**Multirate Digital Signal Processing**
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**Lecture – 05 (Part-1)**
**Discrete Time Processing of Continuous Time Signal – Part1**

We begin lecture 5, we will as always do a quick summary of the key points from lecture 4, this class we will look at a mathematical framework for sampling and reconstruction. Again this is a useful tool for us as we move between the continuous time and the discrete time domains, we will look at a specific example of discrete time processing of continuous time signals as you probably have studied in DSP.

There is also a continuous time processing of discrete time signals because you are able to go back and forth between the two domains and we will look at that and as we mentioned having given this framework for sampling and reconstruction, we are now moving into the topic on discrete time signals, so just to sort of change the pace a little bit, I thought that let us take a look at the examples that we which I mentioned in the last class.

And we will just do a quick run through of the; of these examples and just as an illustrative exercise okay.

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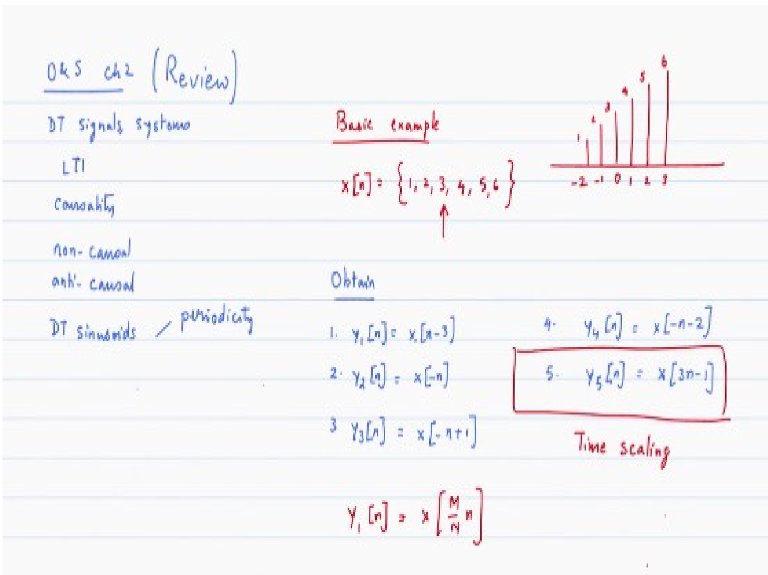


So, I think I will make a small change, there is one more; I will add one more sample to the; okay, so 6 sample sequence, so if you were to think of this as a sketch of the sequence at time 0, you have a sample = 1, time 1, 2, 3, 4, 5, 6 okay, so 1, 2, 3, 4, 5, is the time index, am i right; No, we are told that the sample number 3 is the 0 index, so the 0 index actually happens here, so this is sample number 3; 0, 1, 2, 3, -1, -2.

Please make sure that the arrow is a key indicator, so the height of these are 1, 2, 3, 4, 5, 6 okay that is the; so let me just quickly write down the answers just for discussion and sort of refresh people's memory.

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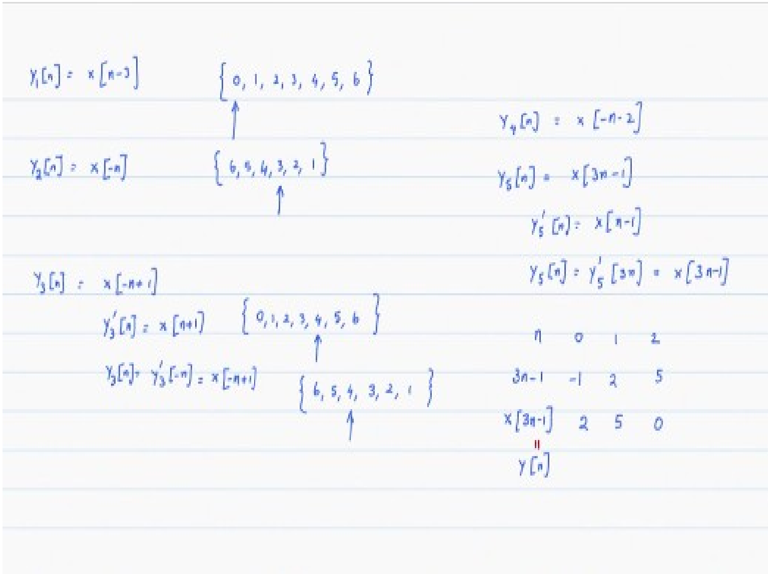


So, y1 of n equals x of n – 3, so if I were to write this down, I would not write down the expression, just write down the sequence 1, 2, 3, 4, 5, 6, the origin has moved 3 places to the left, so the sequence if you were to go look at it, it had 0 valued samples here, now it moved from 0 to -3; -3 was where the origin now lies, so that is the sequence for a sequence number 1, so y2 of n was x of - n basically, a time reversal so that would mean that the sequence will now go backwards 6, 5, 4, 3, 2, 1, the origin remains as before, so basically remains at 3.

