Linear Systems Theory
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Module - 03
Lecture - 01
Math Preliminaries: Linear Algebra 5

Hello, everybody. My name is Ramkrishna. I am from the Department of Electrical Engineering IIT, Madras. So, welcome to the third module of the lecture series on Linear Systems Theory. So far in the previous in week 2, we have covered some basics of linear algebra starting from vector spaces to norm vector spaces to what is the linear transformation and how a linear transformation essentially has matrix representations and all this we need because we want some nicer tools to analyze systems at hand you know which we write in the state space form.

So, we next continue our discussion on the properties of matrices and vector spaces. First we will just look at some very basic types of matrices. Much of this would not be surprising to you; some of them might be even like intuitively clear. So, we just run through those definitions just to maybe make the slides or the or the course of it self contained.

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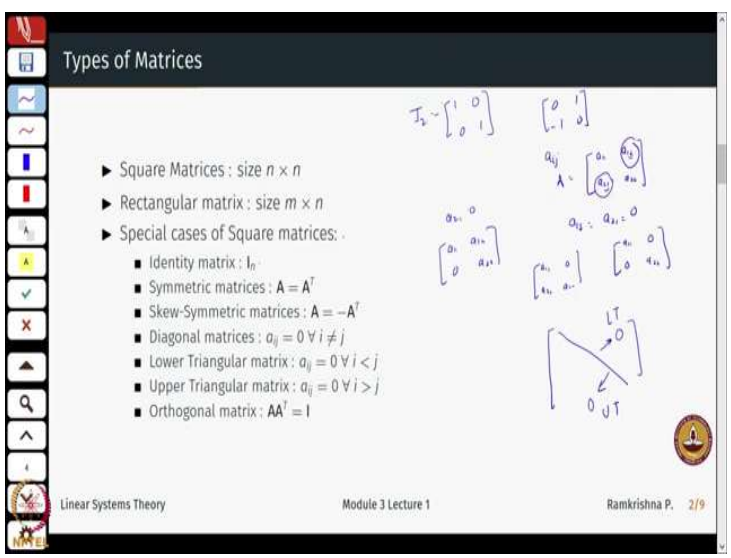

So, we all know about square matrices where the number of columns and the rows are the same. Rectangular matrix would be of size m x n, where m can be less than or even greater than n. So, we can have matrix which are which has more number of rows and columns and vice versa. So, even though much of our analysis would be restricted to square matrices and therefore, we will spend a little more time on looking at properties of those.

So, first is the identity matrix. So, we will denote it as I n which means it is like a n cross n identity matrix. So, typical I_2 would look like $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and then we just write to memorize this notation symmetric matrices A will would be equal to A transpose and I think we all know about the definition of for transpose of a matrix is. Similarly, we have skew symmetric matrices where A is the negative of A transpose A or simple example would be something like this is a is a skew symmetric matrix. You can just look at the substitute and then and look at it in terms of each elements of the matrix.

Diagonal matrices where so, the by definition it says a_{ij} will be equal to 0 for all i ≠ j. So, here a_{ij} essentially is would represent the each of the element of a matrix. Say for example, if i take a 2 x 2 matrix, this would be a_{11}, a_{12}, a_{21}, a_{22} and by the definition of diagonal a_{ij} would be equal to 0 whenever i ≠ j. So, this here i is not equal to j, i is not equal to j and therefore, a_{12} would be a_{212} would be 0, and then the diagonal matrix would just look like this and like very straightforward. It is nothing really to explain much here.

Some more definitions of what is lower triangular matrix again by definition it says $a_{ij} = 0$, whenever i < j. If i just look at this, what are the cases when i is less than j? That is this one. So, a matrix lower triangular would be something like this. Or, in general, if i have larger matrix as a diagonals here so, lower triangular matrix would mean that all entries to the right of this diagonal if you going call this right direction is 0 all other the this is the actual definition.

