Linear Systems Theory
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Module – 03
Lecture – 03
Part 01
Math Preliminaries: Linear Algebra

Hi, this will be the last lecture in the series of module 2 and 3, where we were looking at Introducing Concepts in from Linear Algebra which we will use later in the course.

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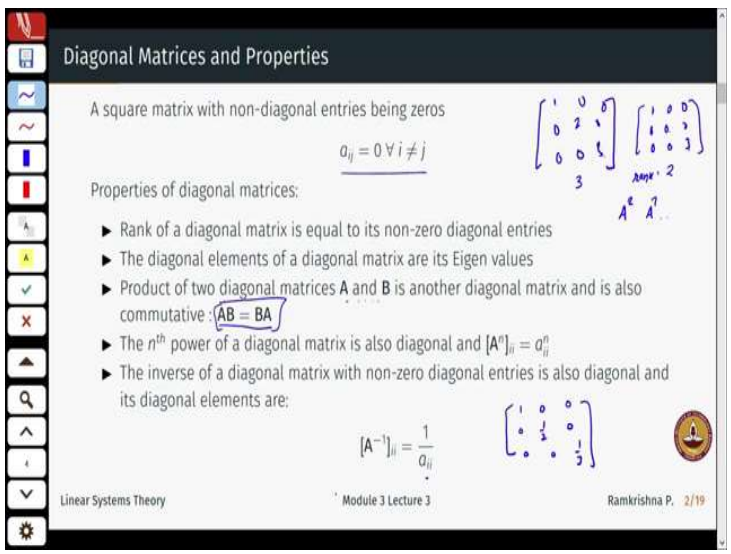

So, we were dealing with a matrix properties and one of the important ones which we would have learned some tricks of how to do this is the diagonalization of matrices or we what is let us first begin by what is a diagonal matrix right by definition it is quite clear and we will deal to begin with we will actually deal with square matrices.

So, a square matrix with non-zero sorry with non-diagonal entries being 0 so, what are the diagonal entries. So, if I look at say 3 x 3 matrix. So, maybe this would be the diagonal elements and all the off-diagonal elements would be 0, that is what is the mathematical expression for this. So, there are only elements in the diagonal and the off diagonals are all 0. So, it is not necessary that all these diagonal elements be non-zero. For example, it you could also have a case like this that this is also a diagonal matrix right.

So, and this and then the next statement which is also kinds of substantiates that you know the diagonals can the diagonal elements can also be 0, they need not always be nonzero. So, what is the rank? Then the rank of the diagonal matrix is equal to it is non-zero diagonal entries. So, for example, here you have three non-zero diagonal entries. So, the rank here would be 3 and here we have two non-zero diagonal entries and the rank would be 2.

Importantly so, this is very easy to verify that if one asks what are the eigen values of a diagonal matrix while the entries of the diagonals are themselves the eigen values, I do not need to need to know to prove this is it is half aligned proof and you could do that by yourself. And, and another interesting property if I take a diagonal matrix A and another diagonal matrix B then the product is also diagonal matrix and the entries would just be there. So, this entry is multiplied by this entry, this by this by this and then so on.

In addition they are also commutative. Commutative essentially means that if I take A and multiply by B that is necessarily equal to if I flip the order at I multiply B by A and this is all not always true right you can just this is a trivial exercise to verify that A times B for a matrix is not necessarily equal to B times A. So, if I take a matrix A well A^2 is will also be diagonal, A^3 will also be diagonal and so on. This is in some sense a product of or consequence of the property which we had had earlier.

Another interesting property is how does the inverse look like. Well, the inverse of our diagonal matrix is also a diagonal matrix. What are the entries? Well, the entries are just the reciprocal of the diagonal entries. So, for example, here the inverse in the first case would just be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. In the second case there will be no inverse because it is of a rank 2 and the determinant will go to 0. So, this matrix will actually not have an inverse.

So, whenever there is an inverse so, the inverse would just be you just take the reciprocal of this of these diagonal elements ok.

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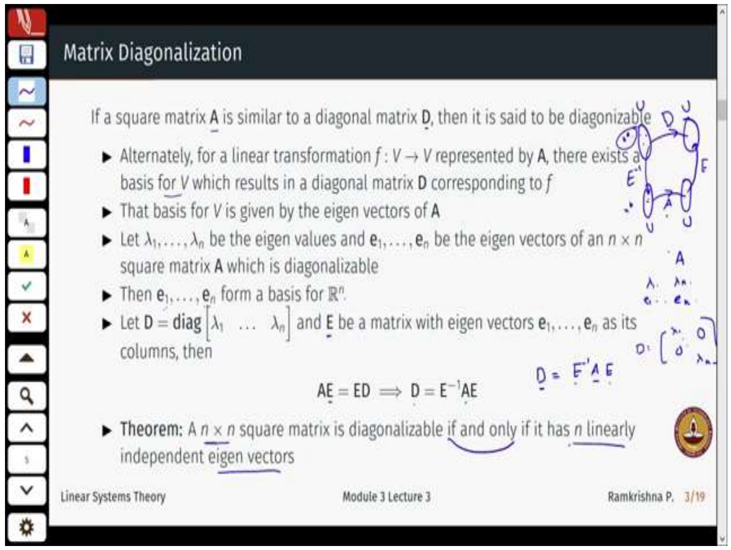

Now, given a matrix well it is not always in the diagonal form, then I ask myself a question can I actually bring it into a diagonal form? Why I need to do this? Well it is to exploit some of the properties discussed in this slide. We will be substantiate or that a little later, but let us for moment ask ourselves a question given a matrix A can I diagonalize it or in other words can I find a diagonal matrix D which is similar to a to it is original matrix A right?

So, when can I do this? Well, if a square matrix A is similar to a diagonal matrix D then it is said to be diagonalizable. So, we had something on the similarity transformations the other day right so, where we started with say I take two subspaces and I have say some transformation A from a subspace say U sorry U to another subspace U that here I had another say transformation again from U to U with say some number D here or the matrix representation let that be called as D.