Then the interesting cases come at 3, 4 and 5 x of - n +1, again there are several ways to did, this is the shift operation, there is a time reversal operation as we have as you always emphasize in the DSP course always easier to do the shift first because otherwise the shift becomes a little bit tricky for us to handle, so the steps that I would suggest is write an intermediate sequence y3 dash of n which is x of n + 1, so you have taken care of the shift.

So, this will be a sequence n + 1 means, you are advancing the operation by 1 so, 0, 1, 2, 3, 4, 5, 6, okay and zeroes be outside of that range basically, the advance operation says that the entire sequence moves to the left, so my the origin now becomes 4 from originally, it was a 3, the whole sequence has moved by 1 unit to the left and then the next step you get is y3 of n is y3 dash of – n, so that you can verify is actually x of - n +1.

But it is much easier to flip y3 of; y3 dash of n, so basically the sequence would be 6, 5, 4, 3, 2, 1 and with 4 being the origin, so basically you time reverse the sequence with that okay and y4 of n is x of - n -2, again I will assume that this is a just a direct extension of the previous case you shift 2 that means you shift the sequence to the right by 2, you are delaying by 2 units of time and then doing the time reversal should be straightforward, okay.

Y5 of n, this was x of 3n - 1 okay and the way to do it would be define an intermediate sequence which gives you the shift x of n - 1 and then you can get y5 of n with a time scaling of y5 prime of n; y5 prime of 3n which will come out to be x of 3n – 1, it turns out that if you do one operation per step, it is much easier for us to actually keep track of the; so x of n – 1, I am assuming you know how to generate.

Then the key step is the next step is the scaling of time, so rather than work with the sequence, I thought it is helpful to form a small table, so if you say n = 0, 1, 2, the argument of the function that we are trying to compute with the time scaling and the shift is 3n – 1, so this would be -1, 2 and 5 and x of 3n – 1, x of -1 is 2, x of 2 is 5 and anything outside that range will go to 0 and this becomes y of n, okay.

So, this is the that becomes y of n now, the I am assuming that these are fairly rudimentary examples which you are quite comfortable with, what I would like to do is just mention and we will come back to look at several examples both in the class and in the tutorials.

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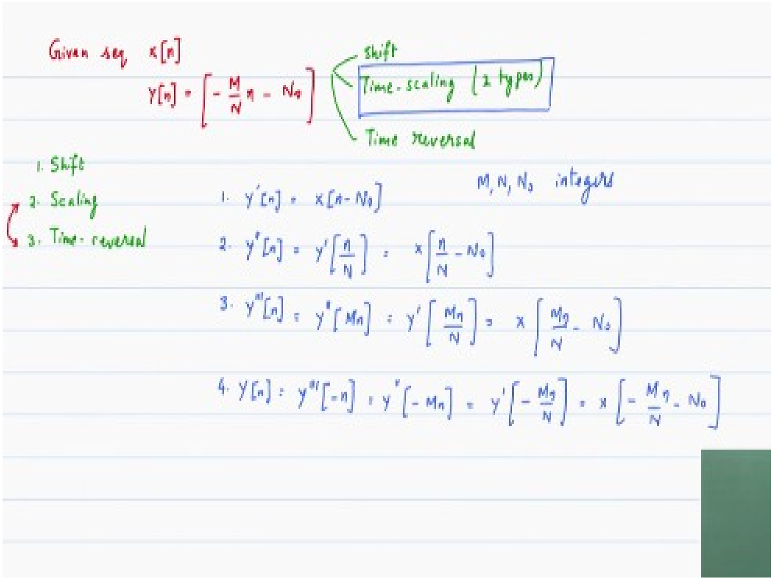


So, this is the general case, if I am given a sequence x of n and I am required to obtain y of n equal to –M by N times n – N naught, okay, notice that this particular transformation has got a shift, it has a time scaling, it has got a time scaling both in terms of compression and expansion, so there are actually time scaling of 2 kinds, okay, 2 types both types of time scaling are present and the third one is the time reversal.

