Linear Systems Theory
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Module - 02
Tutorial
Lecture - 11
Linear Algebra

Hello everyone. My name is Jaydev. I am a PhD student in IIT Madras, working with Professor Ramakrishna Pasumarthy. So, I will be doing this tutorial for week 2 in which we will be covering problems on the concepts that have been covered in the week, which include things on vector spaces norms, matrix. Then, basis span, and fundamental subspace of matrix and even the linear maps and linear transformations.

So, we will do first 4 problems in this tutorial and I will be going through 1 by 1. So, the first problem is we will try to prove the triangle inequality which was discussed while introducing the norms.

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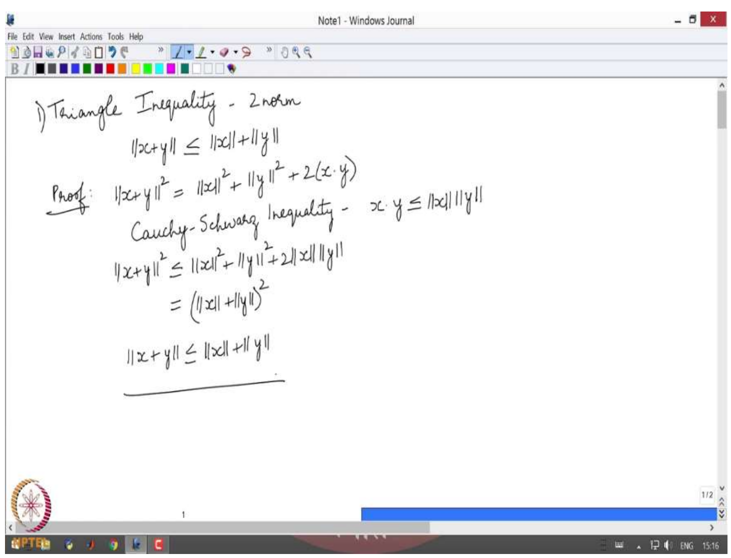

So, it is said that. So, we will try to prove it will strict to only 2 norm for proving this or it can be proven for any other norm as well. So, I will say is the triangle in equality is $||x+y|| \leq ||x|| + ||y||$. So, this is the triangle inequality and we look at the poof.

So, first we will square this right hand side and try to expand that. So, when we take the $(||x + y||)^2$, we get $(||x||)^2 + (||y||)^2 + 2 < x, y >$ ok. At this point, we will introduce an inequality called Cauchy Schwarz inequality which states that the $< x, y > \leq ||x|| * ||y||$. ok.

So, in we will try to substitute this inequality into this above expression and see what we get. The $(||x + y||)^2$, I can now right it as $\leq (||x||)^2 + (||y||)^2 + 2||x|| * ||y||$. So, now these are simple I can write it as a quadratic form, I can say $(||x|| + ||y||)^2$.

Now, if I take the square root on both sides, I will get and assuming we are taking only the positive sign; $||x + y|| \leq ||x|| + ||y||$. So, this proves the triangle inequality. So, now, going to the second problem.

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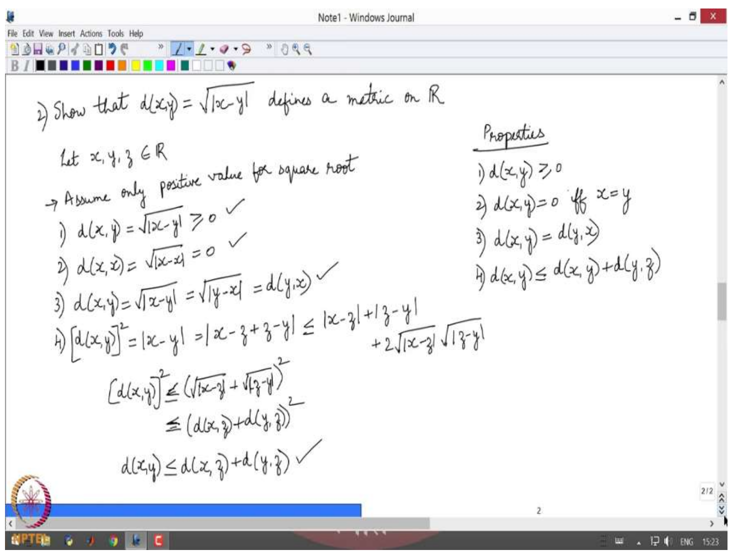

So, this is a second problem. So, we are asked to show that a distance or a metric defined as square root of the absolute value between x and y or the difference between x and y is a metric on the real number space R. So, how can we show that this is a valued metric? So, we have seen that there exist four properties which every metric should satisfy and the first among them is the $d(x,y) \geq 0$.

