Linear Systems Theory
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Module - 02
Lecture - 05
Math Preliminaries: Linear Algebra 4

Hello everybody. Welcome back to this lecture series on Linear Systems Theory. So, we shall continue with our lectures on linear algebra. So, last time what we saw was about how a linear transformation is essentially defined by a matrix.

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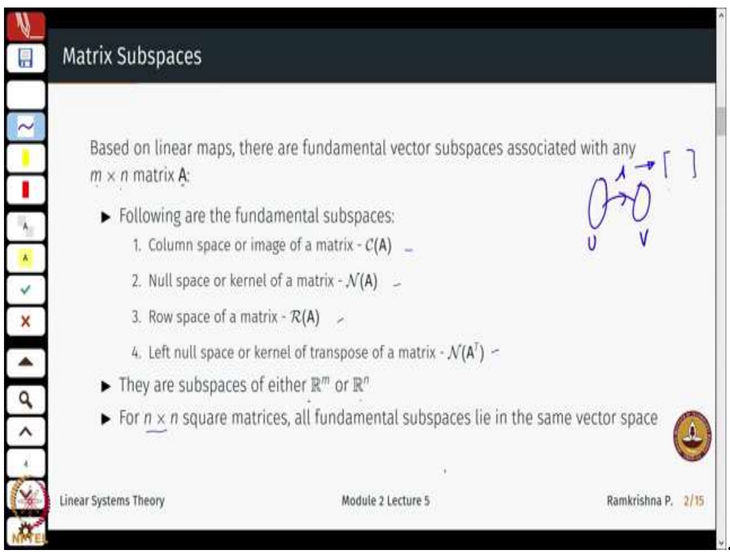

So, if I had transformation say let me call this say A from a vector space U to another vector space V. So, this is a essentially had a matrix kind of structure ok. Now we will do a little bit on the structure of this matrix and if it has some more information that we can we can exploit for ourselves, ok.

So, let us start by this. So, let us assume that I have a matrix A which is of dimension m x n. It could also be square, but we will just do a more general case of m x n and matrices. So, we will define what is known as the column space and the null space, the row space and also the left null space or the kernel of the transpose . What these mean are possibly clear by learning, but we will do a bit of that in detail. So, based on which of these are do they belong to the set 1, 2, 3 or 4. They can either be of dimension R^m or dimension R^n

and for n x n, it is a it is a little bit simpler everybody will belong. So, all these fundamental subspaces will be subsets of or subspaces of R^n, ok.

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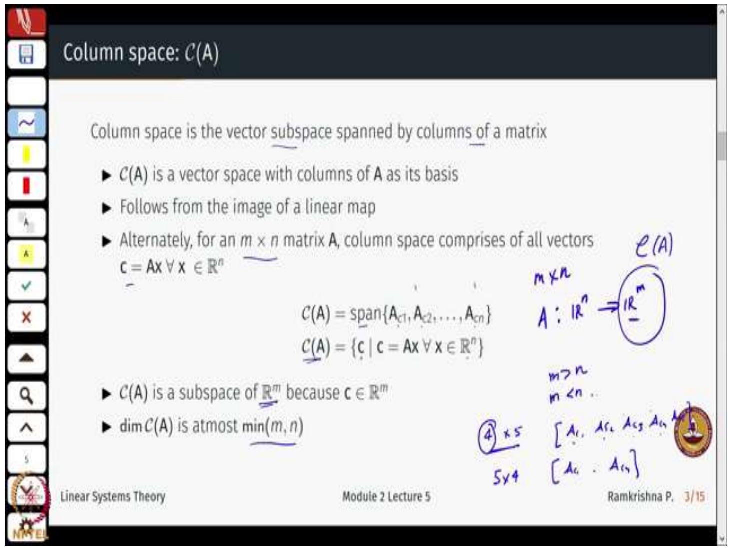

So, first about the column space. So, as the name suggests, it is a vector subspace spanned by the columns of the matrix right and C(A) is then a vector space with the columns of A as it is basis and if you look at it carefully, this also has close connection with the definition of the image of a linear map.

So, what do this means? So, if I have m x n matrix, the column space comprises all vectors c = Ax, but first such that C(A) which is the column space is the span of all these columns of A. So, this is the first column of A, second column of A until you have m x n column. So, you have m x n matrix alternatively C(A) is all the vectors "c" which satisfy c = Ax for all x coming from R^n.

So, your map again is from. So, this map if I look at A is from R^n to R^m, ok. Now this C(A); so, if I just take say the C(A) gives me elements in the space m, right in the space R^m and therefore, C(A) is a subspace of R^m and because you are looking at C(A) is a I am already over here and the dimension of this C(A) is at most the minimum of this either m or n. So, that that depends really if you know if m is greater than n or if m is less than n or so on. So, let us say I take a matrix which is say 4 x 5.

So, this will have 5 rows, right. So, I have $Ac_1, Ac_2, Ac_3, Ac_4, Ac_5$ right. So, this 5 vectors at most I can have vectors which are 4 which are 4 vectors which are independent of each other.

So, whenever I have dimension, this way there will always be the fifth vector which I can write as a combination of the of the other four. So, at most I can have four independent column vectors, right. So, similarly even if I have a 5 x 4 matrix, right so I will have Ac_1 till Ac_4. So, at this I can still have four independent column vectors. I can have 3 2 1 and nothing also if it is just the zero matrix, ok.

