decomposition right.

So, the basis for C (A) would just turn out to be the first r columns of U; I will not do the write on the details, but I think you can do this similarly the basis for the null space of A will be the last n - r columns of V and so on right. So, this you can just verify as a very small exercise right ok.

(Refer Slide Time: 18:25)



So, this kind of concludes the linear algebra tools that I would want to introduce as a some basic building blocks or basic tools for the course. And next time we will start directly with state space models how do we compute solutions of a state space representation of a system that could be. So, we will again deal with linear systems which could be time invariant and also time variant, so that will come up soon.

Thanks for listening.