The $d(x,y) = 0$ iff $x = y$ and $d(x,y) = (y,x)$. And finally, the triangle inequality and $d(x, y) \leq (x, y) + d(y, z)$. So, for a metric to be valid, we said that these four properties should be satisfied. Now, we will see if this particularly defined distance satisfies all these properties and whether it can be called a metric or not.

So, the first property if you can see ok. So, let us say x, y and z exist in R ok, the first three properties can be actually very trivially verified. Because if I take I am assuming that ok, so one assumption is assume only positive values for square root. So, we restricting ourselves only to the positive roots of this number.

So, when we say that $\sqrt{|x - y|}$; $|x - y|$ is always a positive number the square root is definitely defined. So, we can say that the first property that distance between x and y which is equals to the square root of absolute value between the difference between x and y is always greater than 0. And similarly, if $x = y$; then, I can say $d(x,x)$ which is $x - x = 0$.

So, this satisfy the second property. Now, going to the third one distance between x comma y is square root of absolute value or difference between x and y, it is also equals to the absolute value square root of absolute value of difference between y and x because it is an absolute number, it does not matter whether it is $y - x$ or $x - y$. So, I can write it as a distance between y , x.

So, now we have 1, 2 and 3 satisfied. Now, the fourth one is a bit tricky one. So, we will try to concentrate on that. So, what we will do is we will again take, we will follow very similar approach which we use to prove the triangular inequality. So, we take the square of the distance metric and the square is nothing but $|x - y|$ and this, I can write it as $|x - z+z-y|$ and which is always $\leq |x - z| + |z - y|$ because just absolute numbers you can always verify that this inequality holds.

So, what we will do is this number is greater than this. So, I can always add another term, a positive term to this and this inequality will still hold. So, I will add this term. So, you can see that this is one square root; this is one square root as we assume that we will assumed we will consider only the positive values for the square root. Two times of these 2 square roots will give as a positive number and this whole thing on the right hand side will always be greater than or equals to this absolute value on the left.

So, now I can say that; so, now, I can write it as $(\sqrt{|x-z|} + \sqrt{|z-y|})^2$. So, I just converted this into a complete square and now, I can say that this is the distance between x and z. This is the distance between y and z whole square. Sorry. This will be an inequality.

So, finally, by taking the square roots, I will get d(x,y) ≤ d(x,z)+ d(y,z) which is my fourth property of the metric. Therefore, this shows that the distance metric which was defined as a square root of the absolute value of the difference between 2 numbers x and y is a metric on R. So, this completes the second problem. We will go to the third problem.

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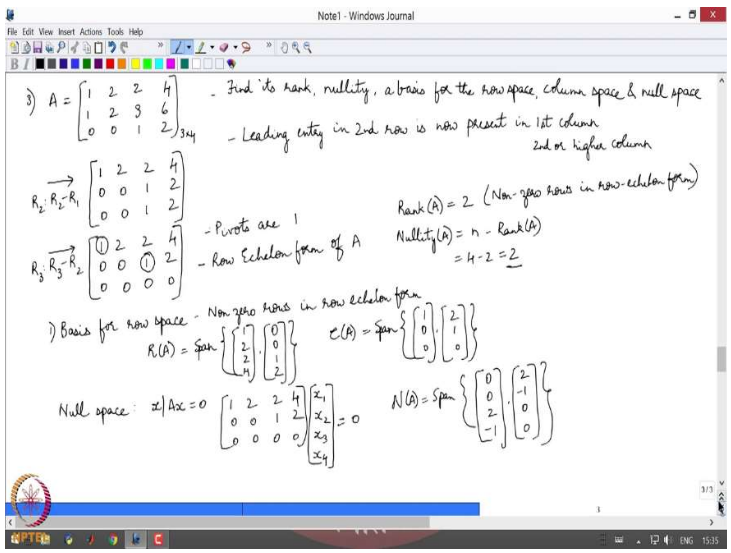

So, in the third problem, we have given a matrix A which is a 3 x 4 sized matrix which has elements as follows ok. So, we have given a 3 x 4 matrix A with the following elements. So, now, we are asked to find it is rank, nullity, a basis for the row space and then, the column space and also the null space.