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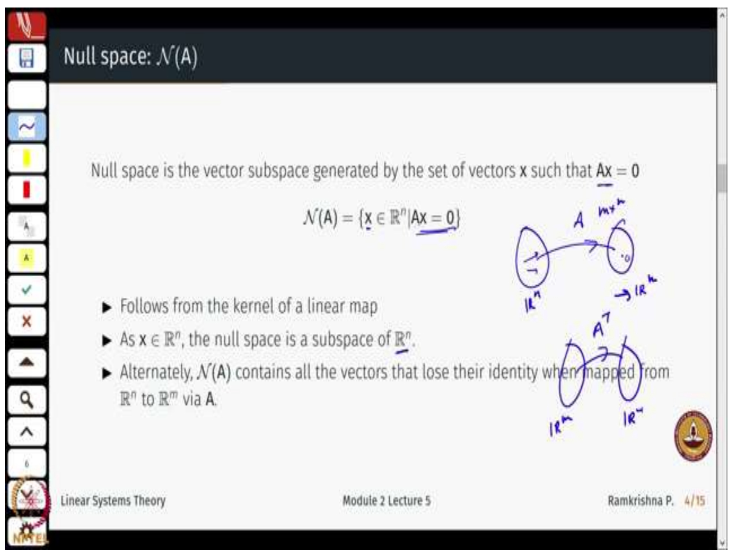

So, similarly the null space is the vector space. It is generated by the set of vectors x, such that Ax = 0 very similar to the definition of Kernels. So, I am looking at all x such that Ax = 0, right.

So, if I take the all of this x, then it will be a subspace of R^n because I am just looking at these xs whereas, in the previous case I was looking at C which is actually A of x or Ax which belongs to the range space and this belongs to the domain ok. So, like what does this mean like practically? So, A consists of all the vectors say maybe I can start with some vectors here. We just under this transformation A just go to the zero element here or they just lose their identity or you know their vectors of dimension zero, right.

So, that is about the column space of A or C(A) or the null space of a transformation which is denoted by the matrix A.

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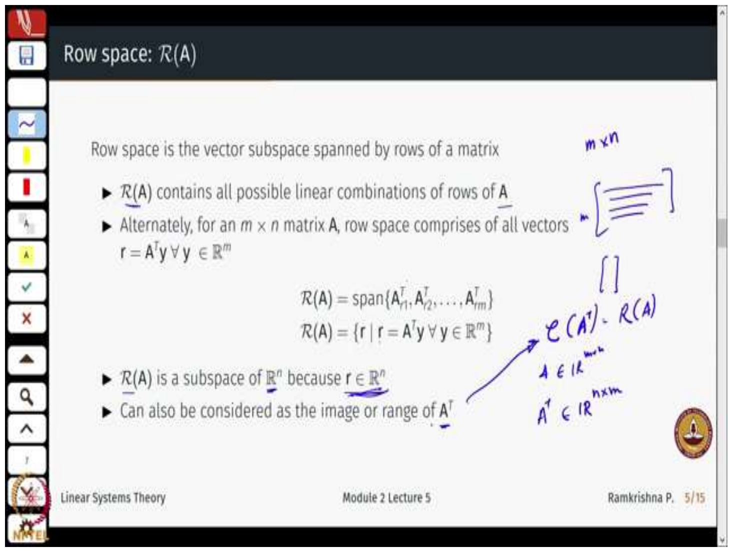

Similarly, I can also define what is a row space. Now it should be kind of kind of obvious by what we by what we defined earlier, right. So, there R(A) consists of all possible linear combinations of rows of A. All possible linear combinations would be the entire space that we get right from.

Student: It is row space.

That is the row space, right ok. So, if I look at the matrix representation again starting from an m x n matrix, the row space comprises all the vectors so, because so if I just have these rows, right so I have 1 2 3 4 and so on up to m number of rows. I am just looking at the span of all this of this of these row vectors. So, I just look at the transpose because usually I talk of. When I talk of a basis vector, I usually talk in terms of the columns.

So, I just look at the that row vectors and the span of it especially the transpose of the row vectors or in other words, this can also be considered as C(A^T) is the same as R(A), right ok. So, what is R(A)? Now R(A) will be now a subspace of R^n because of this, right. So, what is R(A)? All "r"s such that $A^T y = r$. So, if A is of dimension m x n, then I am looking at a transpose of dimension n x m and therefore, the range space is now a subspace of R^n right and this can also as I said earlier can also be considered as the image or the range of A^T, ok.

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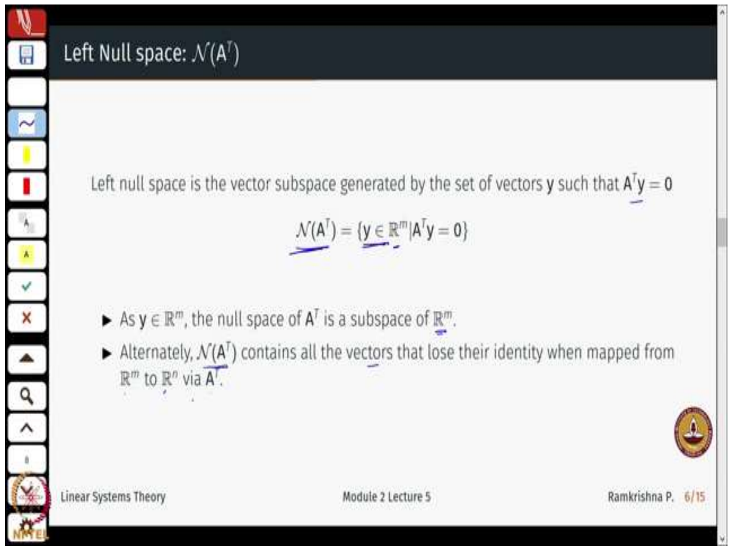

Now, the last definition in this set is the left null space. So, well again as the name suggests, so the right null space if I just look at, I am looking at Ax = 0, right. So, here I am looking at A^Ty=0. So, the null space of A^T is all "y" which comes from R^m such that $A^T y = 0$ again. So, this "y"s are in R^m, therefore the entire set if I constructed will be a subspace of R^m, right.