Similarly, in an upper triangular matrix a_{ij} would be 0 for all i > j. So, this is a case when a_{21} would be equal to 0, and the upper triangular matrix would look something like this a_{12} or 0 here and a_{22} and we are not really worried about what is sitting on the diagonal. And, in an upper triangular matrix, you will have all the entries to the left of the diagonal are 0. So, this is lower triangular and this is the upper triangular.

And of course, the last thing is an is that of an orthogonal matrix where A^T is the is the identity of appropriate dimensions. So, whenever we just the identity, we will make sure that we are working with the appropriate dimensions; specifically when I need n I will say I_n is the n x n identity matrix.

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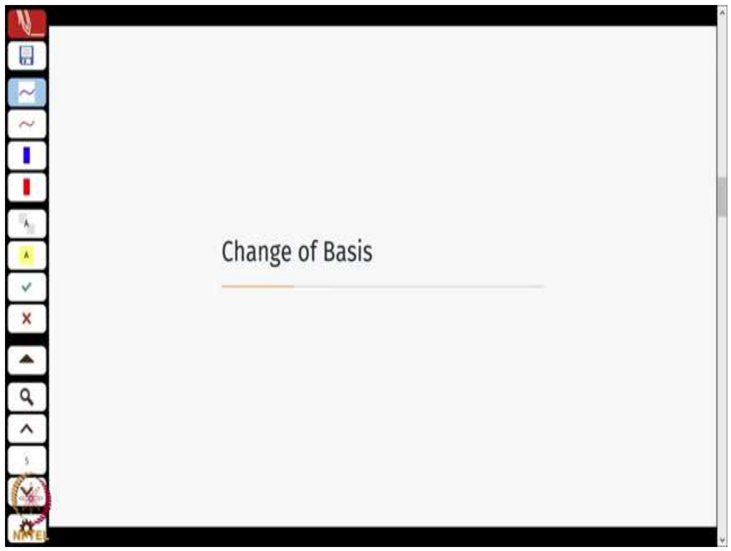

Something which is important and we will revisit this concept lot of times in during the next few weeks of lecturers; one is called the change of basis. If you remember a bit of state space analysis; if you do not remember does not matter, we will do it all over again. Change of base is essentially means how do I change or it is a direct implication on how do I change from controllable canonical form to an observable form or a diagonal form or so on. Or given any state space representation, how do I go about deriving one of those canonical forms and the basis of that or the mathematical tool for that is what we call as the change of basis.

And, what is basis? Well, basis if I look at the standard basis for two-dimensional space it was $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ and what we also did a little example is that given a vector space R^n, the basis may not be unique at and therefore, we have to emphasis how what is the thing that causes the change of basis, how do I go from a set of standard basis to some other or basis right.

So, we will spend some time on that and also as the lecture progresses today and further this week's contents, we will try and slowly relate to the state space representation of systems as and when needed as and when which is it might be a little obvious, just to make a little link the of the kind of tools that we use here and how they are actually going to help us throughout the rest of the course.

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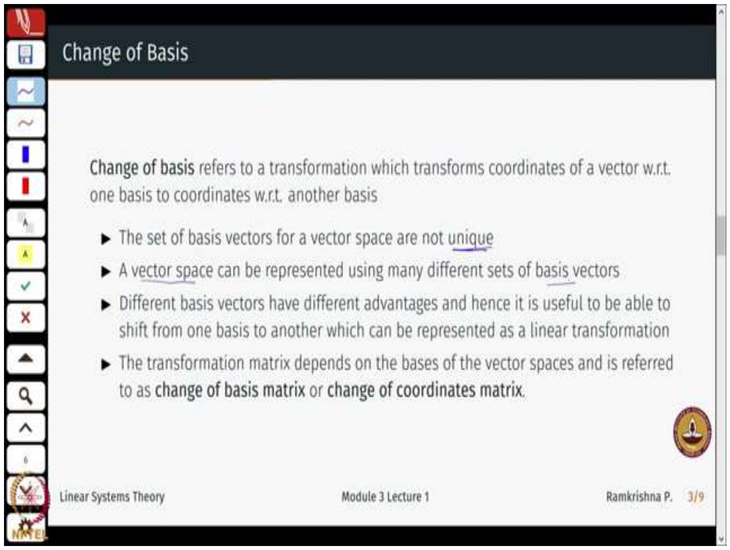

