Multirate Digital Signal Processing
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Lecture – 33 (Part-3)
Introduction to Redundancy and its Implementation in Multi-Rate Framework

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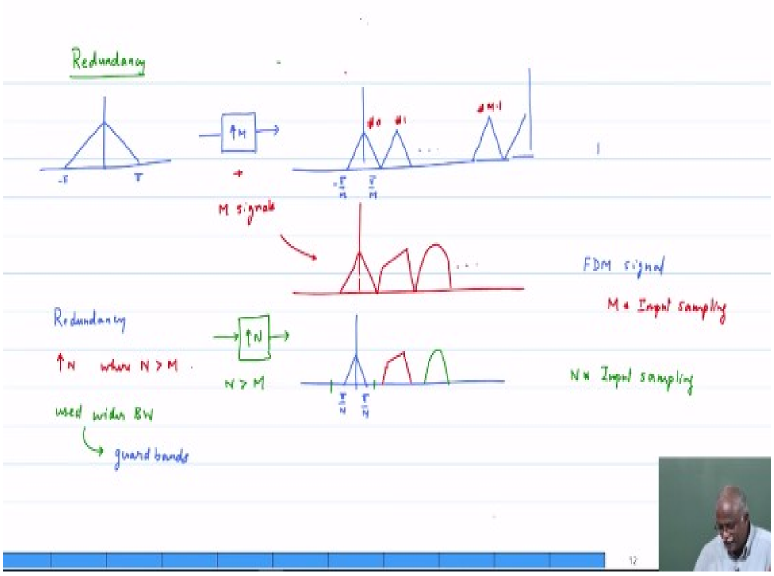

But for now, the most important task at hand is to introduce the notion of redundancy, this is a very, very important concept in engineering, very important concept in communications now, error control coding is considered a form of redundancy, can you explain to me what type of redundancy you are introducing in error control coding? It is a form of redundancy, do you agree, yes what is the redundancy?

You are transmitting some additional bits, so that the information bits can be protected okay, so here is so redundancy can come in different ways and here is where I want you to sort of think of redundancy in a multi rate context and today, we will be talking about this multiple times and we will come back and look at it and visualize it again but let me just sort of give you a flavour for what the element that we are trying to introduce here.

Now, redundancy in this notion supposing, I had a signal spectrum which was went from – pi to pi okay and if I up sampled by a factor of m, I know that you will; we know that you will get a signal of the form which is contained between - pi over m to pi over m and there will be many copies of this signal, okay so this is copy number 1, copy 0, this copy number 1, this is copy number m – 1, okay.

And the limits would be - pi over m to pi over m that is what they would be and let if and if they were m signals, this would be the conventional trans multiplexer where you could think of 1 signal, another signal, a third signal and so on okay, so that would be the way that you would achieve, so if you use the up sampling by a factor of m and we combined it with m signals then this is what we would get, okay.

There is no gaps in the spectrum so basically, you use the information or the resources optimally, now very, very important answer, think about this question, I have only m signals but I want to introduce this notion, I want to up sample by n where n is > m, okay, what am I trying to do; I have only m signals, so which means that my spectrum will now be not pi over m, it will be pi over n, right and it will be narrower, pi over n.

So, which means that every one of these representations okay, can actually be visualized as narrower than the band, so in other words I have separated out the signal at least conceptually okay because when I up sample by a factor of n, I have made the spectrum more compact but I have only m signals, so potentially I mean in the total bandwidth of 2 pi, I will have some gaps okay, so I want you to start thinking along those lines.

So, now what have we done in the same spectrum 2 pi, I have now fitted m signals but because I up sampled by a factor of m, I able to create some gaps in the spectrum okay, so the notion of redundancy in our context, so there is the notion of redundancy that means, some additional resources, so that we can process our desired signal in a more efficient manner, so redundancy, so in our case the redundancy is coming by up sampling by a factor of n, where n is larger than m.

But where did I use more resources, it is still sitting between 0 and 2pi, where did I use more resources, it is a higher sampling frequency, sample rate is higher that means bandwidth was higher, yes 2pi is the discrete time frequency but actually, it represents so in other words, we have effectively used wider bandwidth which means that effectively for m’s if only m channels had to be transmitted, we can create guard bands, okay.