So, alternatively loosely speaking this contains of all the vectors that lose an identity when mapped from R^m to R^n via A^T. So, just if A is a mapping, so this is an m x n going from R^n to R^m, then A^T would simply be a map from R^m to R^n ok. So, this is pretty basic, right just the definitions that we will follow ok.

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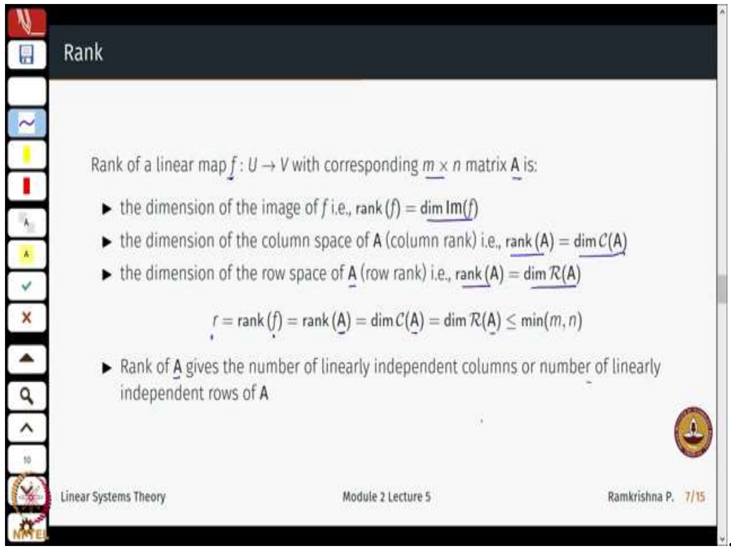

So, what do we do once we define this right? So, first definition is something that will be useful to us is the rank of a linear map, a rank of a linear map F which goes from U to V with a corresponding matrix representation which is A which is of dimension m x n. So, the dimension of the image of F or the rank of F is simply the dimension of the image of F. That again comes from some you know some basic definitions over here coming again from the column space till the image and so on, ok.

So, the first definition is the dimension of the image of F or the rank of F is simply given by the dimension of that image of F. Similarly for column space I can just write that the rank of A is the dimension of the column space of A. Similarly I can write for the dimension of the row space of A is also the rank of A which is dimension of R(A). So, just to summarize I have the rank of a map.

So, this is my map say this is equal to "r", this is also equal to the rank of the corresponding matrix A, this will also equal the dimension of the column space or also the dimension of column of the row space and add this, it can be of dimension m, n. I think you can just write down a little example or the little details, but these are these are pretty straightforward, right.

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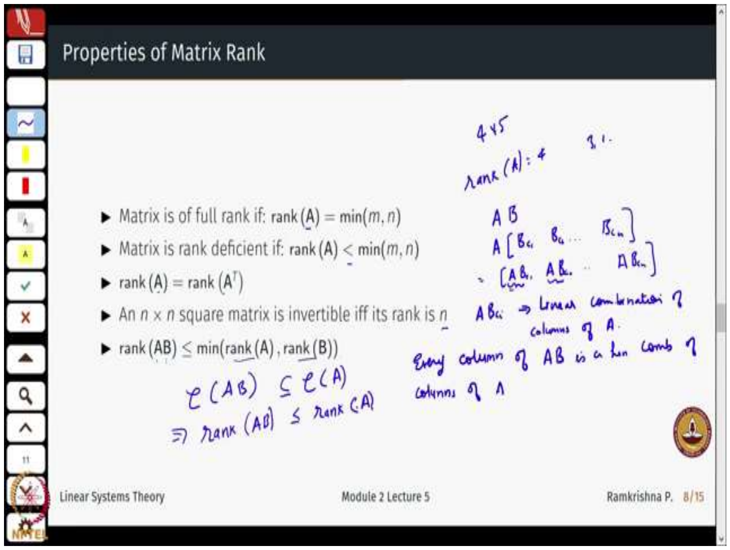

Now what does the rank of A give me? The rank of A gives me the number of linearly independent columns or number of independent rows of A, right ok. Now just to summarize right; so, if so or just to carry forward of those definitions matrix of full rank is a full rank, if the rank of A is a minimum of (m,n). So, if we have say at this take a 4 x5 matrix, if the rank of A is 4, then I say it is full rank; if it is 3 or 2 or whatever, then its rank deficient for the next statement says this is again straightforward to verify that the rank of A is the rank of A^T.

Nothing really to work out here, but you can just substitute here and check for yourself. Lastly as n x n square matrix is invertible. If it is of full rank, we would have learned this in some form or the other in one of our earlier courses and lastly interesting properties. If I take two matrices and multiply them by each other of A B, the rank of it is the minimum of the rank(A) and the rank(B), ok.

Let us quickly do a bit of a proof for this. So, let me take a matrix A and matrix B, alright. So, each matrix will have a set of its rows and columns and rows and columns and so on. So, I can write this as A with say let us see B. B is our this the columns of Bs can be written this way.

So, this will give me A$[B_{c1}B_{c2}]$ so until n ok. Now if I take an element here or this or anyone of this, so each A multiplied with some i^{th} column of B will be some kind of a linear combination of columns of A right. I think this as should be should be easy to check and therefore, every column of AB, this is each every column these this is a linear combination of A. Now if you follow the basic definitions, the column space of A B will follow this. This implies if I look at the rank(AB) ≤ rank(A).

Now again I come back to this and say well let me erase a part of this.