From the point of view of multi rate signal processing this is of a lot of interest to us but it usually happens in the context of shifting and occasionally, time reversal as well, so this is a form of transformation that we are very much interested in the study of multi rate signal processing, so here is the general guideline the sequence in which it is easiest to do is to do the shift first then the scaling.

And then finally, the last stage would be time reversal, it is easiest if you do that minimizes any possibility of you know, careless mistakes of course, these 2 can be interchanged without too much difficulty because they are processes that do not really you can do it in either sequence but the sequence of shifting scaling and time reversal is a very, very robust way you will not make mistakes there.

So, these steps that we would like we would do here, so first step would be define y prime of n x of n – N naught, then the second step is to do the sampling rate expansion, y2 dash of n is y dash of n by M, okay so as we will define this process, this is the process of expanding the sampling rate and this will also end up inserting a lot of zeros, you do not lose information in this step, so always better to do this step first.

I think we divided by N, I am sorry, this should be N okay, so sampling rate expansion comes next and of course, you can substitute and verify that this will give you a signal which is x of n by N – N naught, by the way M, N, N naught are all integers that is the assumption in terms of the basic transformations that we are looking at so, y triple prime of n is y double prime of M n, this is the sampling rate reduction, you retain one out of M samples.

This corresponds to y dash of M n by N that corresponds to x of M n by N – N naught, okay and the last step is a time reversal y of n is y triple prime of – n, I have done the scaling, so the last step will be the and this will give me y double prime of – M n that will give me y prime of – M n by N and that will give me x of – M n by N – N naught, okay and of course, there are several simple examples that we can try out but this is the most general case and good for us to keep that picture in mind, okay.

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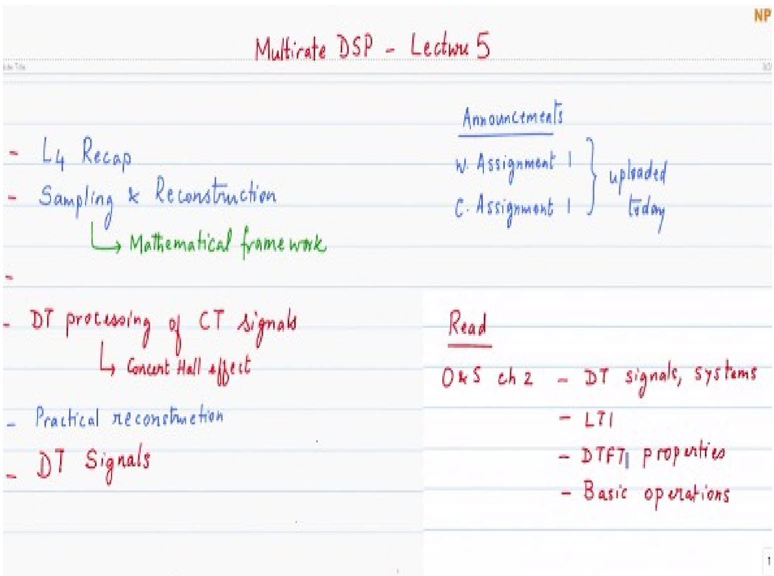


So discrete time signals do take a look at Oppenheim and Schafer chapter 2, basic properties, basic operations and also built into it the understanding of the DTFT and its property, so again that would be a quick review, assignment number; written assignments have been uploaded, do take a look at Moodle, if there is a difficulty in accessing Moodle, please do indicate to the TAs or indicated on the sign on the attendance sheet, okay.

Now, we would like to sort of revisit our sampling and reconstruction one more time again to build on the mathematical framework and let me just take that as the next step that we would like to do okay.