So, we find that the notion of redundancy was what multi carrier modulation was using and we now say that okay yeah, now I can think of it as where do I allow more resources, I have to allow more bandwidth, where is the bandwidth going to come in, if I increase the sampling rate more than I need to okay, so in other words the when I have m signals and up sample by a factor of m, then the FDM signal, so in this case the FDM signal frequency division multiplexed signal.

This is has the minimum sampling rate that is required so basically, it is the output signal is running at m times the input sampling rate, so that is at without any redundancy but in this case, what we are seeing is; it is going to run at n times the input sampling rate which basically means and I am going to utilize more resources and therefore, the notion. So, now how do we create these gaps, how do we operate it?

This is a conceptual present; presenting the concept but now, the key challenge is how do we build this and leverage it for the as we are; as we move forward okay, so if you are comfortable with this then, we want to quickly take this particular thought process forward. So, the first step in a multi-carrier modulation will be; yes, **"Professor – student conversation starts"** yeah I am sorry, so if you think of it that I will throw away n - 1 copies.

After the up sampling, if you see that in this first case, when I up sample by m, I have m - 1 copy which I throw away m – 1, I keep 1, throw away m – 1, likewise each of those gaps that were created is filled in by another signal with the same amount of bandwidth now, what we are doing is; we are doing, we have only m signals so in other words, if I divide the original signal as per the bandwidth 2pi over m, I have to fit in one copy.

But that one has been more compressed in the next channel, I want to in fit in one more, so like I said the center frequencies and other things are I mean this is more to just start you thinking as to where the redundancy comes in, so in other words in a multi rate system, the minute I up sample by a factor n, which is larger than m somewhere redundancy has kept in, so I just wanted to and one way that you could visualize the redundancy is like this.

But this is not exactly, the way we will exploit redundancy in fact that is what the significant part of today's lecture is actually going to be the understanding of a redundancy and the implementation of redundancy okay. So, basically, the up sampling by a factor of n gives you room for adding n signals but you are going to keep m signals which is M < n so therefore, there is potential for gaps okay. **"Professor – student conversation ends".**

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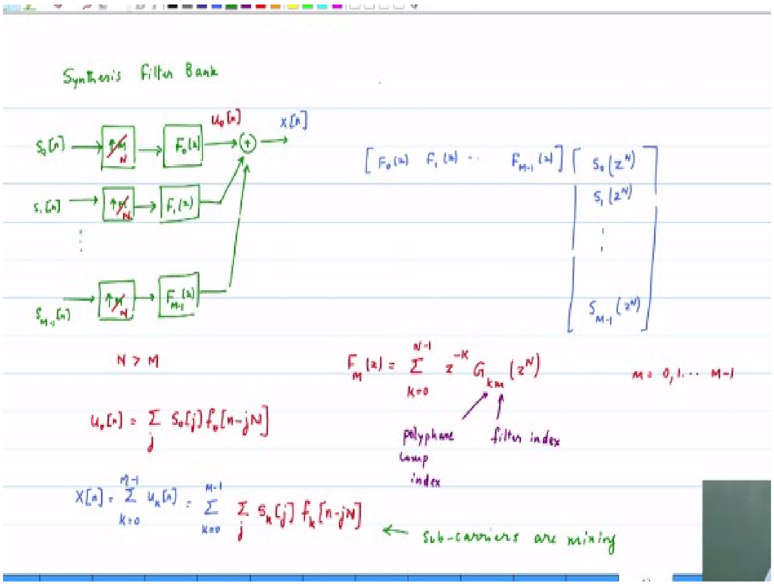

So, the first step in a multi-carrier system is the synthesis filter bank, the step of synthesis okay, so synthesis filter bank basically says, I have m signals up sample by a factor of m, this is a traditional way to do that pass through a filter f0 of z and add it together, so this would be s0 of n, okay, the second channel s1 of n up sampled by a factor of m passed through another filter f1 of z similarly, all the m – 1 signals up sampled by m, s of m - 1 of n, f of m - 1 of z, okay.

So, this is the conventional way that you would combine a in a trans multiplexer type structure, this is what we would do but the first thing that we are going to say is okay, this is not m any more it is n, up sample by a factor of n, where n is > m, so that is our intent as far as the redundancy part is concerned okay, now the way we represent a typically, a synthesis stage is as follows.

Because we want to combine it to produce a single output, so the way this is represented is you write it as a row vector f0 of z, f1 of z, fm - 1 of z and then a column vector which is the inputs which will get combined in this case, it will be s0; let me just write it in the context of the up sample by a factor of n, this be s0 z power n, s1 z power n and last one would be z m – 1 z power n okay.