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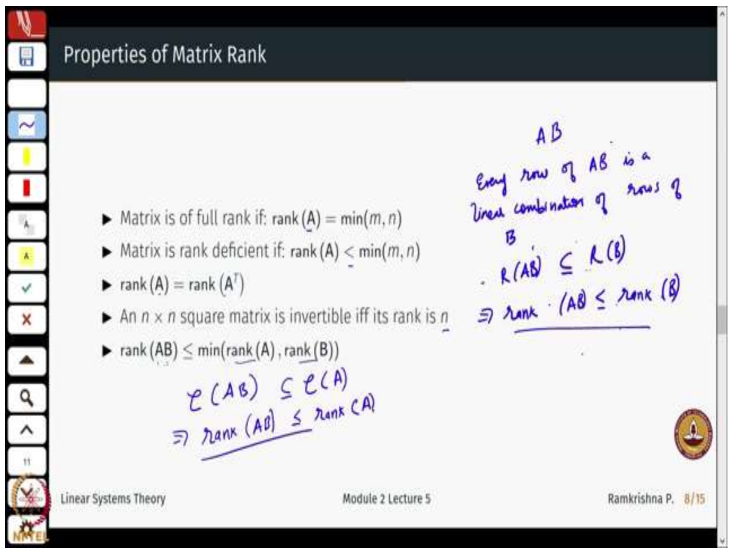

So, let me look at the product AB right and then every row of A B is a linear combination of rows of AB, right and then following similar argument such as before I can get this inequality here which implies that rank(AB)≤rank(B) and therefore, the result follows right combining this expression and this expression that that if I have two transformations with that matrix representation as A and B, the product of the rank is the minimum(rank(A),rank(B)), ok.

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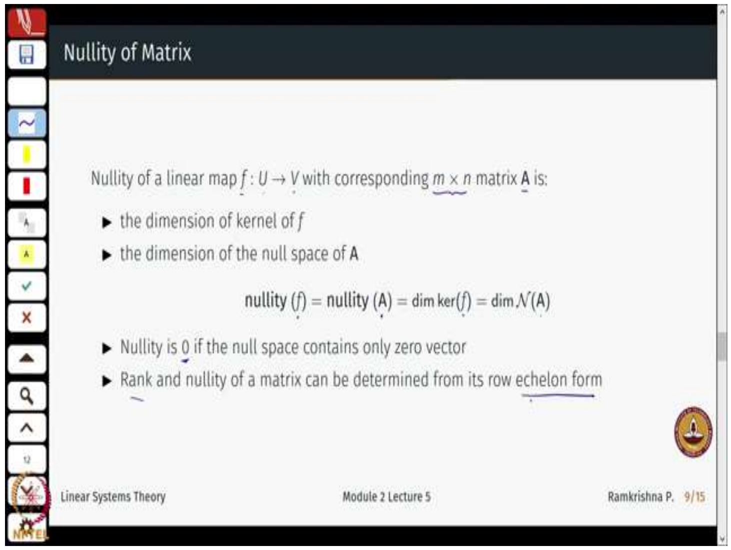

Now, what is the nullity of A matrix? So, we spend some time looking at what is the rank of a map or the rank of A matrix which is a equal representation of a linear map, ok. So, the nullity of map F which again goes from U to V with a metric representation of a of certain dimension m x n is a dimension of kernel of f. So, again it is not really difficult to check that the nullity of f is the nullity of A is by definition the dimension of kernel or the dimension of the nullity of A.

So, nullity is zero; if and only if the null space contains only the zero vector. The zero vector here maps to the zero vector there. Nothing else maps to the zero vector on the other side right and of course so if we remember some for undergrad training on matrix algebra, I can say the rank and the nullity of a matrix can be determined from its row echelon form. So, later if we get some time we will spend some time re revisiting those of how to derive the echelon from given certain matrix maybe during a tutorial class, ok.

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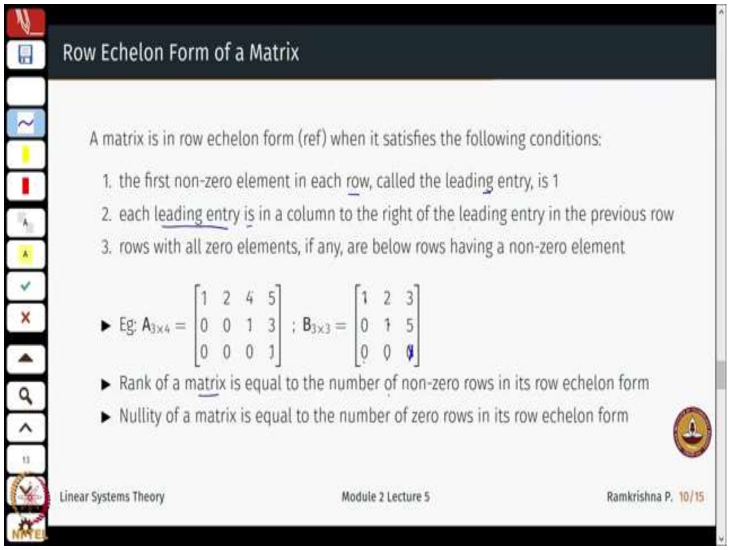

So, just to brief before that I think so, the definition or how I echelon form goes is when it satisfies the following conditions right, the first non zero element in each row called the leading entry is 1. It says 1 here, 1 here, 1 here and 1 here ok. So, the second rule says that each leading entry is in a column to the right of the leading entry in the previous row. So, for example I have 1 here. So, the next one can occur only in the second row and if there is a third one, it can only occur here and then nowhere else. It can occur maybe here or nowhere else, not here or not here.

So, of course we will we will leave the derivations for later, but what do we do with this form. So, the rank of a matrix is equal to the number of non-zero rows in its in this echelon form, ok.