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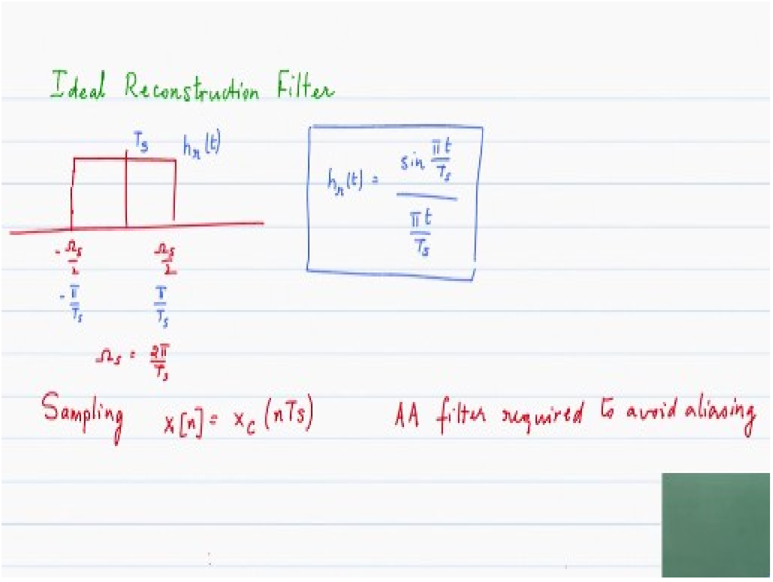


So, the sampling process is given by in the expression in the lower half of the screen basically, you are doing periodic sampling and there is an understanding that we must ensure that there is an anti-aliasing filter to be in order to minimize the risk of aliasing; to avoid aliasing completely and of course, the ideal reconstruction filter which we have been talking about is a brick wall filter from - omega s by 2 to omega s by 2.

We can also relate it to the sampling period – pi by Ts to pi by Ts and the actual expression is a sinc function which we derived in the last class okay, now I would like you to follow along on one of the aspects of the mathematical elements.

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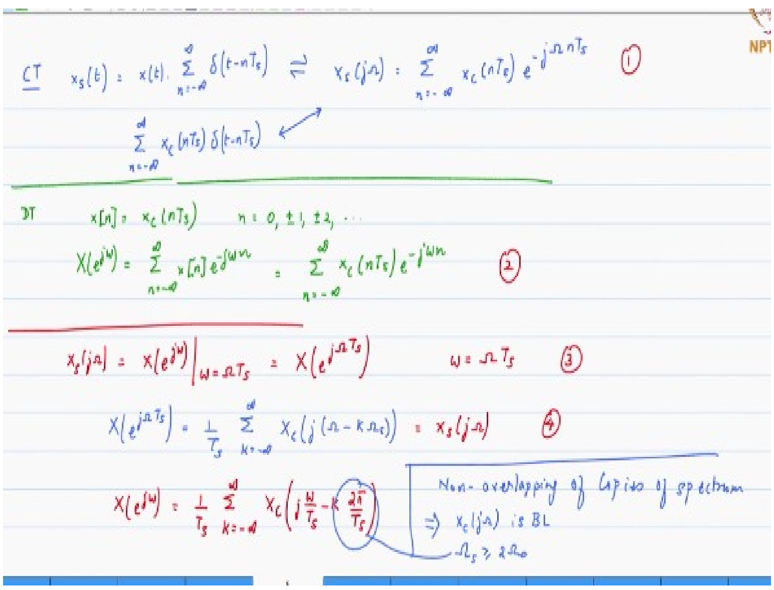

So, let me first write down the continuous time portion and then relate it to the discrete time xs of t, the sample signal, this is x of t multiplied by the impulse train, n equal to - infinity to infinity, the Dirac delta t – n Ts and this has a Fourier transform which we have derived which is xs of j omega which is given by summation n equal to - infinity to infinity basically, if I were to write down the x; do the Fourier transform of this.

You will find that when I apply the integral to the Dirac delta what the expression that I will get is summation n equal to - infinity to infinity xc of n Ts, right that will be the value of the function and e power – j omega n Ts, the integral will leave you behind with just the summation, okay so basically I am taking the Fourier transform of xs of t, okay which is the expression on the right hand side, okay.

We have looked at it from a different angle where we said you are multiplying, we multiplied the Fourier transform; we convolve the Fourier transforms of x of t and the Dirac delta train, this is another perspective of it okay. So, basically this comes directly from the following observation that this summation is nothing but n equal to - infinity to infinity xc of n Ts delta of t – n Ts basically, the sampling property of the Dirac delta and then if I want to take the Fourier transform.