Now, if I want to do any optimal implementation of the up sampling, I should use the noble identities and the noble identities require me to split it into polyphase components which are dependent on the up sampling factor, so which means that now when I write the polyphase component matrix, I have to write it with n polyphase components and not m, so if I were to split the polyphase components in the following fashion.

I am going to introduce a very specific notation, please note that and then we will just, so these filters f0 of z, I want to do again because eventually, I want synthesis filter bank requires delays, so I am going to use type 1 polyphase decomposition, so type 1 polyphase decomposition says summation k = 0 to n – 1, notice it is now already in terms of n, no longer m and the; it will be z power - k Gkm z power n okay.

Now, 2 indices are there basically, let me just do this, fm of z, m; where m can be 0, 1, through m - 1 okay that is the filter index, this the second one represents the filter index, so the filter index is say I am talking about f0 or f1 or f2 or fm – 1, so that is the filter index and the second one is the polyphase component index, so polyphase component index. The polyphase component index will go up to n - 1 poly phase component index, okay.

So, the range of k is from 0 to n – 1, the range of m is between 0 to m - 1 okay, so now if I were to write this, write the expression for this, please help me with it just verify that you have the correct expression.

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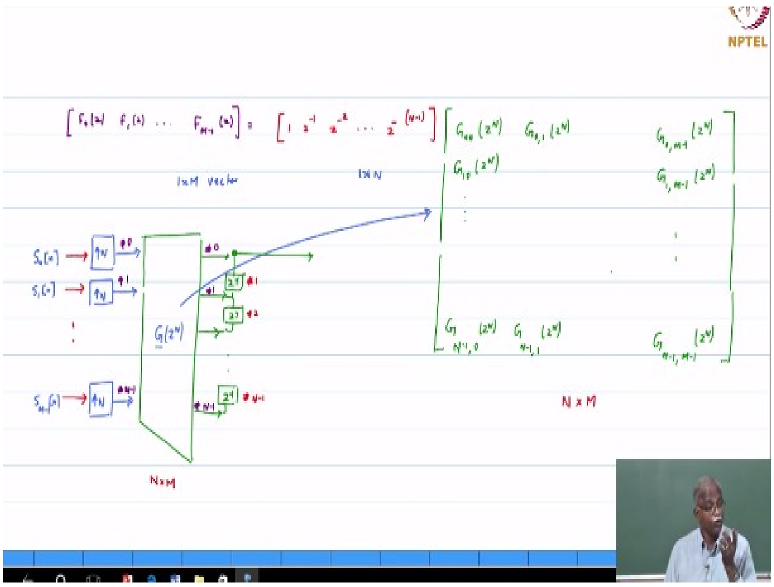

So, I want to write down an expression for f0 of z, f1 of z, fm - 1 of z, this can be written as a row vector; 1 z inverse z - 2 z power - n - 1 notice this is n – 1, so the one on the left hand side is a 1 cross m vector, okay this is a 1 cross n vector, okay. The next part of this are the poly phase components, all the poly phase components of f0 must be along the first column because only then they will get combined by the delayed chain.

So, this would be G00 of z power n, G10 of z power n dot, dot, dot G m -1, 0 of z power n, not m – 1, there are n - 1 poly phase components, so it should be n - 1 okay, so these are the poly phase components of f0, if you go back and verify that is the representation that we have were given, okay, did I right yeah, Gk poly phase component is the first one, the filter index is the second one.

So, the second index is all 0, it represents f0, the second we just write one more column and then we will finish with that this would be G0,1 of z power n all the way to G n – 1, 1 of z power n and the last column will be G0, m - 1 that corresponds to the m - f of m – 1, okay and the second term will be G of 1, m – 1 z power n and the last one will be G of n - 1, m – 1, z raised to the power n, those are the poly phase components of the filter f m - 1 okay.

Number of columns is m, number of rows is n, so therefore 1 cross n into n cross m gives me a 1 cross m, everything is dimensionally consistent but how are we going to represent this very, very important in our course, we are going to indicate redundancy in the following way again, it is a very, very interesting, a very compact and a pictorial representation, s0 of n is the first entry second input, the m - 1 input.

So, this is s1 of n, s m - 1 of n, each of these will be up sampled by a factor of n, this is where the redundancy is coming in okay, now this is going to be filtered through the polyphase component matrix notice, how I am drawing the polyphase component matrix, m inputs, n outputs, so it is a trapezoid not a square matrix because that basically shows the growth in the signals; signal representation, this is where the redundancy part is coming in.

