Linear Systems Theory
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Module - 03
Lecture – 02
Math Preliminaries: Linear Algebra

Hello everybody. Welcome to this lectures in Linear Systems Theory. So, we continue with our discussions related to transformations and where what we had left as topics to be discussed in this lecture was starting with invariant subspaces and see the concepts of along the concepts of Eigen values and Eigen vectors, ok. So, what is an invariant subspace? And this is a very interesting concept. Perhaps a name I tell you something already and few may be take a course a on non-linear control after this much of the severity analysis would talk a lot about invariant subspaces, even good implications of it while study stability in the linear sense and sense also.

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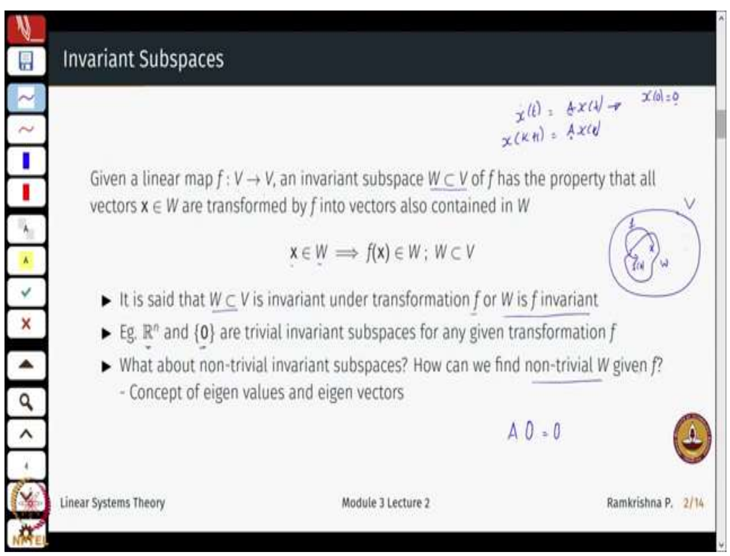

So, what is the definition of this? So, I have a map f from V to V and let me have a subspace W of V. So, let me just draw some some pictures here. And so, this could be my V. I have a little W sitting here. So, I take any vector x, belonging to W. So, it is its not here, right it is inside here. This vector x when acted upon by f or transformed by f will again sit in the same subspace, ok.

So, when this happens that, ok, if there is a subspace if there is V and W is a subspace of V then if something like this happens that I start with a vector x in W, then I have that f of x is also in W, ok. Now, it is then said that W is invariant under the transformation f or W is f invariant, right. It could not be, there could be vectors which come start from here and jump here and then vice versa, but this specific class is called that W is f invariant.

Trivial examples if map from R^n to R^n, where it is always you know R^n is it a is a trivial subspace, right and then the other one is the origin or the 0 vector, right. You take 0 and you multiply anything by 0 so to speak will always get a 0 because the transformations are usually of the format is, if I take the 0 vector and I have the transformation A as always equal to 0. This is also to do with, so what we call as the equilibrium, right.

So, if I say well what is the property of equilibrium? Well, if I am at equilibrium I am I am always at equilibrium assuming that there are no external perturbations. So, the equilibrium, so say for example, if I just write $\dot{x(t)}$ = Ax t or in a discrete time as x(k+1)= A x(k), ok.

So, what is equilibrium mean? That this transformation A, if apply to the equilibrium the system will still be at equilibrium. What does it mean in terms of solutions? So, let us what is the solution of this equation starting from an initial condition x(0) = 0, it will always be in 0. Any other initial condition the various system behaves will depend on this matrix A, right. So, similarly for here, right. So, if the initial condition is 0, this transformation will be at 0. So, the origin is also an invariant a subspace, ok.

This is this might look trivial subspace, but in general it is not. And we will see why it is actually a non-trivial subspace even though, here mathematically it might just see that, if I am 0 I am multiply 0 by something I will always be at 0, ok. But there could be other things like which are like non-trivial subspaces, so when can we have find non-trivial W given an f and this is where we begin now with the concept of Eigen values and Eigen vectors.

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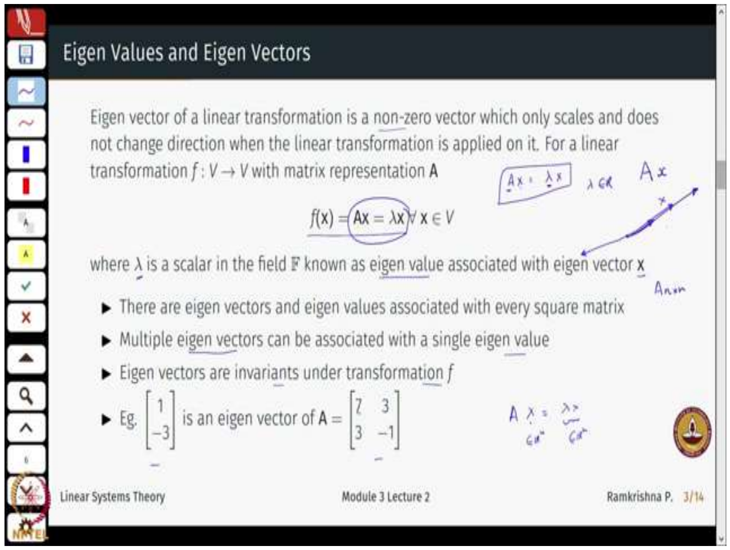

So, first, ok. I am sure we would have read this somewhere, but we will slowly try to revisit and see what could it mean bit geometrically, ok. So, the statement here says that Eigen vector of a linear transformation is a nonzero vector which only scales and does not change direction when this linear transformation is applied on it, ok.