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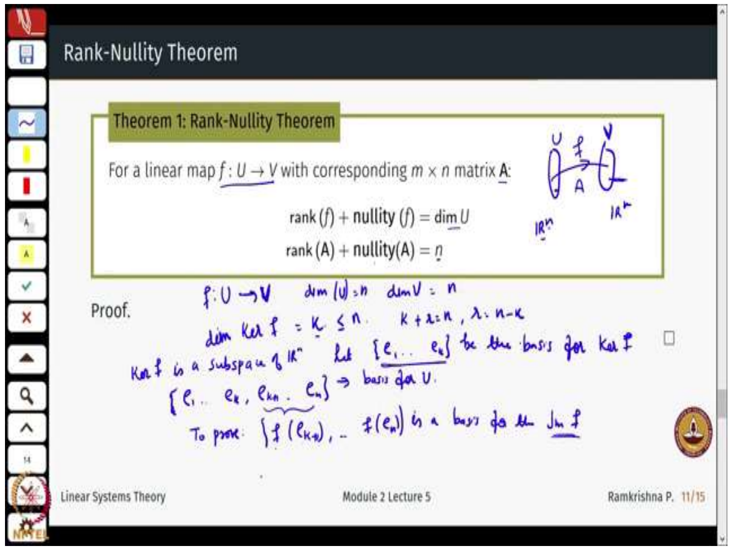

So, this leads us to one of the fundamental theorems that that we will require through the course of in into deriving concepts or on controllability or observability. So, what is the theorem say? So, given a linear map f which goes from U to V with the corresponding matrix representation A as always the rank(F)+ nullity(F) is the dimension of U or the rank (A)+ nullity(A) = n.

So, what is the dimension say? So, see in this case U is R^n, V is R^m and I have a map f which has a matrix representation of A. So, rank(F) + nullity(f) is has the dimension n, ok. Now we will just quickly do a sketch of the proof ok. So, I can start from f going from U to V with dimension of U is n dimension of V is n, ok. Now let us assume for simplicity that the dimension of the kernel of F is some number k, it could be 0 also right. So, that does not. So, k which will be less than or equal to n and let us say that the difference is such as k + r= n or we define a quantity r = n- k, you can also be 0 or any value between 0 and n.

Now what I know is that the kernel of f is a subspace of R^n because the kernel is defined here in U, ok. Now if this is a subspace, it will have a certain basis of dimension k. So, let I will call that say let us say e_1 till $e_k$be the basis for kernel of this map F, ok. Now this U is an n is an n dimensional space. So, let us say that the total basis of that would be e_1 till e_{k+1} till e_n and this is a basis for U. Now what is to be proven here is if that if this is the basis for the kernel that the remaining is a basis for the image of F, right. So, we have to prove f (e_{k+1}) ,,,f(e_n) is a basis for the image of f because. So, why? Why I am doing this f because I am when I am talking of image I am already in this in R m I am already in this subspace v, right.

So, what is to be proven now is that f (e_{k+1}) till f(e_n) is a basis for the image of f, ok.

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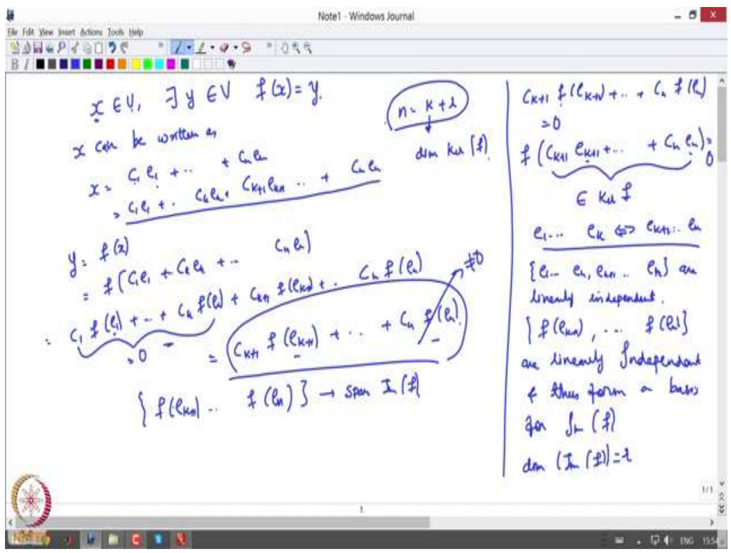

So, just assume that for some x in U, there is a corresponding y in V such that f(x)= y now x can be written as some coefficient c_1e_1+ ,,, c_ne_n, right. So, because e_1 till e_n are the basis for U and x will have some of its own coefficients or its or its or its coordinates. So, this will also include $c_1e_1 + c_ke_k + c_{k+1}e_{k+1}$ till c_ne_n, where n was k + r where k we called as the dimension of kernel of f ok. Now y = f (x) right.

Now, how will these numbers or this vector transform under f? Well this is f. This entire thing c_1e_1+,,, c_ke_k + all the way till c_ne_n I will just expand it since this is a linear transformation I can write. So, similar to what we did yesterday in change of a basis $c_1f(e_1) + c_kf(e_k) + c_{k+1}f(e_{k+1})$ + c_nf(e_n) that is readable right or I know my writing here very very strange ok. Now what do we know about f with e_1 till e_k?

So, the assumption we made was that let e_1 till e_k be the basis for the kernel of f, right. So, these all numbers here will disappear it they just be 0 because all these appear in the kernel of f from e_1 till e_k. So, what we are left with is c_{k+1} f(e_{k+1})+ c_nf(e_n). So, here I am just looking at the image of this vector x ok. Now what is to be shown is that this f(e_{k+1}) f(

$e_n)$ that these vectors, this span the image of f right. So, we also in a way have to ensure that they are not linearly independent right.

So, to preserve the dimension, so let us let us assume for a moment that they are actually linearly independent right which means that. So, this vector representation $c_{k+1}f(e_{k+1})$ all the way it will c_nf(e_n) is actually equal to 0 ok. Now since f is linear I can write this as $c_{k+1}f(e_{k+1})$ which are the basis vectors $c_n e_n$ is also equal to 0, right. Now going by the definition of the kernel this would suggest that this particular vector is an element of the kernel of f.