And, if this side has a coordinate representation of say x x^a with it is basis and here this has a coordinate representation with x^b. And, if this transformation from a coordinate representation corresponding to x^a to correspond to coordinate representation x^b is given by the matrix P, this is given by P inverse and this essentially was a the similarity transformation that D $= \text{PA}P^{-1}$ right ok.

So, that is what it means right. So, a square matrix A is similar to a diagonal matrix D right if this happens then it is said to be diagonalizable and let us see what it means in our context right. So, alternatively as explained here for a linear transformation f going from say V to V over here I just use U to U first it is the same thing. So, there exists so, the statement me said there exists a basis for V which results in a diagonal matrix corresponding to here. What is the basis here? That is the basis corresponding to this; these vectors representation x a.

Now, how do we so, given a matrix A, then how do we go to it is diagonal form b. So, I start with A and can I realize D ok. So, what do I know about a well let say that A has a distinct eigen values $\lambda_1$till λ_n and some corresponding eigen vectors e_1 till e_n right. I assume that say that that these are the n eigen values and these are the n and eigen vectors and possibly that is a is full rank also.

So, now, what I can do here is to find. So, the idea is to find. So, if this D can be represented as just to be consistent with the notation let me call this as E and this as E^{-1} right. So, D can I find or let me call this an E^{-1} just again to be a little more consistent. E^{-1}A E it does not matter right because they are all. So, this is always invertible. So, I can always write one as I can be if I can go from here to here I can always go from this point to this point

So, can I find a given A, can I find a D such that D the elements of it are just the eigen values in the diagonal entries and remaining all going to 0. Now, the question boils down does there exist a E; if that if there exists a e what are those E's that will take me from a A to its diagonal representation? So, it turns out I will do I not do the proof of it I will just give you a little idea of what is actually happening here right.

It turns out that this E or the eigen vectors e_1 till e_n they form a basis for R^n right. So, these eigen vectors which actually form a basis for R^n actually is actually the E which will take me from A to D ok. So, what does that mean right? So, let me just write down in simple terms in coordinates of what does it mean here.

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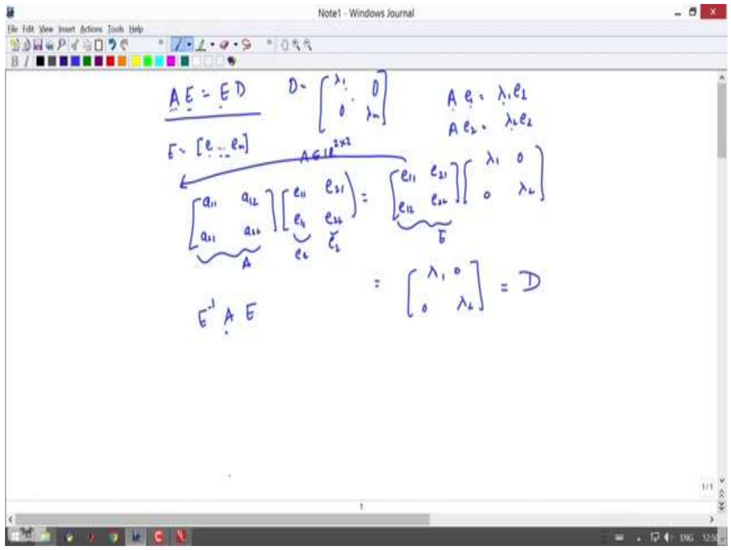

So, so, what I had earlier was I said AE = E D ok. I I know I know what is. So, A is given to me, I want D to be such that all its diagonal elements are the distinct eigen values. They are need not be distinct all the time yeah or they are the n eigen values and what is the E? Then I claim that E is the set of eigen vectors e_1 to e_n which also form a basis for R^n.

So, if I just look at say in the in the 2-dimensional case right say if I have say say A is in R^{2x2}right. So, what is so, can I can actually substantiate this statement? This is not a proof, but this is a little illustration of how we can you know go about doing that. So, my A; so, what do I know of the relation? So, I know that A times sorry Ae_1 is $\lambda_1 e_1$ and similarly Ae_2 is $\lambda_2 e_2$ that given a matrix A. So, I have an eigen vector e_1 which satisfies this relation corresponding to a eigen value λ_1 and similarly, with λ_2.

So, assume I can write A as its independent or its individual entries a_{11}, a_{12}, a_{21}, a_{22}. So, I just I just expand this right and I can write this as. So, e_1 will have it is components e_{11}, e_{12}. So, this constitutes my e 1; similarly with e 2. So, this is my e 2 this on the right hand side I can write this as again e_{11}, e sorry e_{12}, e_{21}, e_{22} with $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

So, well this is a simple exercise to check this $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ I just take the inverse you go here. This is always invertible because I assume that $e_1$till e_n they actually form a basis for R^n. So, I have so, let me call this normal E so, this is E^{-1} multiplied by A E will give me this may this matrix which is essentially the diagonal realization of a matrix A. And, so, this it is a straightforward compute computation process right.

So, so, there is nothing really that the special that is that is happening. So, when can I do this? Well, n x n square matrix is diagonalizable if and only if it has n linearly independent eigen vectors right. So, the reason for that is the invertibility of this matrix A. So, if they are not linearly independent, then you may not always you may not be able to do this ok. What to do when they are not linearly independent is something which we will address later in this in this lecture.