So, typically if I take a linear transformation A and apply x on it, I might get some other arbitrary vector, right. So, this vector could be different in magnitude and different in direction, right. However, an exception happens in this case. So, if I write this statement mathematically it will look something like this, right.

So, just concentrate, just concentrate on this one. The matrix representation of this f is A and this A is such that when it acts on a vector x it gives me λx right. So, here this lambda we can simply assume it to be some some real number, ok. So, what does it do? So, I give you a vector x, this matrix A, so if say may be is something like this is vector x, right, ok. Now, what does the λ do? It can just sorry, if λ is say -1 it will just give me this number, λ is 0.5 it will just give me a vector half the size, λ is 2 it will just give me vector which is will be twice a size, right.

So, what does it do? That is the vector retains it direction and if I just look at the vector representation or the or the geometry of this, this vector x when multiplied by A, if whenever it is equal to lambda x it means is just the same vector just that it is just multiplied by a scalar, right. So, the direction is retained just the magnitude might change. So, only thing it can change direction is this one, right will be the -1 direction, ok. So, this is the a little interpretation of an Eigen vector, ok.

And given any square matrix A n x n, I can always find things like this, ok. And there could be cases, ok. So, in this expression, this λ is what is also usually known as the Eigen value, right. So, for a given relation like this we can call that λ, right, this lambda known as the Eigen value associated with the Eigen vector x. Well, there could be multiple Eigen vectors associated with the single Eigen value and all those variants exist, ok.

And this Eigen vectors are invariants under transformation f. What does it mean? Just is is saying here, right take a vector x coming from some say some R^n or may be whatever I multiplied this by A, I get λx. So, if x is in is in say some R^n n is you multiply a vector in R^n by a number this will also be in R^n. So, it means that Eigen vectors are invariant under the transformation f. And you can just compute the Eigen vectors is a very simple exercise of computing Eigen vector of a matrix which is given by $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$. We will not do the go in to the details of the computation, ok.

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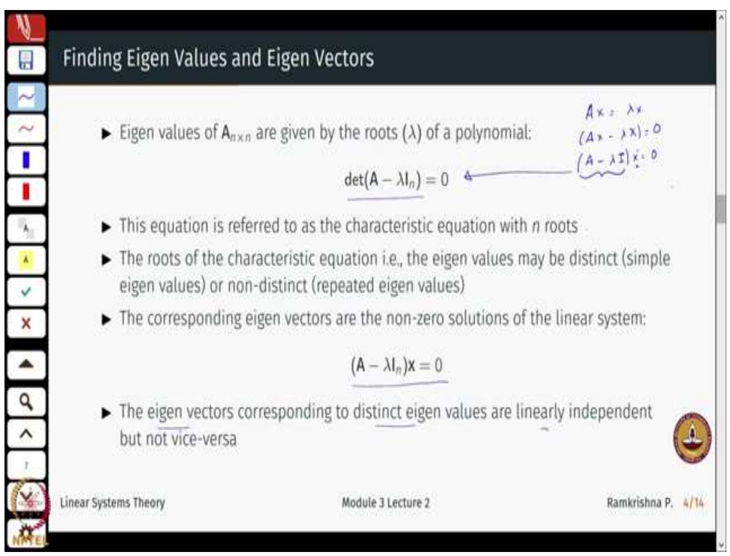

So, where does this expression come from? So, start again Ax = λx which I can write as Ax - λx = 0 or (A –λI), where I is a identity matrix of appropriate dimension is equal to 0, ok. So, what does it mean that x is in the null space of A - λI or the kernel of A - λI.

So, and we are not interested, literally in not interested in. So, we are interested in an in a non-trivial solution of this, ok. So, which essentially means that my A matrix or this matrix A - λI should be a non-singular matrix which turns out mathematically to be is to saying that that determinant of A - λI should be equal to 0. And this equation is also referred to as the characteristics equation of a of a matrix A, ok.

The roots of this equations are called the Eigen values. They may be distinct, they may be non-distinct, there may be real, they may be measure, it might be value 1, it might be value 0 and so on, ok. And the corresponding Eigen vectors are again are the nonzero solutions of and the linear system of equations (A –λI_n) x = 0.

An important property here is that the Eigen vectors corresponding to a distinct Eigen value are linearly independent, but the converse may not be true. We will we will shortly do an example related to this.

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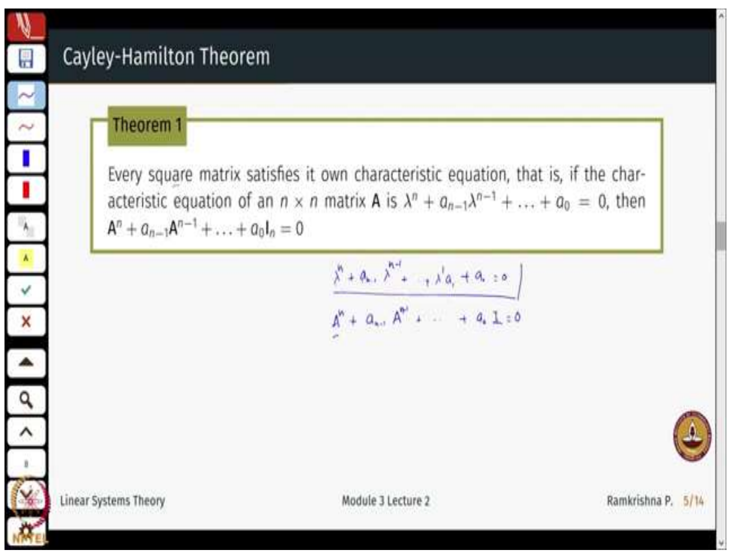

So, there is the theorem here which we will use while we do controllability and another stuff or even computing the matrix exponentials. I will not do the proof of this, but it just says that every square matrix satisfies its own characteristic equation which means that, start from here |A – λI| = 0 will just result in a polynomial of this form, ok. So, I will have $a_1\lambda^1$…so on. So, I will get a polynomial of order n in λ. So, this is the characteristic equation.