I have to take integral of xs of t and then the integral basically because of the property of the delta function gives you the expression on the right hand side okay. So that is one piece of information now, moving over to the discrete time domain x of n is xc of n Ts, okay and of course, n is the periodic sampling where n equal to 0 + -1, + - 2 and so on, for the discrete time sequence.

We define the discrete time Fourier transform xe of j omega as the following; summation n equal to - infinity to infinity x of n e power – j omega n this can also be written as summation n equal to - infinity to infinity xc of n Ts e power – j omega n. I have 2 equations which are very interesting and important, one is equation number 1, second is equation number 2, both are Fourier transform representations.

They are Fourier transform representations, one in the continuous time domain and another one in the discrete time domain of the same signal because the sample sequence xc of n Ts when represented in the continuous time domain gives me Fourier transform given by equation 1. The discrete time sequence represented by the discrete time Fourier transform gives me equation number 2, so they are basically one and the same representation.

So the next step is a very important link between the continuous time and the discrete time which basically, says comparing 1 and 2, we write down the following xs of j omega has to be the same as X e of j omega; x e of j omega evaluated at omega equal to omega Ts that is the relationship between the continuous time frequency and the discrete time frequency. So, this is in the discrete time domain, if you have X e of j omega you are equally.

It is equally valid to write it as x e of j omega Ts, right because instead of the; we can use the relationship omega equal to omega times Ts; upper case omega times Ts, so using that expression so basically, comparing 1 and 3, we write down the last of the equations again, you may wonder why are we spending amount of time on this, it actually is a very important element this link between the Fourier representation of the continuous time sequence; continuous time signal and the discrete time sequence.

So, X e of j omega Ts can be written as comparing 1 and 3 as 1 over Ts summation k equal to - infinity to infinity because X of j omega we already have derived at to be equal to this expression, so this would be Xc of j times omega –k omega s, this is an independent basically, another form of Xs of j omega, so this is equal to Xs of j omega, this is again derived in the earlier lecture, so this also tells us that we can also write the same equation; equation number 4 in terms of the continue; discrete time omega which says Xe of j omega.

How do I visualize it? This is the representation; periodic represent; periodic repetitions of the continuous time frequency with the appropriate scaling of the frequency axis, so it is 1 over Ts summation k equal to - infinity to infinity Xc of j omega by Ts - k times 2pi by Ts, okay, so basically what we are saying is the signal if we view it in the continuous time, gives you a certain insight and representation, you view it in the discrete time it gives you certain insights and representation.

And when we are doing this process of sampling and reconstruction, we are going back and forth and very useful and helpful for us to have a clear picture of what is happening, okay. So, we do have copies of the spectrum which are given in the last equation and the condition that you do not have overlap of the spectra; the conditions for no overlap, so basically if you want to impose the condition that non overlapping of the copies of spectrum, this if you want to impose this condition, this would basically boil down to you can look at it in the; in terms of the pictorial representation.

Or you can again go back to how this expression actually came about, the first thing would be is Xc of j omega must be band limited, right if it was not band limited anytime you have multiple copies they are going to overlap and they also must ensure omega s is greater than or equal to 2 times omega naught , which is the band limit, okay and again this omega s is nothing but this 2 pi over Ts, so once you have ensured that your shift is larger than the; than 2 times the bandwidth of the signal then that is again goes back to the Nyquist criterion.

And then, so we can look at the Nyquist criterion in the continuous time domain which is very straight forward for us but when we look at it in the discrete time also, there is an insight that we can talk about in terms of make sure that you have sampled at a sufficiently high rate and that is very important for us to have this picture as well now, this relationships that we have derived on this page will become more familiar and useful.

Once we start talking about the discrete time processing of continuous time signals which is what I would like to do in as part of today's discussion okay. But before that always interesting for us to look at some interesting examples and so what I would like to do is share with you an insight of one of the very powerful applications of DSP as a baseline and in particular multi rate signal processing which you probably are using in everyday life but may not have associated it with that particular application, okay.

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So, the application itself is in the context of music, so it is in the music and very specifically in playback, okay which most of you I see today even as I was coming to the lecture, it is a lot of people you know have music plugged into their ears while they are coming to class, so you know you probably use it hardly a few minutes back. Now, one of the things that that I am sure you enjoy you know using in on your music players is the graphic equalizer.

You set it to the way you like to hear the music and one of the options that you will have in high end players is to give you certain sound effects, it can give you the sound effect of a jazz setting or you can have a rock setting or you can have a concert hall setting.