So, what is change of basis? It refers to a transformation which transforms coordinates of a vector from one basis to another basis. Its transformation I am talking of linear spaces and this transformation therefore, essentially we will look like matrix ok. And, why can we do this or why is this important is because the set of basis vectors are not unique, and this actually is helps us a lot that the set of basis vectors is actually not unique. I mean throughout our coordinate geometry, we just look at the standard basis right and their I am not sure if we ever encountered situations where we have to write in different basis. But, here it is very very important.

So, the think which will exploit or try to analyze more is the property that a vector space can be represented using many different set of basis vectors. As I said easier this have their own advantages especially in the course, this is not just very abstract think that we are that we are going to learn. But, this is something which has direct implications on the control related concepts that we will do later in the course.

So, what represents now the change of basis or how do I go from one basis to other?

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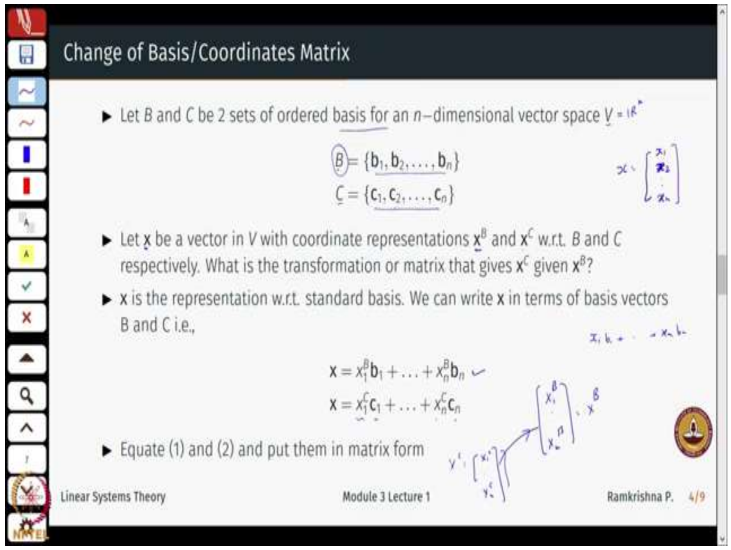

It actually turns out to be a very very simple exercise. So, let us start with two different basis. So, how do two basis relate well I just take two arbitrary basis: one of them could be the standard basis for R^n, that could be any of those right. So, so the transformation exists independent of what will. So, let us let me just take two basis B and C with their you know their individual basis being from b_1 till b_n that the n vectors which represent or which are set of ordered basis for this vector space V. Here it is like because we have n this could be like R^n.

So, b_1 till b_n; so, this means at any vector in R^n can be written as a linear combination of this vectors b_1 till b_n. Same vector can also be written as a linear combination of this basis vector c_1 till c_n and if you remember what we did for the basis vectors right. So, we were talking of linear dependence, linear independence and b_1 to b_n will form a set of basis if any vector can be written as a linear combination of this vectors.

So, let start with x vector x with its own coordinate representations x_1. So, what was the coordinate representation to given x? I will have if I can write this as x_1, x_2 all the way till x_n. This is a coordinate representation of a vector which can also equivalently be in that I am writing this as x_1 say the basis vector is b_1 till $x_n b_1$ ok. So, let x be this coordinate vector sorry b a vector with coordinate representations X^B and X^C at the same vector can have different coordinate representations depending on what basis we choose ok.

What does it mean in when I write the write it on explicitly? So, this vector x with X^B as a coordinate representation with respect to the basis B can be written like this right x_1^b, where x_1^b till x_n^b are the individual coordinates for this X^B; similarly, with X^C. So, this x can be written as x_1^b b_1 till x_n^b b n. The same vector in the basis C can be written in this way where ok. So, what changes? This is the basis change the coordinates will also change. So, I will have X^C written as x_1^b all the way till the x_n^c.