What does the Cayley Hamilton theorem say is a (Refer Time: 11:34) represent this, loosely speaking this matrix by A. So, this replace λ by A, I just have $A^n + a_{n-1}A^{n-1} + \cdots + a_0 I_n$ this is also true, but the matrix a satisfies its own. So, when I says this characteristic equation what I am looking for? I am looking to find solutions λ. And if you just replace λ by A, the equation still holds. We will postpone the proofs till little later. So, when next time when we encounter the use of Cayley Hamilton theorem we can just take a little detour, do the proof and comeback, ok.

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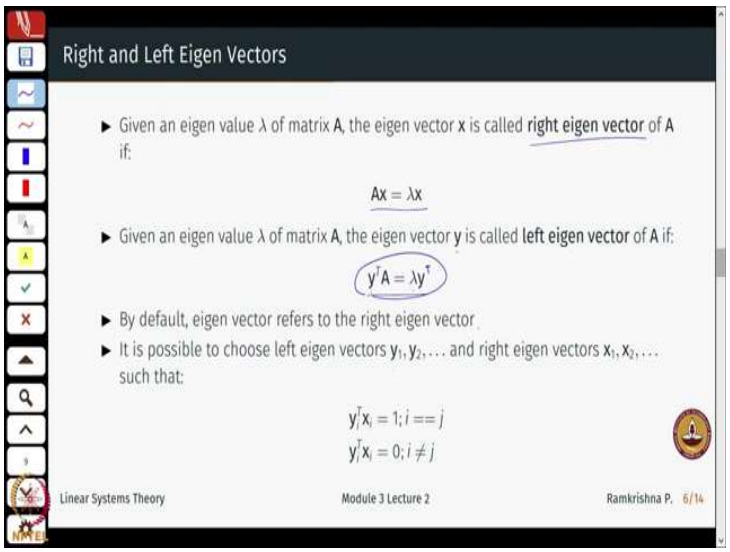

There are some other properties of this Eigen values. So, given, so this is a standard way Ax = λ x is usually called the right Eigen vector of A and whenever we say Eigen vectors it by default is a right Eigen vector. However, there could also exist what is called the left Eigen vector, associated to the to the Eigen value λ which is represented by this one y^T A = λ I. Again this transpose and all comes for a obvious reasons, right, ok.

It is also possible to choose these ys, y_1 , y_2 and so on which are the left Eigen vectors and the right Eigen vectors such that they either, so such a way that y^T x = 0 or y^Tx = 1. And this again so, how do we do this? Well, we can look at those transformations which we did earlier, ok. So, so there should be transpose here, ok. Sorry about that, ok.

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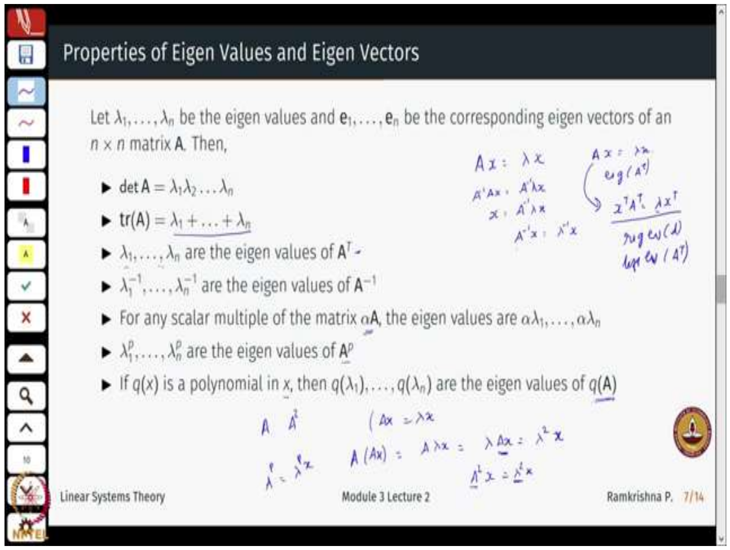

Some little more properties of Eigen values and vectors. Some them will do the proofs some of them, I will just leave it to you as an exercise. So, if λ_1 till λ_n are the Eigen values, so determinant of A is just the product of all the Eigen values.

Now, there is another quantity called the trace of A. For a square matrix it just turns out the trace is the sum of all the Eigen values of system, ok. Then if A has Eigen values λ_1 till n, A transpose will also have the same Eigen values. I think that the proof here should be should be similar.

ah So, if λ_1 till λ_n are the Eigen values of A, A^{-1} will have the Eigen values corresponding to λ, Eigen values has ${\lambda_1}^{-1}$ till ${\lambda_n}^{-1}$, well I just look couple of this quickly. So, I have Ax = λ x

So, I am looking at say the inverse, right. So, here assuming that that the matrix is invertible so, A^{-1}Ax = A^{-1} λ x. So, this will be x is A^{-1} λ x or I can rewrite this as A^{-1}x = λ^{-1}x. Here assuming that you know none of the lambda are 0, otherwise this will not hold. Similarly, I can find out what, so that the of this statement, right. So, Ax = λx. So, I can find out what are, so how do this relate to the Eigen values of A^T.