So, we know that given a matrix, we can always find all these elements like the rank, nullity, a basis for the row space, column space and the null space. So, during the lectures of this week, we have covered all these concepts. So, I will not be going into the definitions. I will be directly applying them to find it. So, during the lectures, it was said that the easiest way to find all these values or rank, nullity and basis is to find the row echelon form of any given matrix ok.

So, now, how do we convert this given matrix A into a row echelon form? So, if you recall the definition of the row echelon form, it says that the first or the leading entry of every row should be 1. So, I can check that all the leading entries. So, leading entry is the first nonzero element in a given row and all of them are 1. So, it satisfies the first property.

So, but if you go to the second property, the leading entry in a given row should be present in the column next to the leading entry of the previous row which means so if I take the leading entry in the second row is now present in the first column. So, which is not right as per the definition of the row echelon form. It should be present in the second or higher columns, but not in the first column. So, what we have to do is we should make this value 0.

So, how can we make this 0? So, we will do the following matrix operation. I will take R_2 and replace it by R_2-R_1. R_2 is nothing but the row 2 and R_1 is row 1. So, what will I get? I will get something like this. So, now, you can see that the leading entry in the second row is now present in the third column which satisfies the property of the row echelon form. But now, if you go to the third row the leading entry is in the third column, but they if as the leading entry in the second rows in the third column, the leading entry in the third row cannot be in the third column. It can be only in the fourth column or higher or it need not be even present.

So, we will do another operation as follows. We will take R_3 and subtract it from R_2 and replace it. So, this is what we get ok. So, now, if we look at this matrix, it satisfies all the properties of a row echelon form. So, all the leading entries, so this is a leading entry; this is a leading entry. So, we call them as pivots. So, all the pivots are one and that satisfies the first property and the pivots are present in the succeeding columns as the rows gone. So, that is also satisfied and any 0 rows are always present at the bottom most thing. So, that is also being satisfied.

So, now, I can say that this is a row echelon form of given matrix A. So, now while defining the rank we said that by looking at the row echelon form, the number of nonzero rows in the row echelon form will give me the rank of the matrix. Because they are the only present independent rows in the row echelon form. So, if you look at this matrix A in the row echelon form, we have just 2 rows which are non zero. So, I can say that my rank of A is equals to 2.

So, which is nothing but the non zero rows in the row echelon form and then, how about the nullity of A? For nullity of A, I can use the rank nullity theorem which I can write it like this. Rank nullity of A is n -rank(A), where n is the number of columns of A. So, I can say it is 4 - 2 and it is equal to 2.

So, now we find out the rank and nullity. So, now, how do we get a basis for the row space. So, the basis for the row space is very easy to find out from the row echelon form. It is just given by the span of the non zero rows in the row echelon form. So, basis I would not say even span, I will just say it is the basis are the non zero rows in the row echelon form.

So, this is given by or I can say that my the row space is the span of. So, this is my row space. Now, coming to the column space we need to identify. So, we know that they are only 2 existing independent columns in this. So, we need to identify which of these 2 columns in the row echelon form are independent, I can we can clearly see that the first 2 are not independent.

So, we can go with the first and the third. So, I will say that the columns space of A is spanned by the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$. So, now, going to the null space, how do we find the basis for the null space?

So, the definition of the null space says that all x such that Ax should be 0. So, if I take this A, I want elements x 1, x 2, x 3, x 4 such that this product is equals to 0. So, I can find out 2 linearly independent vectors. Because I know that the nullity of a is 2 I can always find 2 linearly independent vectors which satisfy this equation and so I will right down the null space of A is spanned by the vectors $\begin{bmatrix}0\\0\\2\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}$.

You can always substitute these two in here and you will find out the product will be 0 and you can also verify that any linear combination of these 2 vectors which are the basis of the null space, if you find any linear combination of them and they substitute there if they are substituted in this x, they are always result in 0. So, that is as per the null space definition. So, this brings us to the end of the third problem.