I went in with m signals, I come out with n signals so basically, this is a n cross m matrix and that is exactly what we have represented here, so this is nothing but G of z n the matrix and it is nothing but the matrix that we have written down here now, how will these signals be combined, these outputs are combined through a delay chain, so basically this is a summing node, I have a delay chain here, another delay chain, another delay element and so on until we have the last delay element.

So, let me number the delay elements this is delay element number 1, this is delay element number 2, this is delay element number n – 1, remember there are n branches coming out, so there are m going in, so this is 0, 1, number 0, number 1, number m - 1 going in, I have number 0, number 1 and number n - 1 coming out and I am combining it through a delay chain and this is a representation; polyphase representation of a multi-carrier system.

We have not yet built the full system we only built the transmit portion of it where we said that we are going to do up sampling by a factor of n, we are going to introduce some redundancy and then we are going to see if we can exploit it in the; in a way that is advantageous to us okay. Now, very important question, can I apply noble identity here, if every entry of your matrix is a function of z power n yes, of course I can apply which is the case.

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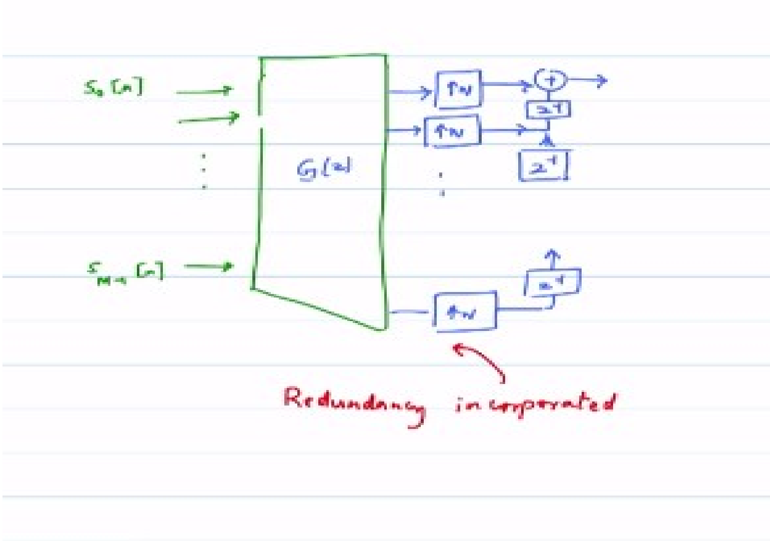

So, if I apply the noble identity, what is the structure that I get so basically, it will be the m inputs; s0 of n all the way to s m - 1 of n now, it is still a trapezoid but it is a function of z, G of z basically, all these polynomials that where G of z power n have become G of z and on the right hand side, we have the up samplers, up sampling by a factor of n dot, dot, dot up sampling by a factor of n being combined through delay chain of n units okay, indicate the arrows, so that there is no confusion.

So, basically we have achieved a very similar structure to the synthesis filter bank with the polyphase representation except that of 5 minutes; except that we now have the; it is now a system where redundancy has been introduced okay, I hope the aspects of redundancy you can kind of see it coming in, redundancy incorporated. Now, how exactly we are going to exploit this incorporated however, exactly we are going to leverage this that is the that is what we are going to be studying, incorporated above okay.

So that is where we are in terms of this aspect now, let us develop it a little bit more in terms of the mathematical representation, I would like you to because as you know just like we are building the process up, this is also very, very important for us. Now, if you go back to the synthesis filter bank, the synthesis filter bank basically says that I am up sampling by a factor of n filtering by f0, correct okay.

Can you tell me how to write down the expression, if I call this as u0 of n, can you tell me how to write down that expression basically, u0 of n can be written as summation over j f0 of summation over j, okay let me see if I am making this writing this correctly yes, summation over j s0 of j, please check if I am writing it correctly, s0 of j times f0 of n - j n correct, we wrote down that we would write down the; we could write down it; write it down in this fashion okay.

Now, if the combined signal is labelled as x of n, can you write down a mathematical expression for x of n, x of n = summation k = 0 through n - 1 of uk of n which means, it will be summation k = 0 to n – 1, am I correct, no, there are only m – 1, always have to check that because this redundancy part is yes, we have to be careful with the writing the; so this is m -1 and this is also m - 1 okay, summation k = 0 to m - 1 of uk of n.