So, let us start from here, take the transpose, so this will be $x^T A^T$ = λx^T. And if you look at it this is similar to this expression. So, it means two things here, one is that one is that the Eigen values remain of the same of A^T and whatever was the right Eigen vector of A, is now the left Eigen vector of A^T.

This is this is not surprising, at this, even this expression tells us the same. Yeah, but the important property is that the Eigen values of A are equal to the Eigen values of A^T, ok. So, if I take a matrix A I multiplied by A λ the Eigen values will be scaled up by λ. I will not do the proof of this. Similarly, if I look at say given a matrix A what are the Eigen values of say A^2, ok.

So, let us start with A. So, again I have Ax = λx. If I do AAx what is this? This is A λx. This is again rewrite as A, so λAx is what is Ax? From here it is λAx is λ^2x. So, I have one relation which says the following that A^2x = λ^2x which means that the lambda square is an Eigen value of A^2 and I can do till p. So, A p if I take particular Eigen value, so this will be λ^px, ok.

Now, lastly if q(x) is a polynomial in x, then q(λ_1) till q(λ_n) are the values of q(A). And this actually can be viewed as some kind of version of or an application of the Cayley Hamilton theorem, ok.

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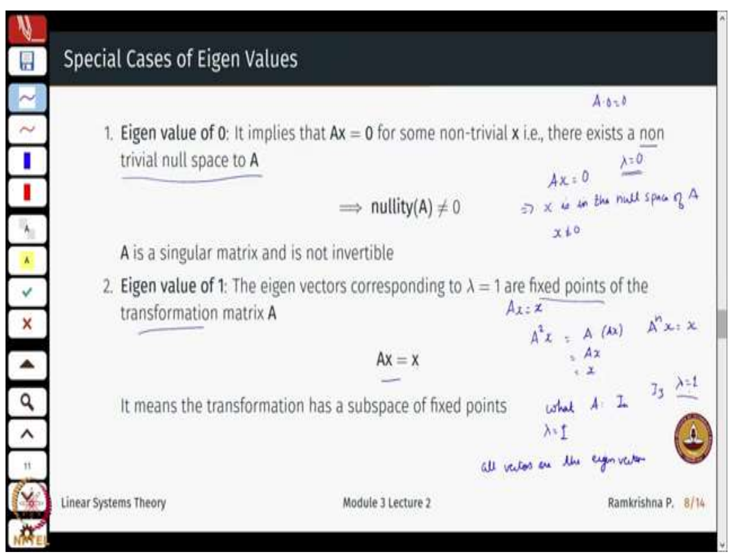

So, some other important thing I will agree this special cases of Eigen values. So, if Eigen value is 0, ok. So, this is corresponds to λ = 0, so which essentially means that A x= 0. So, this means that x which is the Eigen vector is in the null space of A, and this x is nonzero. If you, so here all the time interested or most of the time interested in non-trivial Eigen vectors. So, which means that there exist a nonzero, so or a non-trivial null space to A.

So, what is the trivial null space? x = 0, right. If x = 0, A times 0 is 0 this is trivial null space that is origin to origin if I look at in terms of transformation whereas, if there is an Eigen value of 0 then there exist a non-trivial null space to A. This also means that whenever there is a 0 Eigen value that this matrix is not invertible or it is also called as a singular matrix or the determinant is also 0, right, ok.

Interesting things also happens for the Eigen value equal to the 1, ok. So, if I have λ= 1, then I end up this Ax which is λx = x, ok. If I look at this, so it can mean few things, right. So, this transformations, so these are called the fixed points. So, what happens if I take A^2x? Right. So, A^2x will be AAx. What is Ax? It is again x. Again Ax is again x. So, similarly you can do A^nx will still gives you x. And therefore, these are called the fixed points of the transformations. So, how many hour times I have applied the transformation I just remain at x, ok?

So, an interesting case is what when A is identity. So, this is n x n identity matrix. Identity matrix will have λ= 1 which is trivial thing. And what are the Eigen vectors? Well, all vectors or any vector, right, all vectors are the Eigen vectors corresponding to λ= 1. You can just check that.

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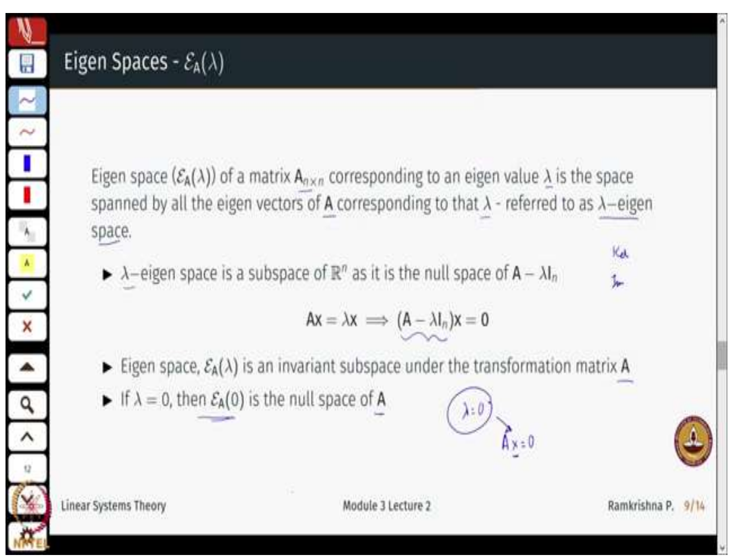

Now, once we have this vectors, always this natural to associate them to a space x. So, if I extract 2 or 3 vectors and I will just be interested in what kind of space do this span. So, that leads us to defining the concepts of an Eigen space, ok.