Now what do we know about elements of kernel of f from starting from the definition when from over here that well they can any element can be written in terms of its basis e_1 till e_k. Now I claim that they can also write it in terms of e_{k+1} till e_n which means that there is some kind of a linear dependence between e_1 till e_k and e_{k+1} till e_n, ok.

Now, what are this e_1 till e_k e_{k+1} till e_n? So, these are these were the basis for R^n right which means a this actually is a contradiction that I can I cannot write it in this way, right. So, e_1 till e_k e_{k+1} till e_n are actually linearly are actually linearly independent right and therefore, this assumption that well can I write assuming that this vector can equate to 0 is a is a contradiction because e_1 till e_n are the basis of U or basis of R^n and thus supposed to be linearly independent.

So, what we can say is that this can never be 0 right and therefore, f (e_{k+1}) till f(e_n) are linearly independent right and thus form a basis for the image of a f, right and therefore, the dimension of this image of f is “r” which proves the claim of theorem right, so that that the dimension of the kernel plus the dimension of image is n where n is the number of columns of the A matrix.

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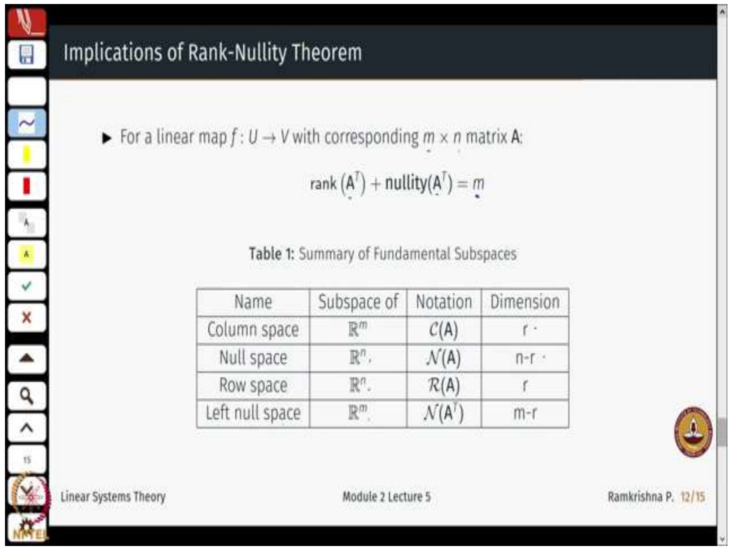

So, as a corollary if I look at in terms of the matrix A transpose, then it is it is obvious now to verify that the rank of A^T plus the nullity of A^T would be m, where m is now the number of columns in the A matrix as a summary. So, if I have again a matrix A of dimension m x n, the column space is a subspace of R^m. The null space is a subspace of R^n row space is a subspace of R^n, the left null space is a subset of R^m.

Say the column space of A would be of dimension r and n - r. So, together they will give me the dimension n. Similarly here I am looking at A transpose. So, r and m - r would give me m and just to summarize what we have so far.

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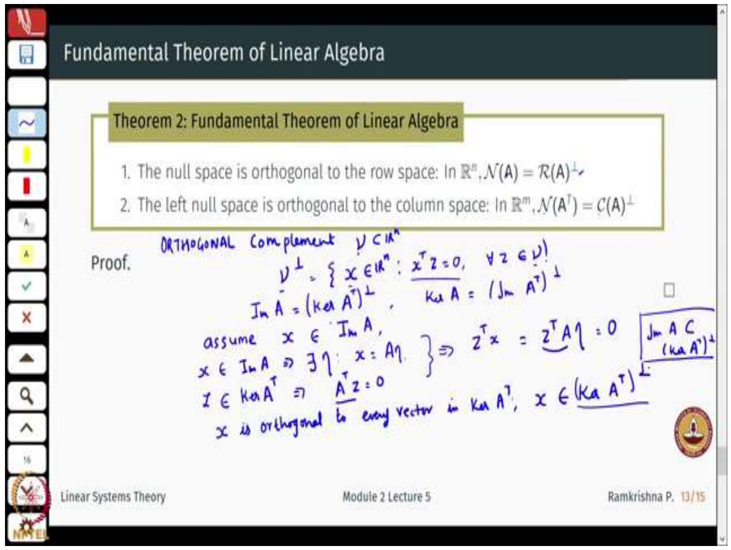

So, the last theorem that we will do in this in this module is something that relates sorry something that relates the null space to the row space and the null space of a transpose to the column space, ok. So, first let us quickly see what this symbol is. So, this symbol is usually called as the orthogonal complement.

So mathematically what it means? So, see given a vector space v, so it is orthogonal complement would be all x. So, so this V is a subspace of R^n. So, $V^\perp$ as we call it is all x in R^n such that $x^T z = 0$ for all z belonging to V, ok. So, so the orthogonal complement $V^\perp$ of v of a linear v is a linear subspace of R^n is a set of all vectors x which are orthogonal to every vector in v. Mathematically, it just means this right that $x^T z = 0$ 0, where x is in the $V^\perp$ and z comes from V ok.

Now, let us just see the proof of this. So, I just write it a little different. So, these two statements can be equivalently written as the image of A is the kernel of a transpose with the orthogonal complement. Similarly kernel of A is the image of A^T with the orthogonal complement, ok. So, this equality should be right between these two statements and these two, ok.

So, we will just prove proof one of this. So, let us assume that there is a vector x belonging to the image of w. So, which means that x is in image, sorry not w it is A ok. Assume that x belongs to the image of A which means x in image of A implies there exists some vector

"η"which is such that this x comes as a multiplied with this eta. Similarly if I take a vector z which comes from the kernel of A^T this would mean that $A^T z$ = 0.