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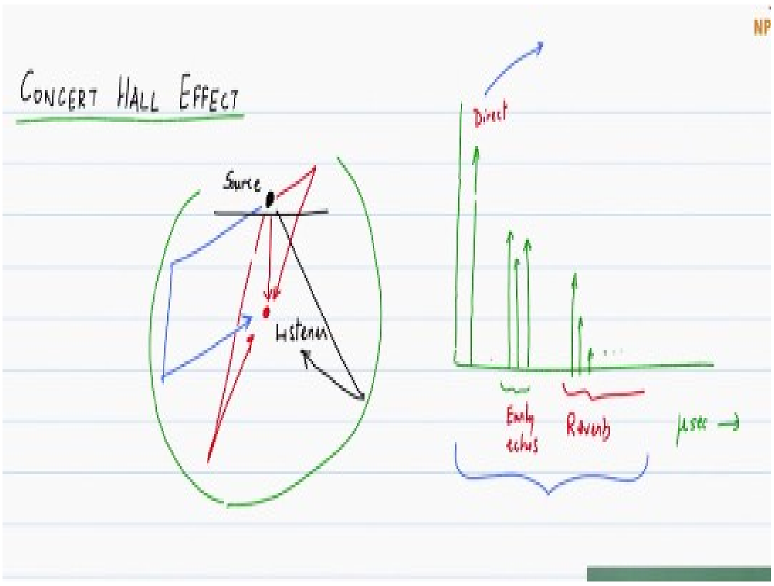


So, I would like to just sort of take you to a concert hall, okay, let us say that you wanted to look at the concert hall setting, okay, I do not know how many of you know that in Chennai we have an absolutely wonderful concert hall called music academy, if you do get to hear it you can sit anywhere in the auditorium, you do not need mics, you can hear it without amplification and that is the beauty of it.

But it is a auditorium that has this sort of an effect, it is a sort of a circular auditorium and the best seats are somewhere in the front, it is somewhere in the sort of the; not exact geometric centre but it is somewhere close to the front, okay, now if you do analysis of this type of a concert hall and see why is it that the best seats are somewhere so basically, if you were to think of it, I will just draw it okay, something of that type, okay.

Now, why is it that the best seats are there, it sounds basically the acoustics is designed such that if you are sitting here, you get some sort of an optimal combination of the different echoes of the signal, copies of the signal, so basically what you get is; one is the clarity of the signal, second one is the feeling of depth that you are in an auditorium because the you do get certain and people have done analysis of this; what produces this type of effect.

So, basically from a signal processing viewpoint, there is a direct path basically, you do get a signal that is coming directly from the source of the music or but there is also multiple copies so, if you were to characterize the received signal, so there is a direct path which arrives okay, this is in microseconds, so there is a direct path and there are several other copies of the signal and what is and basically, if you look at the acoustics, what is the makes it.

What makes a concert hall very good, acoustically excellent is that there are certain copies that come because of what we call, they call as early reflections okay some which has coming from nearby, from where the sound source. So, these echoes are referred to; so this is the direct path, this is the direct path, then there are early echoes and then there are the reverberations, so basically these are slightly longer term echoes and eventually this dies down, okay these are called the reverberation components.

Now, those auditoriums where you get absolutely the perfectly designed and or you know created environment where you have optimal combination of these echoes and reverberation and particularly in locations where you sit and therefore you enjoy the best of music. So but all you have is a CD player and set of headphones and now how do we create this effect so basically, instead of playing only the direct path see if you played only this that will be just a simple CD player microphone.

But if you have equalizer setting which says okay concert hall then you basically, must not play back only the direct one but you must play back the composite signal that we have to develop okay.