The Eigen space of a matrix n x n, so it corresponds to Eigen value λ. So, given an Eigen value λ, so the Eigen space is the space spanned by all vectors of A corresponding to that lambda, referred to as the λ Eigen space. So, in a way that a given λ or given a Eigen value can have more than one vectors. So, in as a trivial example look at say 3 x 3 identity matrix, I_3 with λ = 1. Any vector in R^3 is Eigen vectors. But what is the space that they spans? They spans three-dimensional space and therefore, the Eigen space of this λ = 1 is a is is a R^3. Trivial example, but useful to understand what this statement actually means.

And this is actually a subspace of R^n. Why it is subspaces of R^n? Because well look at this relation, right. This λ Eigen space is subspace of R^n as it is the null space of A - λI_n. And we saw earlier that this kernel and image or actually subspaces in the week 2 lecture, ok.

And not only that this Eigen space is a invariant is an invariant subspace under the transformation A, that is also easy to check. And what happens if λ = 0 for 0 0 Eigen value? Then this space λ A is just simply the null space of A; is that was I can easy to check, right. Ax = 0 was the case when λ = 0. And the set of all x will constitute the null space of A and this null space is the Eigen space corresponding to λ = 0, ok. Just a little concepts defined and then relating to what we studied earlier in terms of vector spaces.

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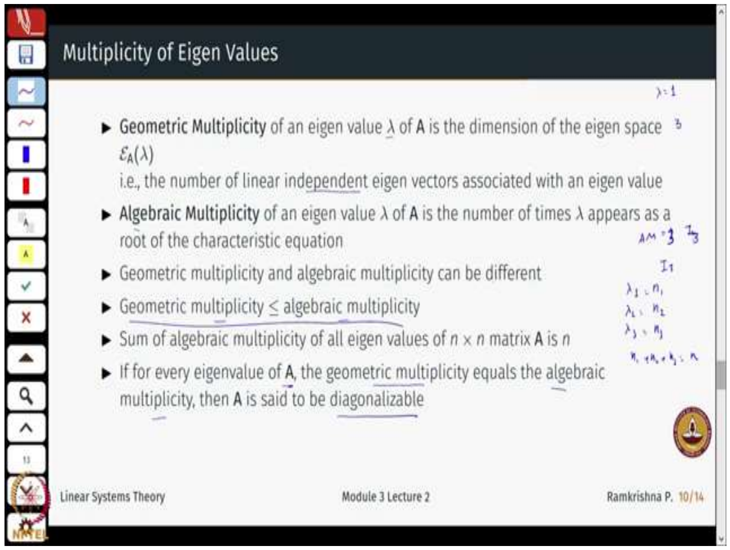

Some just it could be a little dry, but is useful to know bit of this definitions. So, one thing, you start with the algebraic multiplicity. That is, may be little easier to comprehend. So, the algebraic multiplicity of an Eigen value λA is the number of times λ appears in the root of the characteristic equation. Again if I take the 3 x 3 identity matrix the algebraic, algebraic multiplicity is would just be 1, if I look at I_n as my matrix at hand.

So, the geometric multiplicity similarly of Eigen value of A is the dimension of Eigen space, right. Again, what was the. If I look at I_3, let me just, so I_3, the three-dimensional identity or the 3 x 3 identity matrix had an Eigen value of 1. And what is the dimension of its Eigen space? Well, it is a dimension of its Eigen space is also equal to 3, ok. So, its geometric multiplicity was 3, algebraic multiplicity is also 3.

In general, it could be the number of independent Eigen vectors associated with an Eigen value. Though this were independent is important, right. So, what we said for the I_3 matrix is that any vector in R^3 is an Eigen vector. But how many of them can be independent of each other? Which we take any standard basis it will be just be 3, and the remaining vectors can always be written as some kind of linear combination of these 3 vectors. So, the dimension or the number of independent Eigen vector associated to this Eigen values would at best be 3. And so, this is like it trivial examples, right where the algebraic multiplicity coincides with the geometric multiplicity, ok. So, I should write a 3 here, ok.

So, in general, they can be different. But what can happen is the geometric multiplicity can be at most equal to the algebraic multiplicity at if this is 3 this can never can be 4, it can at best be 3. And of course, the sum of algebraic multiplicity of all Eigen values of n x n matrix is n.

So, let us say I have say λ_n n with the multiplicity of n_1, λ_2 with the multiplicity of n_2, λ_3 with the multiplicity of n_3. I am talking of the algebraic multiplicity. So, in a an n dimension space this $n_1 + n_2 + n_3$ will always be equal to n. This is not surprising, right because I am just counting all the number of Eigen values, ok.

So, an important property whenever we talk of Eigen values or in general of matrices is about diagonalizing the matrix. So, we spend lot of time doing that in our math 1 or 2 courses I mean if people had followed books like Grewall or something that is the lot of things on diagonalizability of matrix which is also a kind of use to solve equation by elimination methods, ok. So, the thing is we never ask ourselves can we always do that, right. It just given a problems, I just apply the standard techniques and try to diagonalize it.

Now, because when can I actually do that? Right. So, if for every Eigen value of A, if the geometric multiplicity equals the algebraic multiplicity, then A is said to be diagonalizable. So, there is also then a counter question what if the algebraic multiplicity and geometric multiplicity are not the same, ok. We will keep that question for little later, may be in the next 1 or 2 lectures we will try to answer this question. But this is a very important property to know. We always were talking of diagonalizable matrices without really understanding if I can always do that. We will do examples and counter examples to this in may be also one of the tutorial classes.