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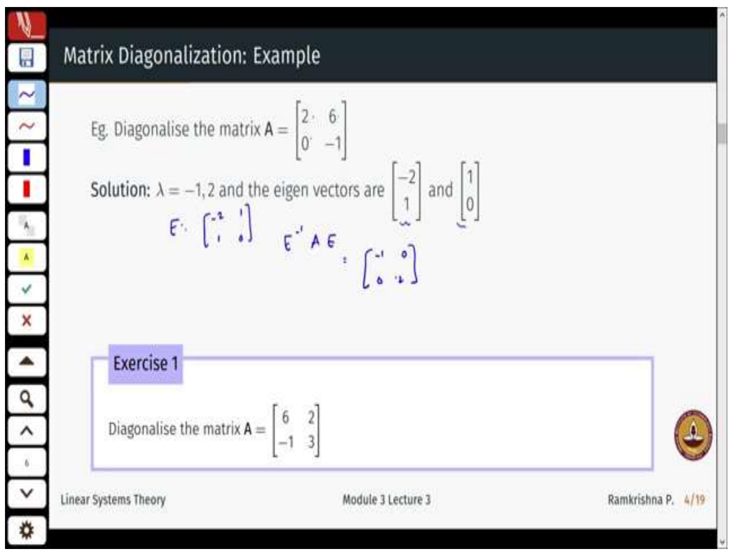

So, there is some simple examples. So, I have a matrix given by $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ with eigen values λ 1 and 2 and then these are the corresponding eigenvectors this is a simple way to do this and I can easily find as that ok. So, my matrix E here would be $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ and I can compute E^{-1}A E to be I will just skip the computations we will just can quickly verify that this gives me this number 0 and 2.

So, this is a very trivial exercise. You can just as an exercise compute the diagonally diagonalization or the D matrix corresponding to the A matrix which is given by this set of numbers ok.

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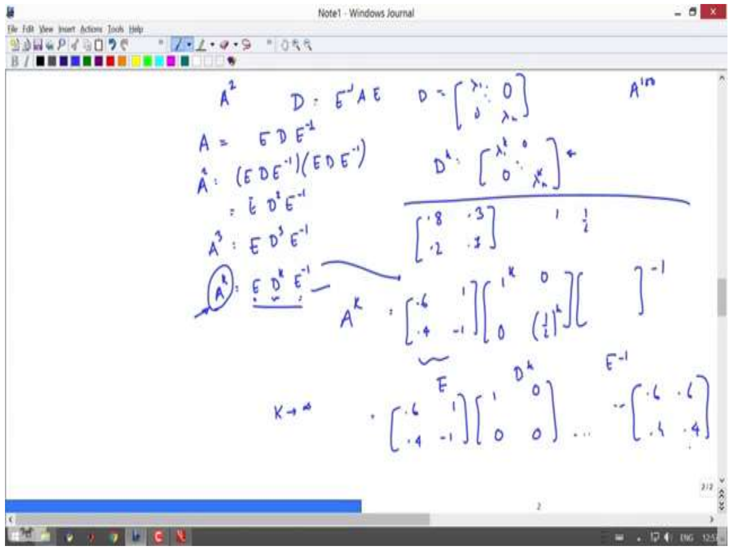

What else can I actually do with this diagonal matrices that I have a beautiful looking diagonal representation. So, one possible application could be give me given a matrix A let us say how do I compute a square. Again it satisfies the conditions that it could actually be diagonalizable that it has n independent eigenvectors. So, let us say I can say that A; so, what was the expression we had D =E^{-1}A E. So, I can equivalently write this as A equal to so, this will be E D E^{-1}. So, what is A^2? A^2 would be (E DE^{-1})(E DE^{-1}) ok. So, this is ED^2E^{-1}.

Similarly, I can write A^3 to be E. So, this should be an inverse somewhere here. So, it is here E D^3E^{-1} and similarly any k-th power can be computed this way ok. Now, this is this is an easy matrix multiplication right. So, what do I know is given a diagonal matrix D with entries λ_1 till λ_n with the all the off diagonal being 0, D^k would simply be the eigen values to raised to the power k λ_n^k. And, all the others being 0 and I just have to have to have this to add addition matrix multiplications are multiplying to the left by E and to the right by E^{-1} ok.

So, one may ask a question of why do I need to do this when actually I can put this into a computer and compute. Well, the answer is it is not it is not that simple right so, why do I need to learn these techniques. So, all of all much of the problems which we deal here would have lots of a computational issues and the idea would be to use as less computational power as far as possible.

So, if I just do a matrix multiplication just take an arbitrary matrix say it is of large dimension and I just do a standard, I just write a standard code for matrix multiplication like you know the rows and the columns and so on. It will for sure take a larger computing time than just doing this because here I think I so, if you just look the computations here are pretty straightforward right as long as I know what the eigen values are and let us have to do two basic matrix multiplications right.

So, once I know the eigen values this is tricky to compute this is very easy to compute. It will just have n computations; n computations and then I just have two matrix multiplications than as opposed to having a k matrix multiplications. So, what we are doing is not a trivial exercise, but if I were to write this up as a as a code it up I would possibly use this technique then as a than using this one and just to save my computing power.

So, well what else can I do? So, if I come back to the to the may to the mark of matrix example which we had in the previous lecture 0.3, 0.7 and then you are interested in what to compute the value of A power 100 right. So, let us let us revisit this and see if we can verify the results that we had earlier ok. So, this had eigen vectors of the form $\begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix}$ and if I. So, this will have 1 was my so, eigen values for 1 and 1 by 2.

So, here I will have this E E^{-1} A K so, whatever is here and this will be 0 this will be 0 and power K. And, similarly with the with the inverse element here. This will be my sorry my A matrix raised to the power K right. So, this is just rewriting this here. E, this is E, this is D^k and this is E^{-1} or whatever you know that could be. So, what happens as K goes to infinity? Well, as K goes to infinity and I have $\begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix}$, the eigen vectors this will go to 0, this is anyway 0, this is anyway 0 this is a 1 and corresponding number here.

So, I can just compute this to be a $\begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$. This is a little illustration of the kind of computation we did last time to find of how the eigen vector definition actually helped us computing higher powers of a without actually having to do the matrix multiplication 100 times right. So, this is also one of the easier techniques that one could one could adopt.