Each of those uk's are a summation over j sk of n, sk of j, f k of n - j times n okay, just make sure that we got it correct, so this is a place where we can see that the different subcarriers are mixing because this x of n is actually a multi-carrier signal, so this is where the subcarriers; subcarrier are mixing, all these or in other words the sub-channels are mixing, so we have to design our carriers in such a way that these mixing parts are taken care of correctly and then we are able to recover the signal in a very effective manner, okay.

So, any questions on what we have done as far as the redundancy, we kind of sowed the seeds for the notion of redundancy; redundancy means that you have to use some additional resources, the additional resources that we are going to use is in the form of additional bandwidth, additional bandwidth means your sampling rate has changed so that means, your sampling rate is a bit higher.

So, therefore up sampling by n and not by m, now if I want to get efficient polyphase implementations, I must do the polyphase decomposition and apply the noble identities, so now I need to split it into n polyphase components though I have only m filters, so again this is the way in which we have developed the notion of redundancy and how we want to; how do you want to move it forward okay.

Now, let me just add one more element and then we will conclude this section and let us take a short break, so I hope this representation using the trapezoid is comfortable then the application of the poly phase component is also, so now I want to move to the analysis filter bank okay.

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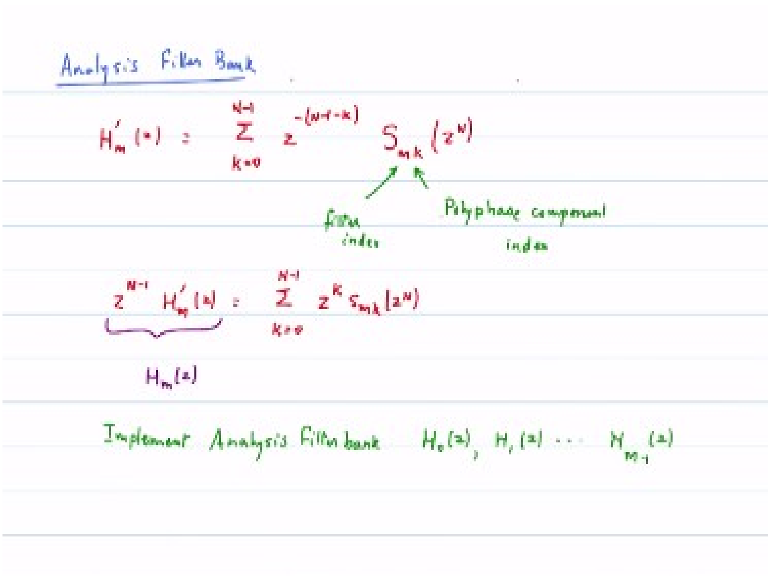

Analysis filter Bank is the one where we are going to split it back, analysis filter bank, if you remember the analysis filter bank is where you do the serial to parallel conversion, there we wanted to have it as powers of z, so here is the representation, I will just introduce it and then we will pick it up in a few minutes, so if I the filters that I have in my analysis bank, if I call them as the filters are H, if I denote them by H.

So, here is a way to visualize it, Hm prime of z, I am writing it as type 2 polyphase components k = 0 to m - to n – 1, again remember we are its redundancy has been introduced to n – 1 z power - n - 1 - k this is type 2 polyphase decomposition, I am going to use the poly phase components Smk okay, notice that there is a swapping of the index notation, the first index represents the filter index, the second one denotes the polyphase component index.

Because this matrix is going to appear in a transposed form with respect to the other one, so polyphase component index okay, now this should be function of zn okay, now again factor out z power n – 1 times Hm prime of z can be written as summation k = 0 to n - 1 z power k Smk of z power n, I am going to call this as my filter Hm of z just for ease basically, it is just a shifted version of it.

And I will now derive the implementation of the analysis filter bank, implement the analysis filter bank with the filters H0 of z, H1 of z dot, dot, dot to Hm - 1 of z okay, now remember this is a type 2 polyphase decomposition and I want to characterize this, so here is the polyphase implementation that we will achieve with this and basically, now we would then put the 2 pieces together to get the complete multi-carrier system.

So, the part that we want to want to build in here is; how do I implement the this particular filter, so the analysis filter bank takes the common signal and then passes it through the filters and then splits it up.