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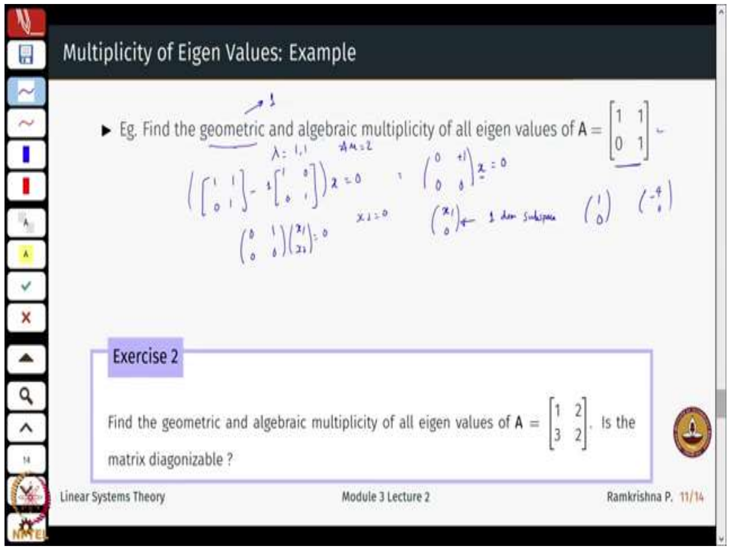

As an example let us do this, right. So, to the question is to find geometric and algebraic multiplicity of all Eigen values of A which is this one. So, just gave the computations, but the Eigen values here are λ = 1,1, which means algebraic multiplicity of λ = 2, ok. Now, what is the geometric multiplicity? So, I will just find the set of all Eigen vectors associated to this Eigen value, ok. So, let us do this. So, I have A which is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ - λ which is 1 times identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ right. So, this is what I look at like (A - λ I) x = 0, this will gives me 0 here, 0 here, minus 1 here.

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Sorry, 1 here and a 0 here, and times x equal to 0.

So, this we are in R^2, so let us simplify this and write this as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ = 0. So, what are the possible x is here or the what is the possible x_2 and x_2? So, the only equation or only thing I can infer from this is x_2 = 0 and therefore, any vector of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ is an Eigen vector, ok. Now, what is the span of this? The span of this will just be a one-dimensional subspace, right, ok.

Now, let us go back to the definition of this. The definition of algebraic multiplicity is this is simple. Let me just say, find how many times λ = 1 occurs, the answer is 2. So, the geometric multiplicity is the dimension of the Eigen space of this one or the number of linearly independent Eigen vectors. So, how many linearly independent Eigen vectors I can get is just 1. So, I can take x_1 or multiples of x_1; just say this is an Eigen vector.

Any either Eigen vector would just be a multiple of this. It could be (4, 0), (-4,0) and so on, ok. So, the geometric multiplicity here is 1 and the algebraic multiplicity is 2. So, this a little example to show that to why did this statement they can be different and, so this is this is this is does not need a proof it is a little easy to understand that the geometric multiplicity can at best be equal to the algebraic multiplicity, ok. So, just a little example or if we remember one of the examples which we did when we are looking at building consensual algorithms in the wireless sensor networks or do they converge to some value and we talked about some properties of matrices and how they converge to certain Eigen vector, ok.

So, I will just do a little not really a proof of what was happening there, but give a little intuition behind why that algorithm converge to a certain value which was the average of all the initial condition. This is just from the from the week 1 lectures.

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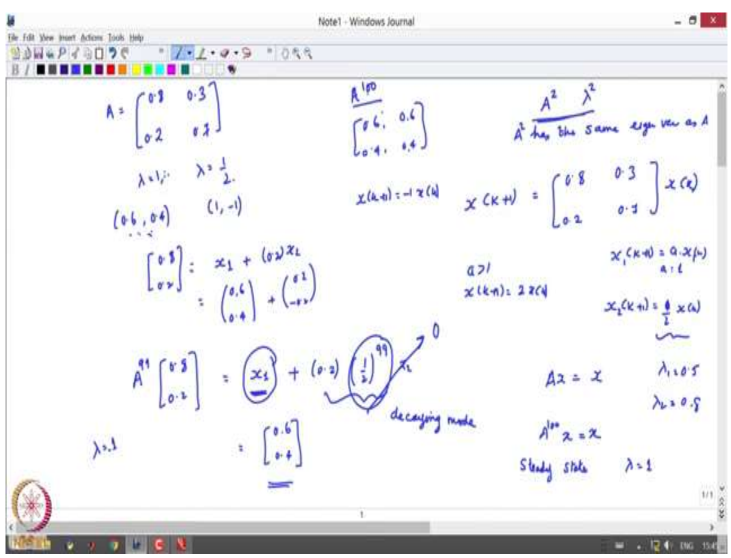

So, this also is there will be an example straight from Gilbert's (Refer Time: 31:38), where he motivates by saying what should I do if I want to compute given a matrix A^{100}. Do I do it 100 times? Right. That will be a computationally expensive.

Now, are there easier tricks to do this? So, you just redo that that example here, so that we understand of where we are going from here. And again at each point of time we will try to give it system theoretic explanation, ok. So, the matrix he uses there is A = $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. So, if you look at it you know the columns add up to 1. It could be a little relevant to what we had the statements we had earlier, ok

So, this has Eigen values of $\lambda = 1$ and $\lambda = \frac{1}{2}$, and the corresponding Eigen vectors of (0.6, 0.4) and (1, -1), ok. So, what was also, so we earlier said that if λ is the Eigen value of A, then λ^2 is the Eigen value of A^2. Well, what is Eigen vector? Eigen vector of A^2 or A^2 has the same Eigen vector as A. So, A square sorry, I will just write it little nicely. So, this may be we did not we did not mention it earlier, but A^2 has the same Eigen vector as A, ok, ok. It Just a little things which we would has miss earlier, ok.