Now, these two are the same vectors if I draw it on the sheet I think they will look at the same. We can just do it said by yourself that these two are essentially the same vectors.

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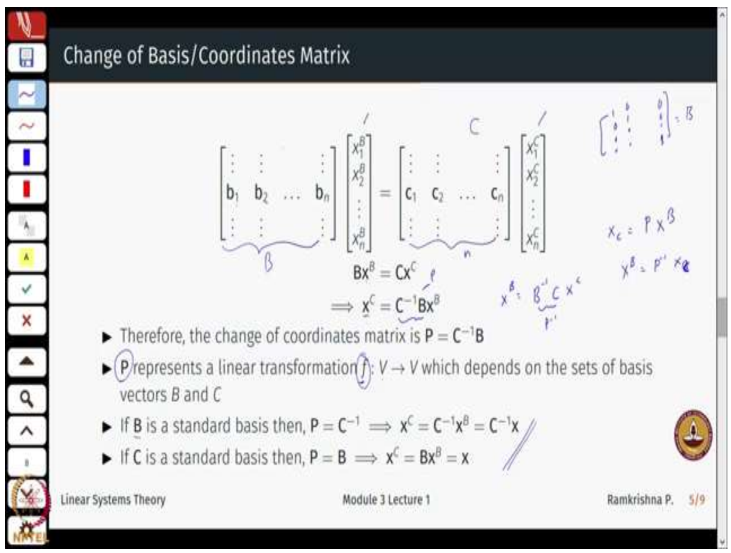

So, in matrix representation I can write this $b_1 x_1$ all the way is of something like this ok; similarly, with this one. Let me call this matrix which already called this by this is B and this is C. So, what I have here is BX^B, the vector X^B is CX^C that in standard if I just write it in terms of matrices.

And, now, look at this matrix, it has n independent columns similarly with this matrix right. It has also n independent columns because b_1 till b_n are linearly independent. Why are they linearly independent? That comes from the definition of the basis. So, now, once this matrices are full rank right so, this the rank of this will be n, the rank of this will also be n. So, I can interchange terms and write X^C as C^{-1}BX^B or I can also do the do the reverse and I can also write X^B as B^{-1}CX^C.

So, what does this answer me? So, I have a vector X^C this one right. So, this things what is the matrix that transforms from X^C to X^B and so, X^C is written in this basis c_1 till c_n; X^B is written in this basis b_1 till b_n. So, what is the matrix or what is the transformation that takes me from one to the to other that is simply this one ok. So, let me call this as P and this would simply be P^{-1} and we all we know of why these are invertible matrices because of this n independent columns. So, what is this P? P represents a linear transformation f from V to V which depends on the set of basis vectors B and C right.

So, so, look at these right. So, so these are the set of basis vectors for B, these are the set of basis vectors for C right. So, this two coordinates of the vectors X^C and X^B can be related via their basis vectors given by this little relationship and it is a straight forward derivation right. A one of them is a standard basis say, let us say B is a standard basis and P is simply C^{-1} because B is just the identity right. So, it will just be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ all the way till $\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}$ this is the standard basis. So, let me call this as B. So, is conversely you see the standard basis then P = B and this is like a straight forward then to derive.

So, what we understood this how to go from one basis to other basis and this is this will be useful lots of also help to kind of memorized at such things actually happen. And, again this transformation f from V to V is has a matrix representation P right. So, this is a this f is essentially has this matrix representation P.