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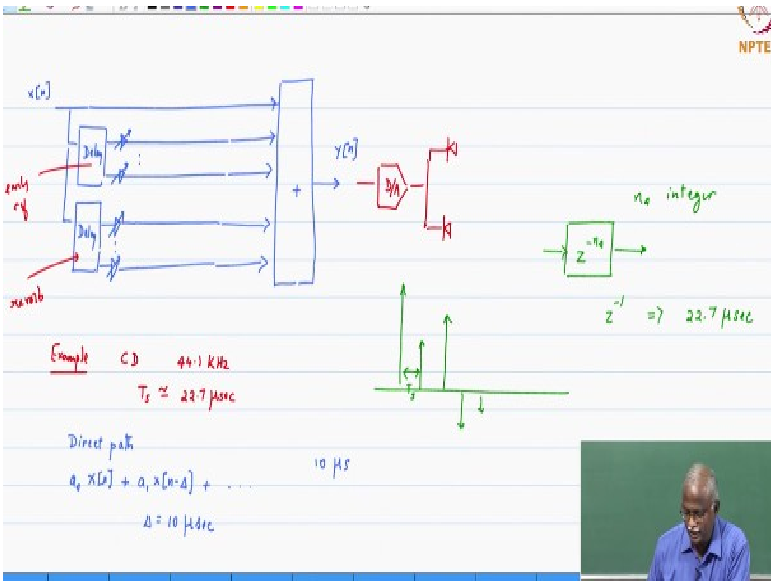

So, the multirate signal processing comes into play when we say that okay, I take the sample signal x of n, there is a direct path and then there is a bunch of delays; different delays from which we will get the early echoes in a hall, you do not have to create this, this comes automatically but when you want to do it on your CD player, you have to create that and then you have a second set of delays.

These are different delay components which you then; these correspond to the reverb signals dot, dot, dot several of them you have these reverb combinations. And all of these if you combine together with suitable phase and lots of the things are there in when you handle acoustics, you must combine them in a suitable fashion, so I will just put a big plus sign here, this produces for you.

y of n which will be similar to what you would have gotten had you sampled the signal that is being received at this red dot, all you have is a direct path but you have artificially created these multiple things. These are the early reflections, and these are the reverb components, okay and this is what will then get passed to a D to A converter and then played back through your stereo headphones of course, you can create some differential signal.

You can create a left ear left speaker signal, right speaker signals differently from using the same combination, so but ultimately the basic principle is this, okay. So, you can actually create any sort of acoustic effect, concert hall rock band place where is a; so basically, you will set the settings, you will set the frequencies maybe you will boost the basses up so but the in effect what we are producing is a very interesting combination of these components okay.

Now, where is the multirate part, I did not see any multi rate, it is a samples which you delayed and multiplied and so that is what will actually happen. If you end up in a situation, so let us take for example that you had a CD's, you are playing back CD music, so CD music is at 44.1 kilohertz sampling rate, sampling period is approximately 22.7 microseconds, the samples of your music are coming at a sample spacing of 22.7 microseconds okay, so some samples are coming, this spacing is your Ts which is 22.7 microseconds.

Now, anything that I do in the form of a delay, if I want to implement a delay in digital domain, it will be z power – n0, where n0 is an integer okay, so I can only get multiples of basically the smallest delay that I can get is z power -1 and if I delay it by one sample period, this would correspond to 22.7 microseconds okay but in the construction of the; of this concert hall effect, you cannot restrict yourself to say no, no give me only delays in multiples of 22.7.

Sometimes there are delay; the signals that are coming at different delays, so as an example supposing I want to combine direct path okay, so that would be x of n, let us call that as a0, some scale factor plus a1 times x of n – delta, where delta is 10 microseconds, okay, suppose, I just want to combine this and of course, in a concert hall effect there will be several combinations and so I am just taking the case of 1.

So, now this is a case where you can say well too bad, either you; so if let us say you want to get 10 microseconds, this is total is 22.7 microseconds, so we say well you know you can either choose to either not delayed or delayed by one unit which will move it to a 22.7 microsecond which okay both of those are not acceptable, so then we say that okay let us see if we can do something simple.

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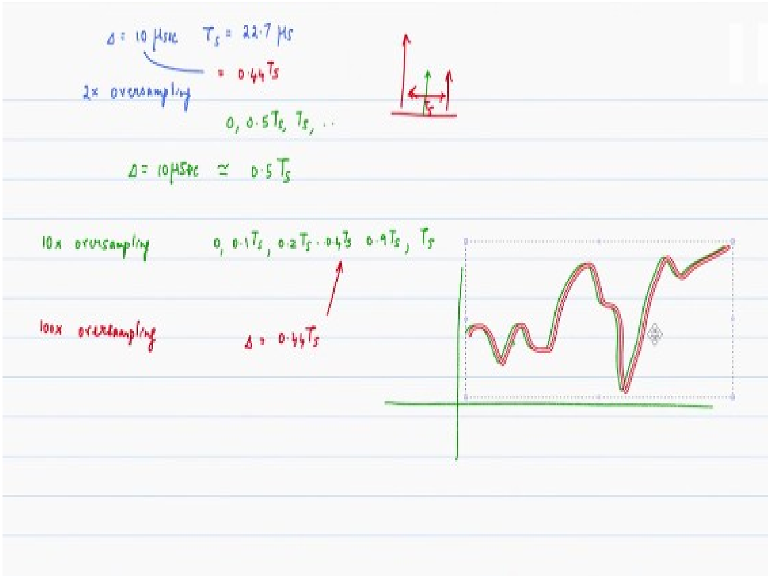