So, now what does A^{100} look like? Well, A^{100} apparently would look close to something like this $\begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$, ok. So, if I just were to make wild guess of how do I compute A^{100} or you know higher powers? That as a as A powers go as a large and large the columns converge to the Eigen vector corresponding to the Eigen value λ = 1 and λ = 1 was also an Eigen value in that wireless sensor network problem, ok.

Now, we will do a little proof for this, ok. So, let us say, ok. So, how do we go about doing this. So, this column $\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ can be represented as well in terms of its Eigen values and vectors as x_1 which is the first Eigen value plus $0.2x_2$, which is simply say 0.6, 0.4 plus the vector x_2 is 1 minus, this will be 0.2 and this will be -0.2, ok. Now, the 100th power of A, the first column would just look something like this A^{99} with 0.8 and 0.2.

All the properties of A^2 and λ^2 A^3 and so on, we can write this as the following. This is still be x_1 because ${\lambda_1}^{99}$ will still be 99 plus $0.2{\lambda_2}^{99}x_2$. Again some thing is rewrite these things it will come show up hear. Now, this is arbitrary close to 0, right, that may be the first digital show of the after some 20, 30, 40 decimal places and therefore, A^{99} times 0.8, 0.2 it simply 0.6 and 0.4, similarly with the second column also, right.

Now, why does it happen this way? Right. So, there are two terms here now if I can look at it. So, if say assume that may be this is this is a system a matrix of a dynamical system of a like this. A is comes from a dynamical system $\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$x(k), ok. Now, I can ask myself of what is the steady state value of x, starting from say some nonzero, non-trivial initial condition. It turns out the following, right. So, I have something here is just be the way it is right, x_1. So, this term does not really change. So, for example, Ax = x when λ = 1.

So, say A^{100}x also equal to x. So, this is in the control language could be called as it is like correspondence to the steady state, right where if I am a steady state I just be there for like forever, right. This is correspond to the Eigen value of λ = 1. Let just say I have a scalar system x(k+1) = ax(k), where a = 1, right. This will correspond to something like this, right. I will just be at the initial condition for all times k > 0.

The second term is if you see with time it just get decaying, you can just visualize this as say I will just call this let me call this x_1. So and let me call this $x_2(k+1)$ is some number which is say $\frac{1}{2}$x(k). As times grow this just decays, right. So, this is usual something like this. So, this is the second term is called usually or this term is the decaying mode, right and at steady state just convert just to values like this, ok.

Now, what happens for some other values? Right. So, we just assume things were very nice here, but what if this number a > 1, if x(k + 1) is say 2x(k), then you kind of grow exponentially. Similarly, if my Eigen value λ = -1 instead of +1, then I have x(k+1) is say –x(k), then I will just be a switching between +1 and -1 and there will actually be know steady state that will exist, right and so on.

These are essentially so, if I were to look at this in terms of poles of the systems this would be called as a marginally stable system in the in the discrete sense. So, we may at the moment not really know the definition or the interpretations of stability, but the way the system behaves at steady state shows the it is a marginally stabled system, at a particular I just reach a constant nonzero value. What if both the Eigen values are less than 0 say λ_1 = 0.5, λ_2 = 0.8 then at steady state both x_1 and x_2 will actually go to 0, because everything will have the decimal number race to a very high power, asymptotically, right, ok.

In literature this is also refer to as the as the Markov matrix and what we see is that, my values converge to the to the Eigen vector corresponding to the Eigen value 1, which possibly the largest Eigen value. Anything larger than 1, I am looking at an unstable system. So, I have not really worry about conversions over there because it is just unstable behavior.

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So, after we are we are done defining Eigen values, Eigen vectors and the corresponding Eigen spaces let us see there is relation between invariant subspace and also similarity transformation, ok.

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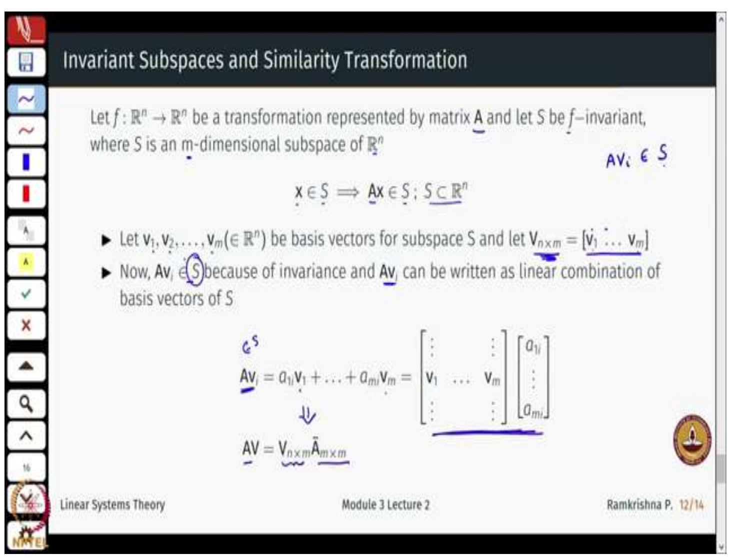

Let us start again with f from R^n to R^n. Let it be a transformation with an equivalent matrix representation by A and let S be f invariant where S an m dimensional subspace of R^n, ok. So, so what does this mean? That if I take an element x which belongs to S is also implies that Ax will also be in S, right and S is a subspace of R^n, ok.