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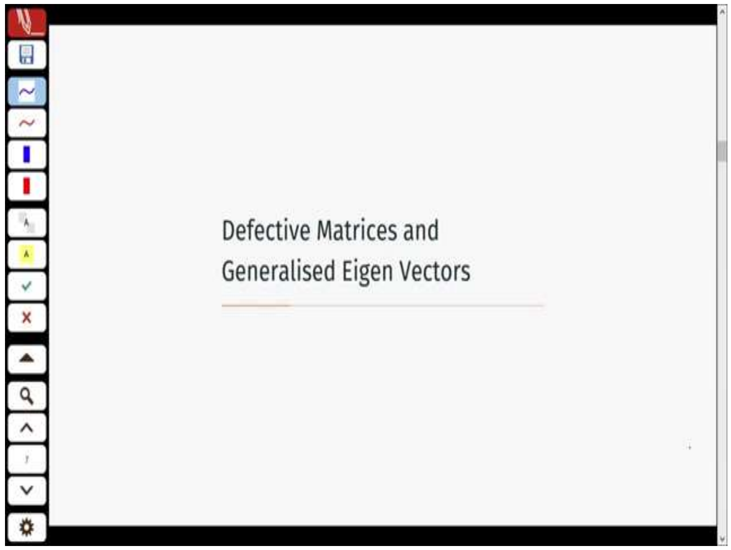

So, all this required so, this statement was crucial right that it actually needs n independent eigenvectors ok.

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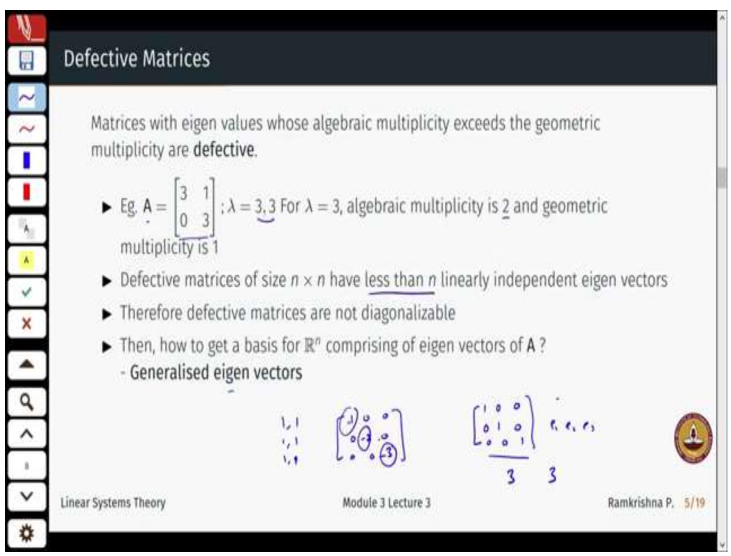

What if that does not happen ok? So, if it does not happen, it means that I have so, what it what it means mathematically is that if I have matrices with eigen values whose algebraic multiplicity exceeds the geometric multiplicity then they are they are defective. A trivial example just to say if I take the 3 x 3 identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ok, here I can always find three distinct eigen vectors say which could be whatever from right any.

So, any any vector in R^3 is an eigen vector and I can always find 3 distinct eigen vectors and in that case I can so, all these three eigen vectors e_1, e_2, e_3 will span the entire of R^3. It may be possible in examples such as this that I that I that there could be. So, the eigen values here are 3 and 3 and the algebraic multiplicity is 2 and you can actually check that the geometric multiplicity is 1, that you can only find one independent eigen vector given these eigen values for this matrix A ok.

So, it could be you could also have examples where these eigen values are distinct right and then so, if you have an example like $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. So, here so the algebraic multiplicity is always equal to the geometric multiplicity everything is one here ok. So, it could also happen where so, here you have a multiple eigen value. So, algebraic multiplicity is 3 geometric multiplicity is 3.

Here for each eigen value it is 1, comma 1. 1 is the algebraic multiplicity one is the geometric multiplicity and. So, on for each of these eigen values 1, 2 and 3. There is a little trivial example, but good in understanding. So, defective matrices so, all these matrices which we can which we considered in this previous sections were such a way that they were of course, is this is a square matrices and we could generate n linearly independent eigen vectors out of those right. Those eigenvectors can span the n-dimensional space.

So, in the case of defective matrices if I compute eigen values, eigen vectors they will be less than n linearly independent eigenvectors. So, they will not span the entire n dimensional space and what was one of the conditions or the necessary and sufficient condition it was an if and only if condition that it should have n linearly independent vectors.

Now, what do I do with matrices who say for example, have n-1 independent vectors? These are called defective matrices and therefore, not they are not diagonalizable in the in the standard way. So, we need to find a way around this. Is there some something that I can do here? Right so, how to get a basis is to use what are called as generalized eigen vectors ok.

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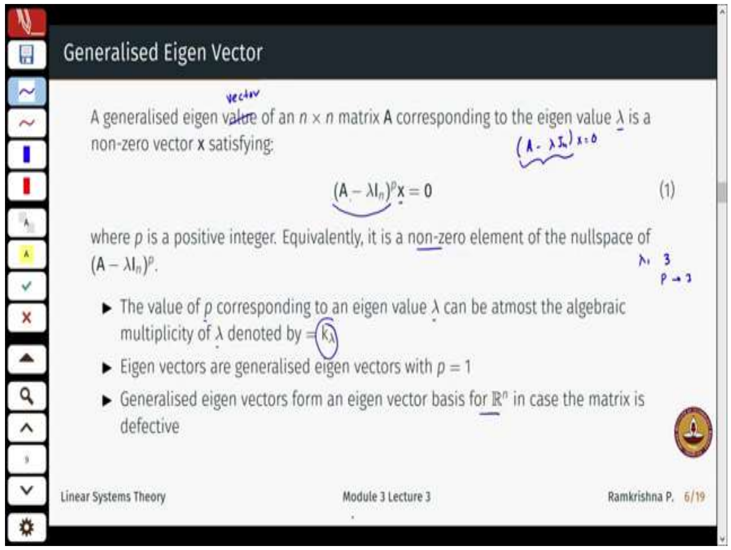

What do this mean? By generalized eigen should be a vector of an n x n matrix A corresponding to a certain eigen value λ is a non-zero vector x satisfying this one right (A-λI_n). So, what was the traditional thing that we did was (A-λI_n)x =0 ok. So, this so, if I do it this way I do not get n independent eigen vector. So, I just look at the higher powers of this matrix (A-λI_n) and a p p is a positive integer. Or equivalently this is a non-zero element.