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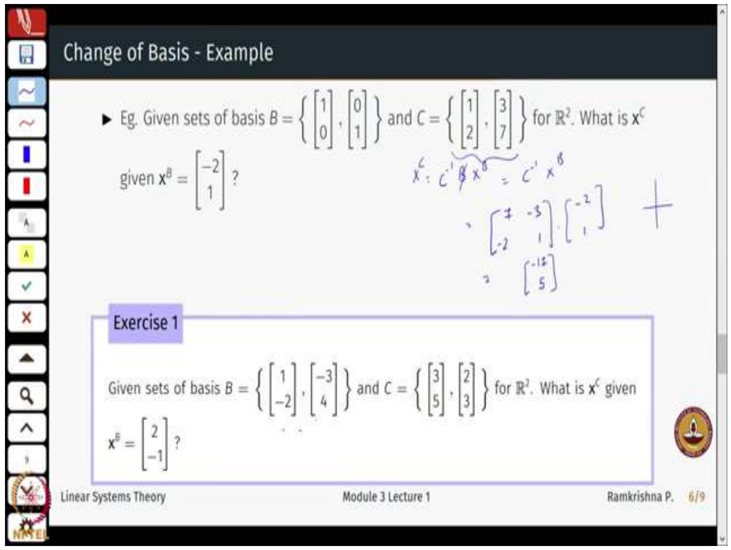

Let us let us do a small example. It is pretty easy if you look at the basis B this is the standard basis as here and C is given by this basis; you can check why these are basis right as a as a little exercise. These are actually linearly independent and so on. So, what is X^C when X^B is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ok? So, we just make use of this formula X^C is C^{-1}BX^B. Whereas, this B is just the identity is I am just looking at $C^{-1}X^B$. C^{-1} can be computed easily to be$\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ and multiply this by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

So, this will be -14 -3 is -17 and this will be -4 plus 1 is -3.

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And this will be 4 plus 1 is 5 and this is very very trivial exercise and you can just draw this two vectors on a on the graph and check, they are actually the same and you can do a little more not really difficult, but something which involves computing a one more inverse. So, instead of B being identity I just give you B matrix which is like this. So, there is nothing really special in this in the slides or in this in this example.

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Let us do something which we would come across a lot of times and this concept called the similarity transformation.

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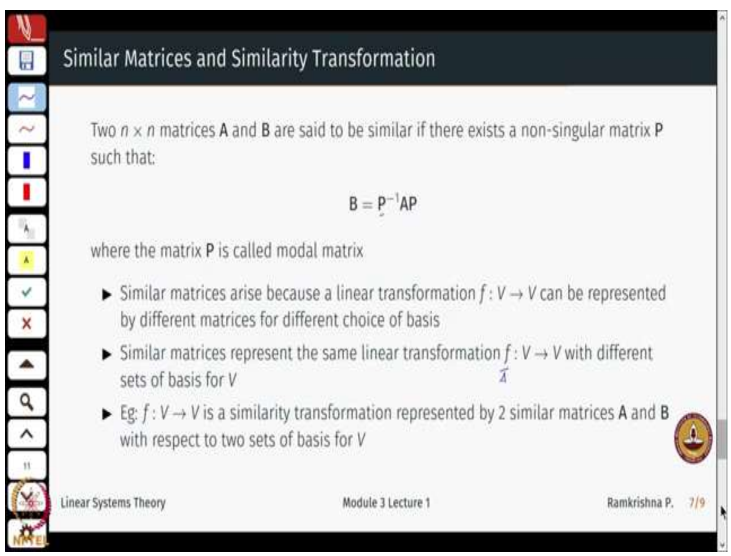

So, let us start with two square matrices two n x n matrices and by definition and we will see why what this definition means and where this possibly even comes from. So, two matrix are said to be similar if there exists a non-singular well be need that to be invertible matrix P such that B is P^{-1}A P, but P is called the modal matrix where we will see how that comes from.

Now, this arise because a linear transformation can be represented by different matrices for different choice of basis as we saw here right. So, we had two basis and you know. So, say for example, here if this coordinates had a basis given by the matrix B, this coordinates or this vector has a had a set of basis vectors represented by this matrix C. Such a thing we are doing some kind of a special case of this one.

So, we so this similar matrices represent the same linear transformation. So, whenever I say linear transformation, I am just talking your of some matrix A with different set of basis B.