Now, if S is an n dimensional subspace it will have n independent or let V_1 1 till V_m be basis of vectors for the subspace, right and let me denote this V_{nxm} matrix in the following units. So, I will just collect all the basis of vectors V_1 till V_m and call this matrix as V_{nxm}, ok.

Now, any V_i will be an element of S because of it is invariance and each Av_i can be written as a linear combination of basis of basis elements of S, right. So, let us see again this. So, I am taking one basis vector v_i, Av_i will always be an element of S, right because of the invariance property of S.

And we know that each element of S can be written as a linear combination of its basis vectors, right. So, Av_i which is an element now of S can be written as linear combination of this basis vectors V_1 till V_m, ok. And I can just write an equivalent matrix representation of this like this, ok.

And I can in general now write for all v, right, so for this entire thing here AV = V$\bar{A}$, ok. So, this V, so similarly like here, right. So, I can just generalize this to write it as AV is V which is an n x m matrix coming from here and a matrix A which is an m x m, m matrix. You can write that down and it is a pretty obvious relationship to derive, ok.

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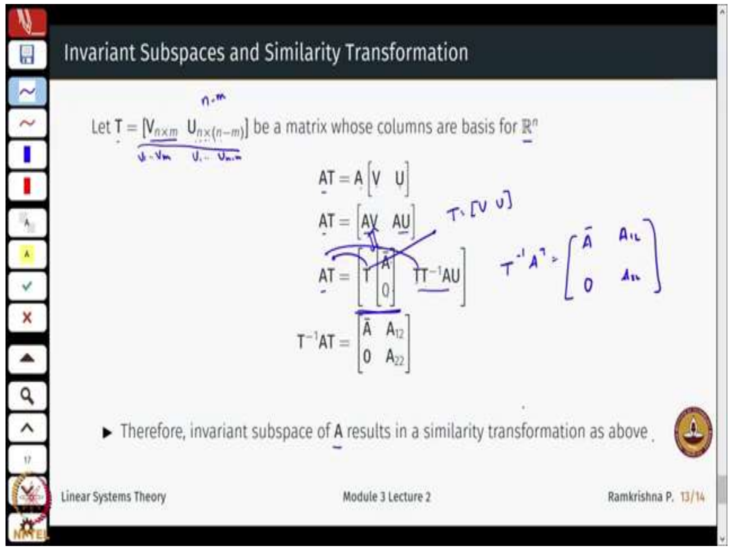

So, let me now construct a matrix T which comprises or its first m rows are V_1 till V_m which form a basis for S, ok. Let me always, let me write down some n - m vectors in such a way that this set of V_1 till V_m and, let me call this say U_1 till U_{n-m}, let this be a matrix whose columns are basis for R^n. I can always to do this, right.

You give me say m or m independent vectors, I can always write the remaining n - m vectors such that they are linearly independent of each other, ok. And then AT can be written in this way A[U V], but T is T is actually [U V],right, this one, ok so, AT is now a matrix which as components AV and AU, ok.

So, A times T, now I go back here and expand this relationship. AV is V$\bar{A}$, ok. This can now be written as AT = T$\bar{A}$ and 0 and TT^{-1} I just just multiply by T and its inverse. And this is obvious which I because T has a representation of V and U, ok. So, I can equivalently write this as this, ok.

So, now, what I have is AT = $\left[T\begin{bmatrix}\bar{A}\\0\end{bmatrix}\ TT^{-1}AU\right]$ ok. So, now, from this I can write $T^{-1}AT$ so I just get this on the other side. So, what I am left with here. So, this T also goes here. So, I have $T^{-1}AT$. So, just concentrate on this one. This is this is the term of interest.

So, I have $\begin{bmatrix} \bar{A} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. So, this could be it could be whatever, ok. So, what does it mean that, so this in the invariant subspace of A results in the similarity transformation as we just derived, ok. What is the importance of this? The importance of this will be seen when we decompose a given system into its controllable and uncontrollable components. So, that and similarly with observable and non-observable components. So, we will remember this and use this as a basis to generate what is called as a controllable subspaces, ok.

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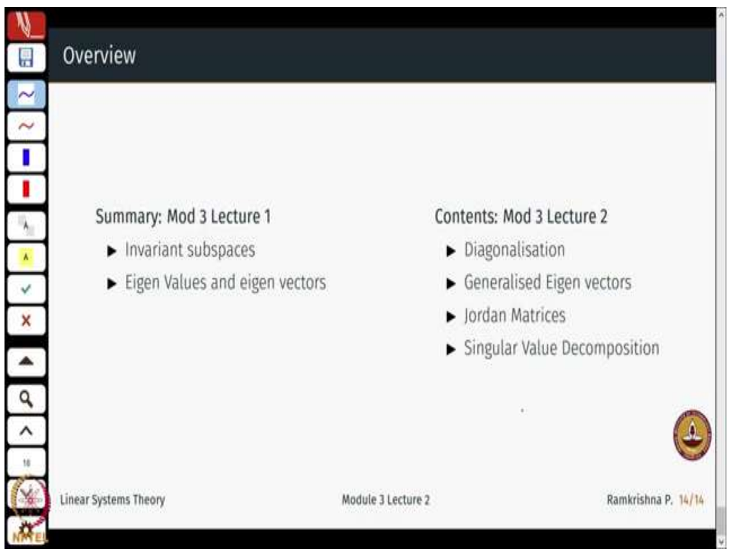

So, so to end this end this lecture, so what we saw was the concept of invariant subspaces, Eigen values and Eigen vectors. In the in the next lecture we will talk about diagonalization, generalized Eigen vectors, the Jordan form and the singular value decomposition. These are also would be instrumental in simplifying complex matrix computations. So, that is coming up in the next lecture.

Thank you.