So, I am not interested in a 0 eigen vector yet, I am always interested in finding non-trivial solutions of this and therefore, if (A-λI_n) was invertible that was not it was not always exciting for me right. So, I always was looking at a of a minus lambda I and being a singular matrix and it should not be invertible ok. So, it is a non-zero element. So, this is important of the null space of this matrix A- λI times p $(A-\lambda I_n)^p$=0.

So, the value of p corresponding to an eigen value can be atmost the algebraic multiplicity and let us say is denoted by some number k i which means that if I have an eigen value say λ_1 occurring 3 times, then this p can at atmost be 3 not more it can never be greater than 3 right. And, we possibly will not get more eigenvectors if you do that and well, the standard eigen vectors are of course, generalized eigen vectors or a special class of it with p being equal to 1 ok.

Now, these generalized eigen vectors now these vectors they will actually form a basis for the entire R^n space in the case when matrix is defective when it is not defective I can just find n independent vectors and this will form the basis for R^n. What if the matrix is defective then I look for the generalized eigen vectors.

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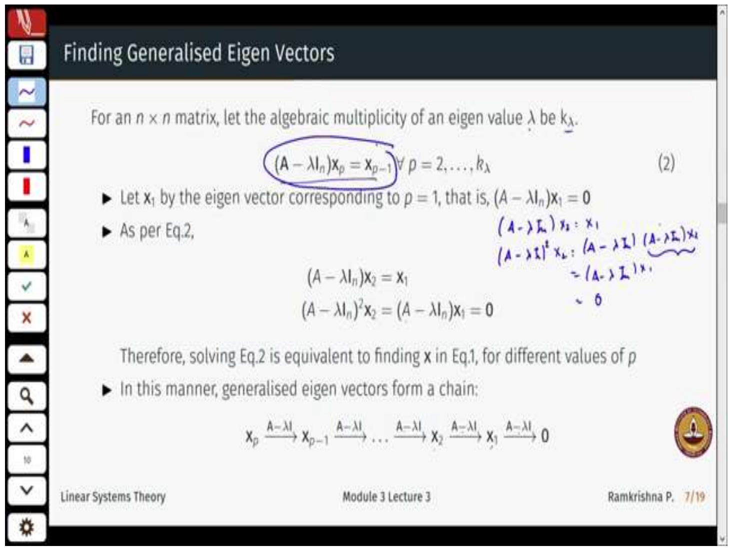

So, how do I compute this? Right. Let us start with which are n x n matrix with an algebraic multiplicity of λ with the lambda occurring k λ times that is how we define the algebraic multiplicity ok. Now, to find the generalized eigen vector I just defined an expression which is which is like this and I am again; so, looking at solutions non-zero solutions to an equation like this equation 1 ok.

So, if I go about doing this way so, what I get as an x is the first expression is A power sorry (A-λI_n) x_2= x_1 ok. And, this will result you know in an expression which is$(A-\lambda I_n)^2 x_2$ this I can write as (A-λI_n) (A-λI_n)x_2. Now, what do I know about this guy? This guy is (A-λI_n)x_1 and this, I know that x_1 is already an eigenvector right so, far so from here so, this will be 0. So, this is how I compute the generalized eigen vectors right.

So, then similarly you can do for the cube and so on right. So, you can just visualize them as a chain right. So, that what I have is (A-λI) x_1 = 0 and similarly A power λIx_2 sorry (A-λI)x_2 will give me x_1 (A -λI)x_3 will give me x_2 and so on and this will give me say k λ different eigen vectors ok.

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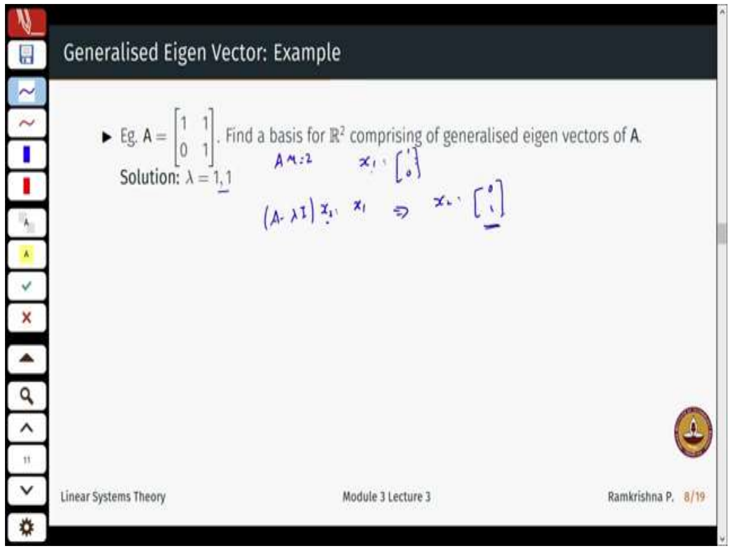

As a simple example let me go with this matrix right: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. So, it has a eigen value 1 and 1 with much algebraic multiplicity being 2 and the only possible eigen vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ right though if you do it in the standard way. Now, I can I find I the second eigenvector right? So, can I write can I find the solution to this (A-λI) x_2 which is x_1 ok?

So, now I know what is x_1, can I find what is x_2 from this and I can just solve this to find x_2 as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ok. So, this $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are generalized eigenvectors of this matrix A corresponding to this eigen value 1 ok.