Now, look at these two combined. So, if I take z^T and multiply it with x, I get the following right. So, I have $z^T A\eta$ ok. Now this is also necessarily equal to 0 because of this A^Tz =0 0 will also mean that $z^T A = 0$ ok. So, what does it mean right? So, x this vector is orthogonal to every vector in the kernel of A transpose ok. So, so we have not proved yet right. So, what this actually means that x is orthogonal to every vector in the kernel of A transpose which mathematically which means that x belongs to the kernel of A^T with the orthogonal complement.

So, x is orthogonal to every vector in kernel of A^T which means x belongs to this space here, right. Now where does x come from? X is the image of A. So, therefore if I take or the set of all these x, I can only say that let me write it in a box here that this image of the image of A is a subspace of the kernel of A^T the orthogonal complement. That is all I can say. I still cannot prove the equality, ok. So, to prove equality we will invoke the rank nullity theorem which we had seen in the in the previous slides, ok.

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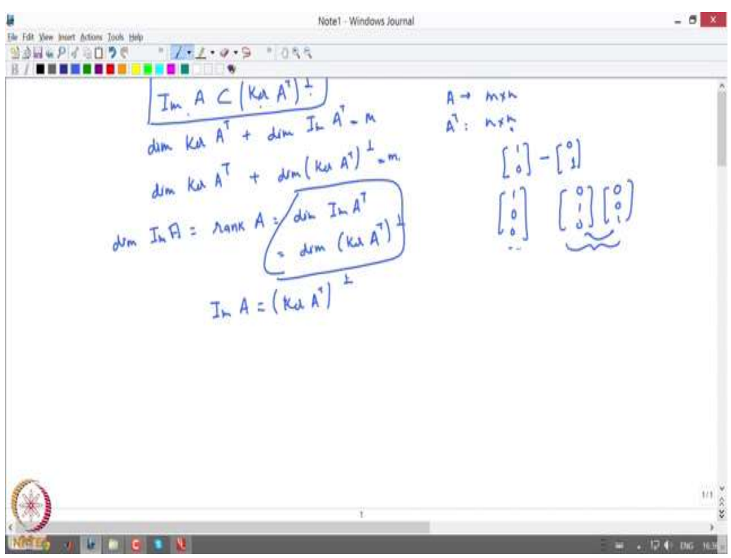

So, what we know so far is that the image of A is satisfies this the kernel of A^T with the orthogonal complement ok.

Now, to show that these are actually equal we need couple of more steps. So, first thing we know from the Rank Nullity Theorem is that dimension of kernel of A^T plus dimension of the image of A^T is m because here A is an m x n matrix and A^T will be n x m. So, the dimension was the number of columns. Second thing that is if you are writing the first time and it is easy to it should be clear, right. So, dimension of kernel (A^T) + dimension of the kernel(A^T) with its orthogonal complement is also equal to m say like m equal to 2. If the if the kernel of A^T looks like 1, 0 then this orthogonal complement will necessarily be 0 1 and hence the total dimension this is like 2.

Similarly, if I have kernel of a transpose in three dimensional space looks like this, then its perp or its orthogonal complement will have either $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or any linear combinations of these two. I can just swap these two and say this is the kernel of A^T, then this will be in its orthogonal complement. So, this is this is like intuitively, clear.

So, from these two things what we can conclude is the following that the dimension of image of A is also the rank of A is also equal to the dimension of image of A^T. This is also equal to the dimension of the kernel of A^T with the perp, right. So, so what I want? So, what I know so far is these two that the image of A is a subspace of the orthogonal complement of the kernel of A^T and additionally what I know is that these two spaces are such that the dimensions are equal, right and this is possible if and only if there is a strictly equality between the image of A and the kernel of A^T to this perp ok.

That is a that is that is one part of the proof and the second part of the proof is kind of straightforward from this as a as a little simple example.

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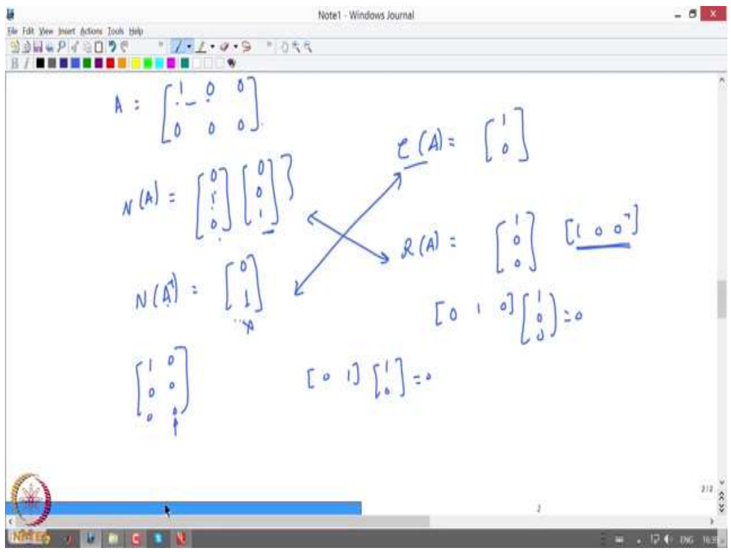

So, let us take a matrix A which looks like this it says $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ok. So, the null space of A would be the span of these two vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or any linear combination of these two elements. The null space of A^T would look like $[0 \quad 1]$ ok. So, you so what it means is that you take a this matrix A and multiply it by N(A), then you get 0. Similarly with the A^T where A transpose looks something like this $\begin{bmatrix} 10 \\ 00 \\ 00 \end{bmatrix}$ you may since here 0 you may as well get rid of this row and then if you, we can easily check that this is the is the null space of A^T, ok.