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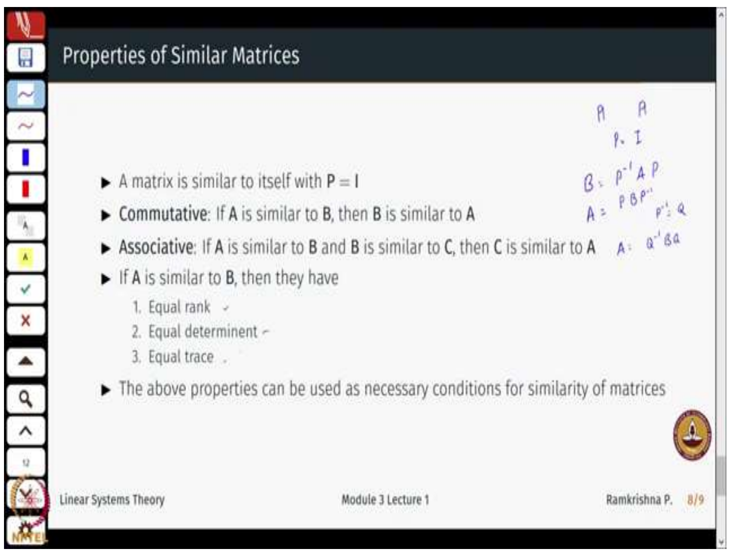

So, so what how do we like write derive this a matrix is similar to itself right so, what is a similarity transformation between P and P? So, I like I say that this is just the identity right so, that P equal to. So, take a matrix A and a what is P the P turns out to be identity this is just trivial ok. So, second is if A is similar to A then B is similar to A. So, let us say that B =P^{-1}AP, that is you can even write A as now P BP^{-1} ok.

If I just want to write so, I will just call this P^{-1} to be Q and then, I can write a due relation like this Q^{-1}BQ. Then all those properties have associatively if A is similar to B, B is similar to C then A and C are also a similar. So, A is similar to B then they have equal rank, of course, they have equal determinant they also have equal traces right. So, so we just do some things here so, why this is important first from the control point of view.

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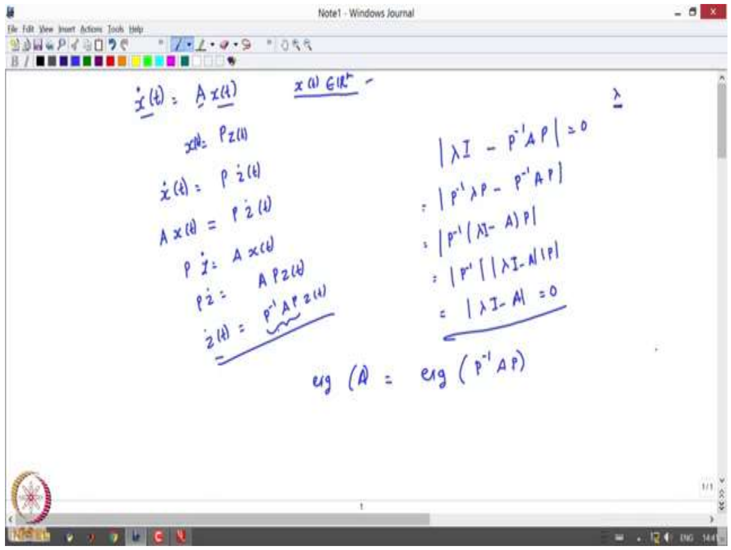

So, most of our of our examples would look like this. So, if I just say I have a system $\dot{x}$(t) is Ax(t) a standard state space representation that we will substantiate a little more later. Now, suppose I were to write this in some other; these are all coordinate representations right $\dot{x}$(t) which have the states. These are essentially coordinate representations where x(t) belongs to certain say n-dimensional space.

Now, let me just call something that x = P z ok. So, I this is the omit the arguments in time. So, these are all this is P, this is t then $\dot{x}(t)$ =P$\dot{z}$(t). Now, what is the $\dot{x}$? $\dot{x}$ is Ax(t) = P$\dot{z}$(t). So, now, what I want to say is a given my system dynamics in x how would they look in z? So, z dot and this is always an invertible transformation that is what that was a necessary condition.