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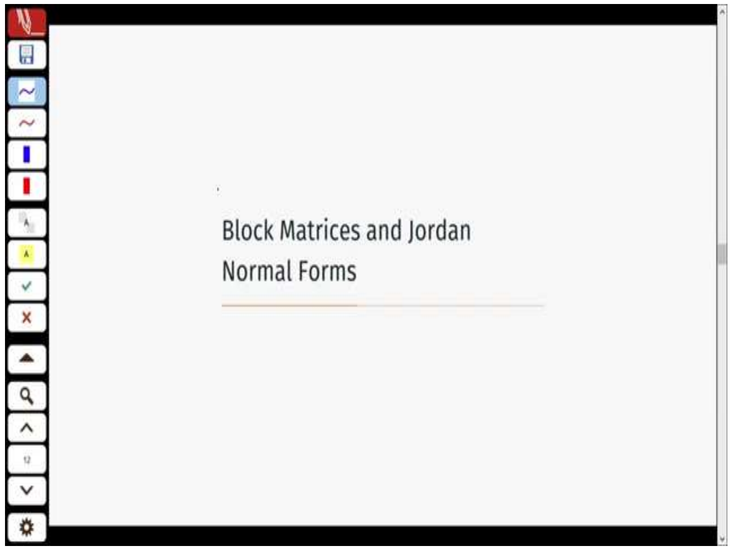

So, before we even talk of diagonalization of what can I do when I have generalized eigenvectors or when I cannot or the matrix A does not have an independent eigenvectors.

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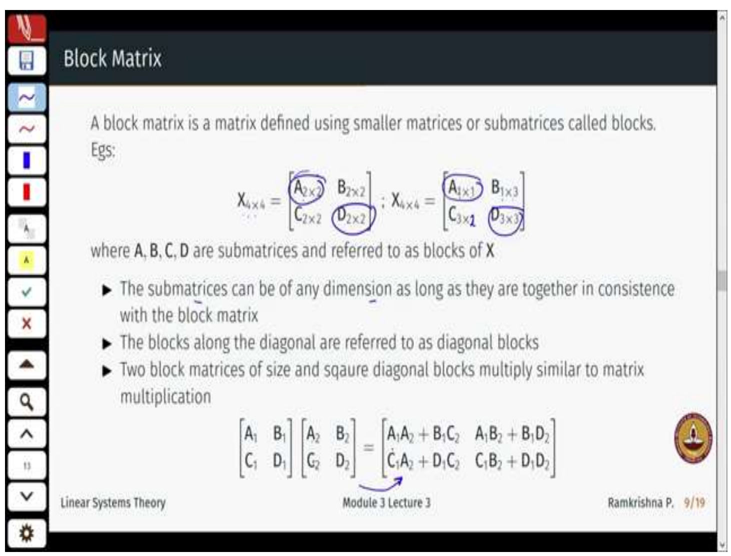

So, let us first define the concept of a of a block matrix. Well, I think from the definition it is or from the word it should be intuitively clear, but let us just write down mathematically. So, let us say I have a matrix of dimension 4 x 4. I can split it into smaller sub matrices right this and this, this and this or I can also do it there are several ways of doing this and so, this could just be a 1 x 1 and this could be 1 x 3 and so on.

Just and that the overall dimensions should agree. So, this A, B, C, D in both these cases are referred to as certain blocks of X and this A, B, C, D which are sub matrices they can be of any dimension as long as they are together in consistence with the block matrix. Say, if we can I can have a 1 x 1 here, but I possibly cannot have a 3 x_2 here right. So, that that dimensionality should be should be consistent ok.

So, the blocks along the diagonal so, this A and D in both these cases are referred to as the diagonal blocks and a nice property of this is the following right. If I have blocks of A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2 they just multiply according to the standard multiplication norms of $A_1A_2 + B_1C_2$ and so on. So, this is pretty standard and that is a pretty nice looking thing that that will help us.

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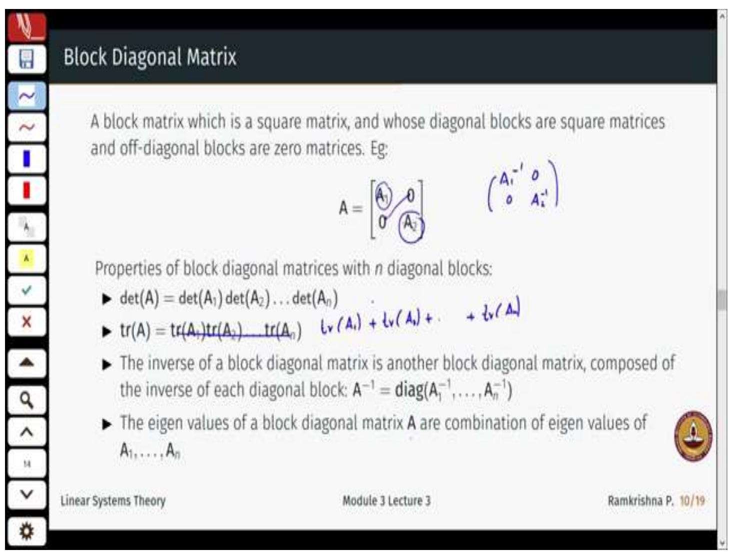

So, a special case is it is a block matrix right so, or a block diagonal matrix where I just have blocks in the diagonal or so whose diagonal blocks so, here A and D were the diagonal blocks. So, these diagonals blocks are square matrices and the off-diagonal ones are always 0. So, this being square is also a little important here like here these are actual they were square matrices here.

So, now this one side I have diagonal block diagonal matrices, it has some neat properties, again that the determinant is of this matrix A would be the determinant of A_1 multiplied by the determinant of A_2 and so on if I have n blocks and similarly, with the trace the trace will just add up sorry is a typo over here this is trace of A_1, trace of A_2, so on until trace of A_n. This is again these are all very very easy to verify; similarly, with the inverse side. So, A inverse would simply in this case be A_1^{-1}, it will again be a block diagonal matrix with A_2^{-1} right.

And, eigen values what are the eigen values of this block diagonal matrix A? Will just be the set of all eigen values comprising of eigen values of A_1 all the way till A_n ok.