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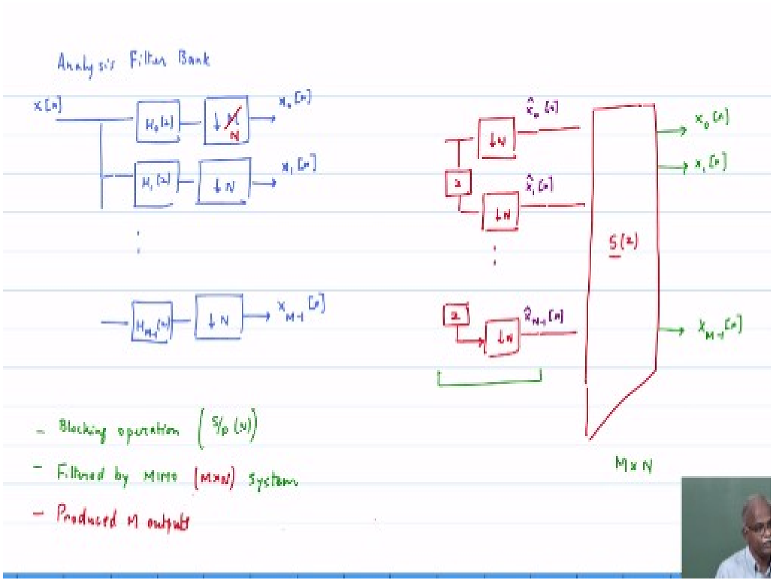

So, here is the visualization of the analysis filter bank, this is the one that splits it back to the sub signals, I will take x of n whatever is coming on the channel pass it through a filter H0 of z and I must down sample it in the other original case, I would have down it; done it by m, now I have to down sample by a factor of n and this signal now, becomes x0 of n okay. Now, I have to do this for all the filters H1 of z down sampled by a factor of n.

This is x1 of n dot, dot, dot, the last branch is m - 1 H m - 1 of z that is the last filter again, down sampling by a factor of n to obtain x of m - 1 of n okay, now write down the matrix expression that we have done and we should be able to show the following implementation again, I will pick it up from here after the break so basically, write down the poly phase component matrix, apply the noble identity, you should get down sample by n, an advanced operator z, down sample by a factor of n dot, dot, dot you have n such parallel branches.

The last one is followed advanced operator followed by down sample by n, so there are n branches coming out, we will let us label them as x0 hat of n just to show you how many signals are there, x1 hat of n, x n - 1 hat of n okay, now the filtering part; the polyphase filtering part it is a trapezoid but it input more number of inputs, fewer number of outputs, so I denote it by a trapezoid in the other direction.

So, call this as s of z matrix polyphase component matrix, we will write down the structure subsequently this then producing m outputs that we are interested in which should be the signals of interest x0 of n, x1 of n, xm - 1 of n okay, so what we have done is; whatever we did for the synthesis filter bank we have been able to do something very similar except that type; it is now type 2 polyphase decomposition, so that you get a string of advanced operators and not delay operators.

And you have done the; moving the down sampler to the extreme left, so that you can apply an implementation; efficient implementation, this matrix s of z is not a square matrix, it is a rectangular matrix, the dimensions of S are m cross n, so it takes in n inputs produces m output, so therefore we have shown it as a trapezoid and we have this representation okay. So, if I were to say it in words what did we do here?

We blocked it into the; we did a blocking operation, if you remember this is called a blocking operation okay, so the first step is a blocking operation; blocking operation is also another name for parallel to; serial to parallel conversion, so this is the same as serial to parallel conversion of dimension n okay, so that is exactly what is happening there serial to parallel conversion that is first operation.

Second operation; it was filtered by a MIMO system; multiple-input multiple-output, this s of z can be treated as a MIMO system, it takes in some number of inputs, so it was these signals that were obtained through the blocking operation by a MIMO system, the dimensions of the MIMO system are m cross n, it is a MIMO system okay, once you finish the filtering by the MIMO system, then we look at the output, the output has produced m outputs okay.

The similar thing you can write down for the synthesis step, there what would you have done; you took m inputs, you filtered it through a MIMO system which expanded your signal representation and then you did a parallel to serial conversion and fed it into the channel, okay so that is the counterpart of this, so this is what we have developed so far but with this framework, you can achieve some amazing insights.

And actually, completely developed the multi-carrier system in a multi-rate framework, multi-rate DSP framework that is what we will do in the next session, thank you.