So, this $\dot{z}$ or P$\dot{z}$ is Ax (t) or P $\dot{z}$ is A, what is x in terms of P? So, I will get AP z(t) or in other words $\dot{z}$(t) is P^{-1}A Pz(t). So, this is much similarity transformation right which transforms my dynamics in certain given coordinates x to some other coordinates z. Now, all properties if we remember vaguely of stability or related to the sorry to the matrix A right.

So, say for example, so, here I just said that well these are these are equal rank equal determinant equal trace many one of these would be easy. So, equal rank is like easy to find out. So, a very important concept I would always look at is what is the implication on stability and if you remember stability had something to do with Eigen value, some the basic properties of the matrix had even if I look at in terms of my characteristic equation, the poles and so on.

So, how do the poles of the system relate in x to the poles of the system in z? So, let us let us quickly do this x. So, how do I how do I find those? So, I have |λI-$P^{-1}AP$| is the determinant of this is my characteristic equation in said coordinates ok. Now, this will give me a set of Eigen values and I were to if I were to look at. So, what I want to establish is the Eigen values which come from this matrix P^{-1}AP and how they relate to the originals so to speak A matrix.

I can write this, then as |P^{-1}λP- P^{-1} AP|. We can do this is equal to P^{-1}; so, inside I will have lambda minus A sorry, this two make it consistent and as I just put |λI-A|P ok. Now, this is equal to |P^{-1}||λI-A||P| and this is |λI-A|right. So, the solution to the characters equation would just be do equate this to 0. So, if I equate this to 0, it is as good as equating |λI-A| to 0 and therefore, the Eigen values of the matrix A are the same as the Eigen values of P^{-1}AP.

So, the much of the properties remain invariant right, that means this is kind of intuitive also why this properties should remain invariant right ok. So, now what does similarity transformation or how do this relate directly to a to a change of basis right? Are there is there nice interpretation of this of this $P^{-1}AP$? So, let us do a little scenario here.

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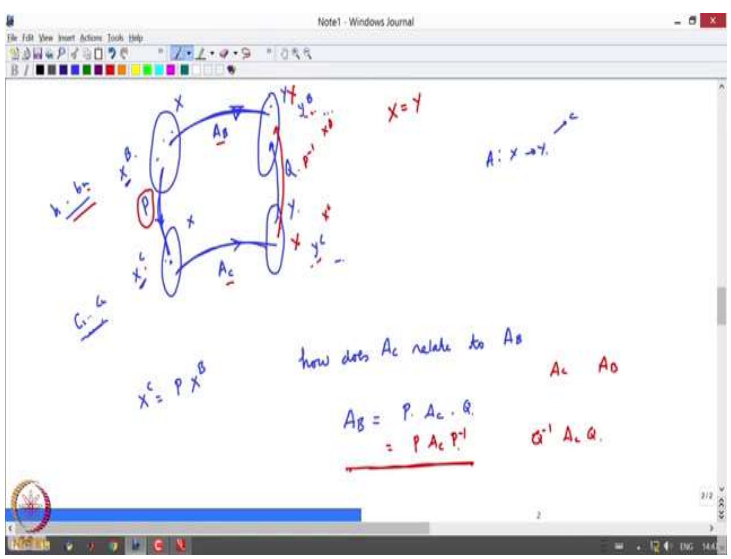

That is why let me take space x could be of whatever dimension and say it has the coordinates X^B with b_1 till b_n being its is basis vectors, the same x has another coordinate representation X^C with being c_1 till c_n being it is a basis vectors ok. So, what we saw is that. So, if I if I just what to rewrite this formula here that X^C is PX^B ok. Let us rewrite that here that X^C = PX^B.

So, it would be mean that this P is a transformation which goes this way the certain on this way is correct PX^B will be X^C here ok. Now, assume also that I have another space say Y which could be of a different dimension say and has its some vector representation here as a Y^C some vector representation here as Y^B with its own set of basis right or whatever they could be you can just write whatever notations you want for that say ok.