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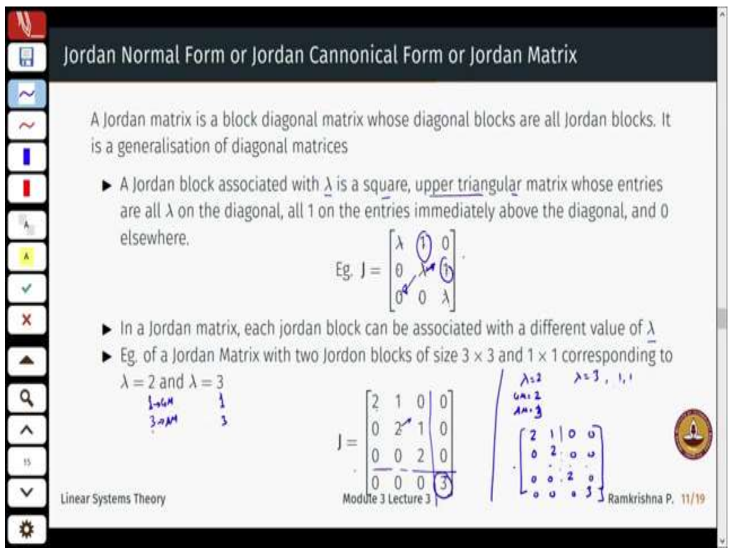

So, one thing that we can answer now which we started with that what if I cannot find n independent eigenvectors, can I actually diagonalize? Well, now you cannot well we can possibly not write all the diagonal elements as a set of eigen values, but there is some somewhere or there is a little trick we can still do and still get something which will look closer to a diagonal form. We cannot write it in a diagonal form because we violate the necessary and sufficient condition.

So, the answer to that is to write my diagonalization as a Jordan normal form or what is also called as the Jordan canonical form or the Jordan matrix. So, a Jordan matrix is a block diagonal matrix whose diagonal blocks are all Jordan blocks ok. What is a Jordan block? So, let us say I have an eigen value. So, Jordan block associated with λ which is an eigen value is a square and this is always be a square, upper triangular matrix whose entries all are all λ on the diagonal and there will be the entries 1 immediately above the diagonals.

So, if I just go to the right I just put a 1, 1 here and a 0 right. So, everything else will be 0 and is these numbers here the lower triangular block will always be 0 ok. So, that is why we say I get an upper triangular matrix. So, you know Jordan matrix each block can be associated with a different value of a of λ ok. We slowly try to understand what this means. So, just as an example just say that I have say a the λ=2 with an algebraic multiplicity of 3 and λ=3 right.

And, this so, if the geometric multiplicity here is 1 and the geometric multiplicity here is 1 that will result in you know in two set of Jordan blocks: one is the 1 corresponding to the eigen value 3, and the second 1 would be the one corresponding to eigen value 2 which occurs 3 times right. This has an algebraic multiplicity 3. This is a geometric multiplicity, this is algebraic multiplicity ok. So, if this occurs then say what do I do I just put a 2, 2, 2 as a this is just says like as a recipe I can just remember this and I just go here and I just put 1, 1 and 0 ok.

So, different things combinations are possible right. You can ask a question of what if I still have λ= 2, λ= 3 and say this λ= 2 has a geometric multiplicity of 2, algebraic multiplicity of 1 and this can just be 1,1 right algebraic multiplicity of 1 and geometric multiplicity as 1. So, this has an algebraic multiplicity of 3 and geometric multiplicity of 2 this is the same as here right. So, I will just increase the geometric multiplicity by 1. So, in this case I will have a Jordan block which will look like this. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$

So, this will constitute my 1, Jordan block because I have I have 3 now in total I have two independent eigen vectors from here and one independent from vector. So, I so, the second so, I will have a 2 again here and I will have finally, a 3 here at and all the others will be 0 0 and 0. So, I can still do it with say what if say I have I will write it on this sheet here.

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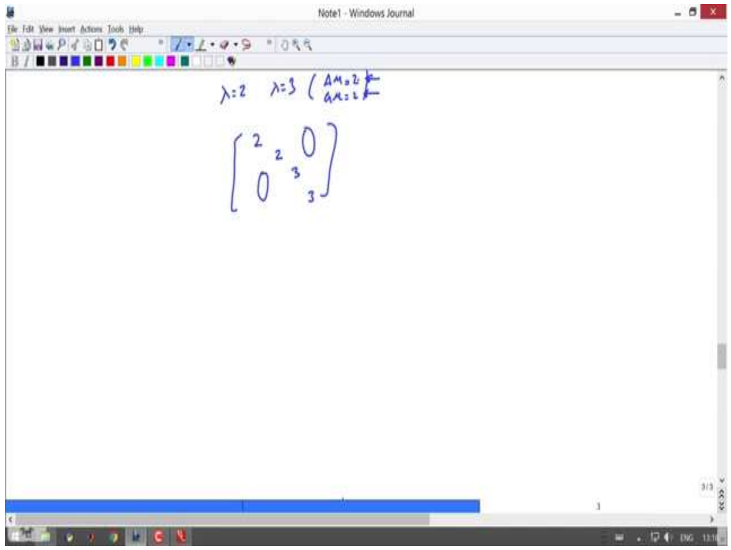

If I just have again λ=2, λ=3 both have algebraic multiplicity of a of 2 and geometric multiplicity of 2. You can rewrite the Jordan block as say 2, 1, 0, 0 and corresponding to λ sorry this will be 3 and corresponding to λ_3 which has again algebraic multiplicity of this should not be.

Student: If both are same, then it is just a diagonal.

Just a diagonal, right.

Student: Maybe 2, 2, 3, 3 and (Refer Time: 37:15).

Yeah, if it. So, in this case so, the algebraic multiplicity and the geometric multiplicity are equal. So, I will just have 2, 2, 3, 3 and the remaining all being 0 and you can do a several combinations of this. So, this is diagonalizable because the algebraic multiplicity and geometric multiplicity corresponding to that eigen value are equal ok.