And supposing, so we say that okay I do 2 times over sampling okay, so the problem statement; I want delta = 10 microseconds, my Ts is 22.7 microseconds so basically, this is approximately equal to .44 Ts, so if I now do 2x over sampling that means, I create this is the original signal, this is a sample spacing of Ts and I know how to increase the sampling rate, so I introduced a new sampling point here at Ts by 2; 2x over sampling means that I can reduce it.

So that means I will get samples at 0, .5 Ts, Ts and so on every .5 Ts I will get so basically, I have increase the sampling rate by a factor of 2, I want to get .44 Ts delay, I cannot get .44, the best that I can do is this can be approximated, so delta = 10 microseconds can be approximated as 1 by 2 or .5 Ts, instead of .44 Ts, I got .5 Ts, okay you say okay that is okay but not quite all right because I am doing very, very high end, I want to reproduce, someone to reproduce a concert hall very precisely.

So then we say okay. I will go to a 10x over sampling, 10 x over sampling says that yes, now I can introduce delays in my signal which is of the form of 0, .1 Ts, .2 Ts all the way to .9 Ts and then Ts, so basically in between 2 sampling points, you have gotten 9 other sampling points, so that is the 10x over sampling okay and somewhere here is .4 Ts and of course, .44 means that the closest one will be .4 Ts, I can introduce a signal of that form, okay.

And of course, if you are very, very picky you can even say okay go to 100 x over sampling not a problem, multi rate signal processing can do the over sampling fairly easily and which means that you can get delta = .44 Ts, if you want it very, very precisely okay, so again it is an interesting way to look at the problem, it says that I want to recreate these concert hall effect which means that I must recreate with several signals of several delays which I need to combine.

I cannot that depends on the acoustical properties and not necessarily on my sampling period and if I want to achieve something that is very close to what I want to implement, what in the ideal case then multi rate signal processing is a very, very useful tool which I think we can use and exploit and take several lots of very good advantages from that. Any questions on what we have said, why multi rate even comes into play?

**“Professor – student conversation starts”** Well you can run it if you have done the over sampling unless you have done the down sampling all before doing it you would have to run it at a higher rate okay, okay, wait, wait that is a very, very good point so basically, what we are saying is; you create so it is like this easiest if I show it to you in a continuous time signal. So, I have some something which is like a music signal.

Now, I must what we are trying to do is; so the important thing is that you have been able to create the signal with the appropriate delay, the bandwidth has not expanded so though, I have gone to a higher over sampling factor, the bandwidth of the signal; only introduced a delay, so basically it is an acoustic effect, I have not changed the bandwidth of the signal, so I am perfectly able to think of this as a signal for which I can go back to the original sampling rate.

So, I may have gone 100x over sampling I might have done but I can still go back to the original sampling rate because without losing the quality of signal and I would not lose the effect of the acoustic effect because this has already been produced now, think of this as the sum of the green signal and the red signal, take the sum of it and then sample it which means, now you have a signal which has the acoustic effect built in and is a tick at the right sampling rate.

Of course, you can need not go down to the lower sampling rate because over sampling actually helps your D to A conversion, so you can actually say that anyway I want to do over sampling to do a good D to A conversion but in principle this is what we are trying to do so, you have actually introduced the effect using the multi rate part you can go back to the original sampling rate if you want to or you can retain it and then go to the detailed process at the higher sampling rate.

But a very good question yeah because we would we will not lose the effect of it because see the effect of the acoustic processing that we have done basically says that I have created the effect of another echo and the combination of these 2 has already been incorporated, so even if I go down to the lower sampling rate I will not use lose the effect of the signal, okay. **“Professor – student conversation ends” A**ny other question?