Now, we know say a suppose there X is some transformation which goes takes from takes me from X to Y and let me call that transformation to be say some A ok. This transformation should also exist here right. So, this both this basis are Y, this is X, this is X and this X the X on the on top has some basis representations b_1 till b_n with a certain vector the X^B similarly over here the X on the bottom has vector representation X^C with some basis vector c_1 till c_n ok.

Now, if I just look at A as a transformation from this X to Y may be over here right with this under this basis. So, we always define the transformation with basis. So, let me call this you just associated to see. So, it will not notationally correct, but let us see how. So, this is how it looks like. So, maybe I will just call it say A_c. There also exists a similar transformation from X to Y under this basis. So, let me call this A_B.

So, the question is how does A_c relate to A_B? So, now I know this I can also define transformation which goes from Y^B to Y^C via some matrix Q. But, I can just do whatever I can even define this where it because this are just. So, if I can write X^C in terms of X^B I can also write X^B in terms of X^C sorry, X^C here not C here whatever is notation right.

So, I can just go from here till here. Now, look at this carefully. So, I can always write A_B as PA_cQ right. So, this is transformation which takes me from this basis till this basis and it is it is a same of me going from this to this via A_B is the same as me going via this route. So, I take go from here till here, then jump from here to here and then jump from here to here. These are exactly the same.

Now, a special case of this could be when I am just take a different color when X and Y are the same. And, so, so this case X and Y can be can also be of different dimensions. So, not need on necessarily be just this is an x^C and n here this can be a different number also. So, if X equal to Y so, what will be this one? So, this is this is X, this is also X. So, so with the similar changes over here.

So, if I can go from x^B to x^C via this matrix P then I am going again so, y^C this is all everything is now X right I can go from here till here via P^{-1} and therefore, I can write this as P$A_c P^{-1}$ and then equivalently I can also write A_c in terms of A_B. So, what we saw here in terms of so first we try to answer the question of what is the relation between A_B and A_c when A_B and A_c are respective transformations from X to Y which respect to some basis function basis vectors.

So, see so, we can also have a little similar interpretation of the similarity transformation. So, P^{-1}AP would also be similar to writing it as P. So, some other matrix $Q^{-1}A_cQ$ right. So, and this is always invertible is what we know all right.

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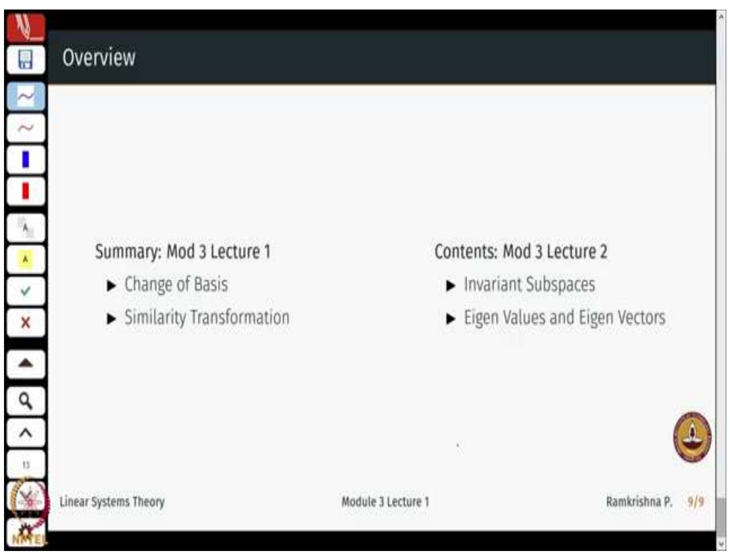

To conclude talked about change of basis and how these are related to something called the similarity transformations and it essentially means writing down my given dynamical system or a control system in terms of in a new coordinates we will see we will use this very often in during the course. So, in the next lecture, we will talk about invariant subspaces and slowly introduce the concept of Eigen values and Eigenvectors. We would have studied this earlier, but we will just give a slightly different interpretation to this, the basics remain the same. So, we will see that in the in the next lecture.

Thank you.