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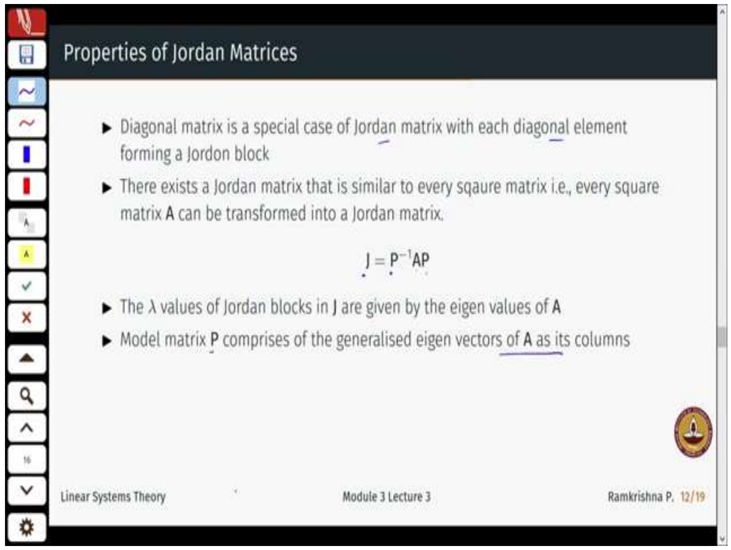

Now, so, how do we how do we construct this Jordan form? So, the diagonal matrix is a special case of a Jordan matrix which each diagonal element forming a Jordan block. So, if I so, as to simply look at the example here I can call each of this to be a Jordan block and corresponding to each eigen value.

Now, the question is can I given a matrix A square matrix can I transform it into a Jordan form and what is the transformation matrix that that does that does this trick for me? So, I see the same as before. So, I can add J as P^{-1}AP where the model matrix P which transforms A to J this will now comprise of generalized eigenvectors of A as its columns.

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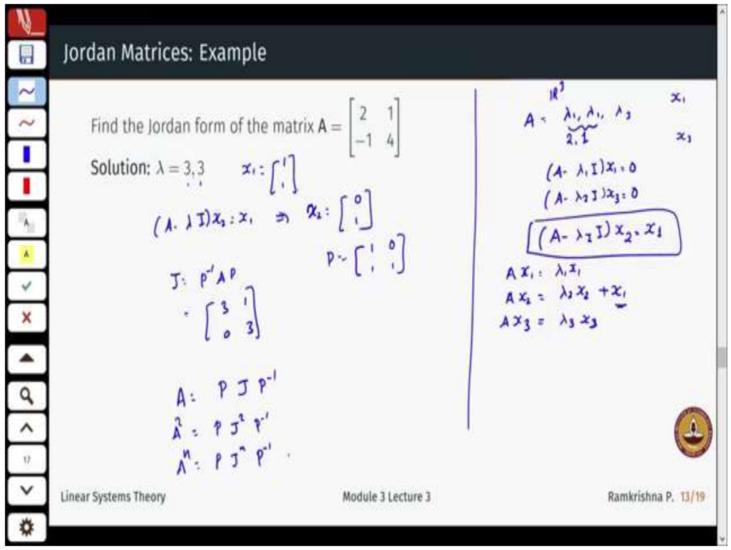

So, before I even go here right so, let us see of why this rule we learned that anything so, there is rule at that I just put the λs on the diagonal and 1 all 1 on the entries immediately above the diagonal elements. So, where does this come from? Again, I will do a little illustration and the hope that will be useful in understanding ok.

Let us say I have again for illustration this is you have to work in R^3 and I have a matrix A which has the eigen values λ_1, λ_1 and λ_3 and I had assume that you know this is this has a geometric multiplicity of 1 and an algebraic multiplicity of 2 right. So, the 2 is clear. Now, what do I know right. So, this I can find two independent eigenvectors, let me call them x_1 and x_3. If I just so, this will come as solutions of (A-λ_1I) x_1 = 0 and (A-λ_3I) x_3= 0 ok.

Now, I need to find another eigenvector. So, that will come from ok. So, let me just re recall this expression of a generalised eigenvector. So, this expression if I rewrite this I will get an equation which looks like this (A-λ_1I) x_2 = x_1. And, if I rewrite these equations again so, in the standard format I will get A.

Student: x 2.

Ok sorry, there is a little typo here. So, times x 2 is x 1 ok. So, I will have Ax_1 is $\lambda_1 x_1$, Ax_2 is $\lambda_2 x_2 + x_1$ and Ax_3 is $\lambda_3 x_3$ ok. Now, if I rewrite this you know in the in the in the standard form I can just see why these entries 1 occur here right. That is essentially because of these additional terms over here which we use which comes from here to calculate the generalized eigen vectors.

So, it is a there is a little illustration of not a proof, but how why we follow this rule of putting this 1s here and you can just you know write down the further details for yourself ok. So, I have a Jordan matrix with the eigen values sorry I have a matrix A with eigen values 3 and 3; I just want to write down it is it is Jordan form. So, first let me calculate the eigen vector. So, x_1 would be $\begin{bmatrix}1\\1\end{bmatrix}$ right.

And, then I just do (A – λI)x_2 should give me x_1. So, this will give me a x_2 which looks like $\begin{bmatrix}0\\1\end{bmatrix}$ and then I do can just do this. So, my P is now $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ and if I just compute P^{-1}AP this will be $\begin{bmatrix}3 & 1\\0 & 3\end{bmatrix}$ and let me just do the computation and check for yourself. And, all the other properties which we had earlier of computing a higher powers of A; so, here if I write A would be P JP^{-1}. So, I can just say similarly A square will be $P\,J^2P^{-1}$ in P P^{-1} and so on. Similarly, with A^n would be $P\,J^nP^{-1}$ and so on. So, those properties still would exist ok.

Let us pause here for a while and then we do we will discuss out what if the matrix matrices are not square. See you in a bit.