Similarly the column space C(A) is the line which passes through 1 0, and the row space is also a line which passes through (1, 0, 0) or $(1,0,0)^T$, ok. Now if you go back to the theorem statement, it says that N(A) is in the orthogonal complement of R(A), right. You take an element from N(A) and that it will essentially be in its in the orthogonal complement of R(A').

So, this is R(A) take any element from here from N(A) say take say $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ I get 0. Similarly with the second element also right also the definition here is that x^T z should be 0. Another important thing or interesting thing to note here is I take the perp and I take the perp again. I get back my original space. Similarly you can verify this also right that the null space of A^T is in the orthogonal complement of C(A), right. So, take this and multiply 0 1 from here and 1 0 from here will give you 0 right. So, so this the relation between these two holds and the relation between these two holds. So, that is a little illustration of a theorem which says the following, ok.

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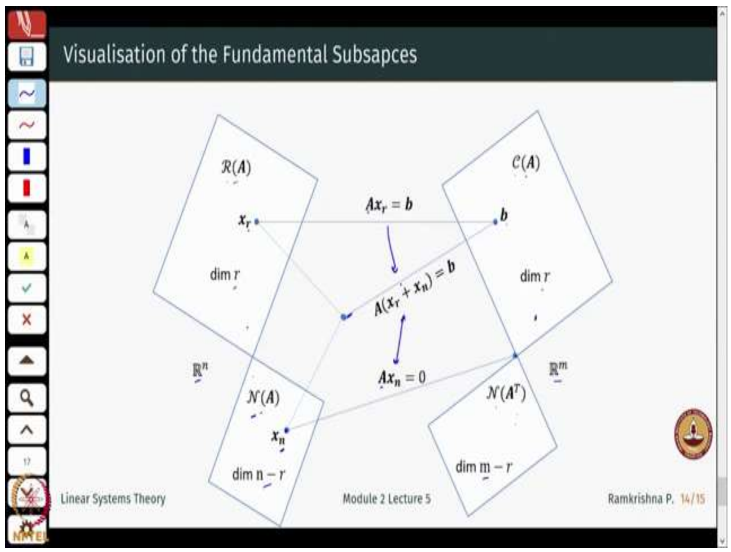

So, how do we visualize these four fundamental subspacs? We have the row space of A, the column space of A, the null space of A and the null space of A^T ok. So, what do we have as relations we know the following that well the dimension of the row space is r. It is a subspace of R^n, ok. So, row space it is a subspace of R^n, it is of dimension r, then I go to the null space of A which is also a subspace of R^n and of dimension n-r, ok.

So, this is the null space of dimension n-r and together the row space of A or the dimension of the row space of A plus the dimension of the null space of A is R^n the fundamental theorems which we had had earlier right. So, so this one the rank (A) + nullity(A)= n, ok. Now similarly I have the column space of A which is a subspace of R^m. I can denote it as the column space of A of dimension r is a subspace of R^m and similarly the null space of A^T is a subspace of R^m now and its of our dimension m-r, right. So, this is where this now. So, if what does this mean? So, so the null space of A means the following. So, if I take A vector x n which belongs to the null space of A, the map A will take it to the origin, ok.

Similarly, X r via this transformation A will go into the column space of A of dimension r and so on. So, what we also know is the following that the null space of A and the row space of A are orthogonal to each other, ok. The null space of A and the row space of A are orthogonal subspaces to each other and together they will again be of dimension n. Similarly with the null space of A^T and the column space of A are also orthogonal to each other. So, this the dimension of the of this plus this would be would be m. So, that is a little pictorial pictorical representation of this and also for what is this map show. So, if I take an element X r from the row space and element X n from the null space, it will still map to here.

So, which essentially means that AX n is 0 so, this is rewriting these two equations together ok. So, that is a little visualization of the fundamental subspaces given A matrix representation of a transformation of which is essentially of dimension m x n, ok.

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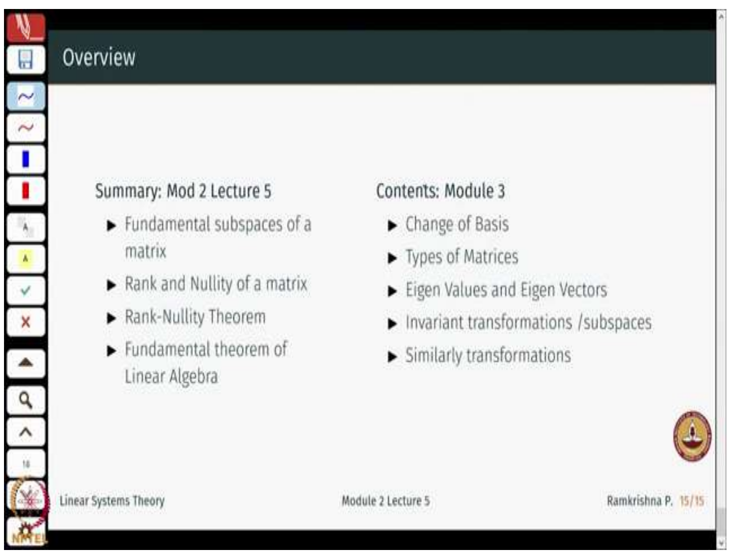

So, what we did here is in this in this lecture or is to define fundamental subspaces of A matrix. We had the concepts of rank and nullity of A matrix. We did a proof of the Rank Nullity Theorem and also the Fundamental Theorem of any algebra together with its geometric visualization.

So, the next module we will talk of change of basis. We will do lots of things on properties of matrices, Eigen values, Eigen vectors, we will also look at invariant transformation subspaces and also end with similarity transformation and how we and how these transformations will make matrix computations a little easy for this so, that that is coming up in module 3 or the lectures of B.

Thank you.