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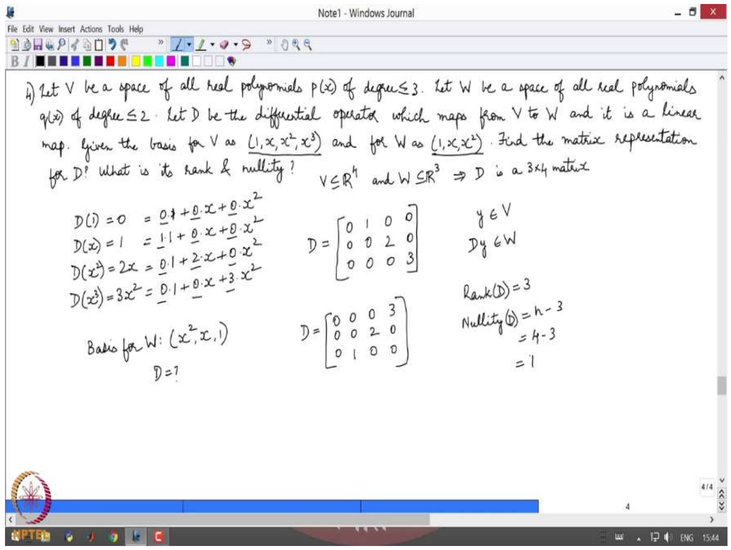

Now, given to the last problem of this tutorial. So, this is the fourth problem. So, we are saying there is a space V which contains all the real polynomials. So, which contains all the real polynomials I mean the coefficients of the real polynomials of degree less than or equals to 3 and similarly, there is another space W which also contains all real polynomials, but which are of degree 2 and less than 2. So, let D be a differential operator which obviously, maps from V to W and it can be easily verify that this D is linear map.

So, now given a basis for V which is 1, x, x^2 and x^3 and given a basis for W as 1 ,x and x^2, we are asked to find a matrix representation for this linear map D and we are also asked to find out the rank and nullity of this matrix. So, now firstly, we can verify that V is a subset of R^4 and W is a subset of R^3 which implies that D is a 3 x 4 matrix or it has a 3 x 4 matrix representation and we are asked to find out that matrix.

So, first what we can do is we can apply this operator D over the given basis. So, I will say D(1) is nothing but 0. D(x) which is the differential of x which gives us 1. D(x^2) is 2x and D(x^3) is $3x^2$ ok. So, these are nothing but the differentials of the given basis.

So, now how can I write these numbers in terms of the basis for W. I can write it as follows (0)*1 + (0)*x + (0)* x^2 which is obviously, 0 and (1)*1 + (0)*x +(0)* x^2 which is equals to 1. (0)*1 + 2* x +(0)* x^2 which will give me 2 x. (0)* 1 + (0)*x + 3*x^2, this will give me three x square.

Now, I will put these as the columns of my matrix representation. So, the first column will be $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. These are the coefficients and the second column will be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. the third column will be $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ and the fourth column will be $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. So, given an element say y which belongs to V, if I apply this matrix over y, I will get an element which belongs to W which is as per this particular basis.

So, it is all aspect this basis. So, before finding out the rank and nullity of this matrix, what if we say that we change the order of these basis. So, I will change just the order of one of these basis. Let the basis for V remain to be the same. So, I will say that the basis for W is now change to the following order x^2, x and 1 ok. I just reversed the order. So, now, what will be D?

So, it is obvious that it cannot be the same. So, we just have to actually reverse this coefficients. So, we will get it this manner. So, as you see that it depending on not just the basis, but even depending on the order of the basis, the matrix representation that you get for a linear map actually changes. So, this is to be very much noted and observed while dealing with linear maps and a given basis. So, now, finally, the finding the rank and the nullity.

So, the rank can be at most 3 because it is a 3 x 4 matrix and you can actually observe that the rank is 3 because all the three rows are independent of each other. So, the rank of D is equals to 3 and if I apply the rank nullity theorem, I will get the nullity of D to be n - 3 which is the number of columns it is 4 in this case; 4 - 3 is 1. So, this is how this problem is solved. I hope you got an idea of how the basis plays an important role in determining the matrix representation of a linear transformation ok. So, this brings us to the end of this tutorial.

Thank you very much.