Multirate Digital Signal Processing
Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture – 16 (Part-2)
Application of Multirate - Part2

Okay now come some interesting the Filter bank elements and some interesting results I hope you will enjoy this part of the discussion.

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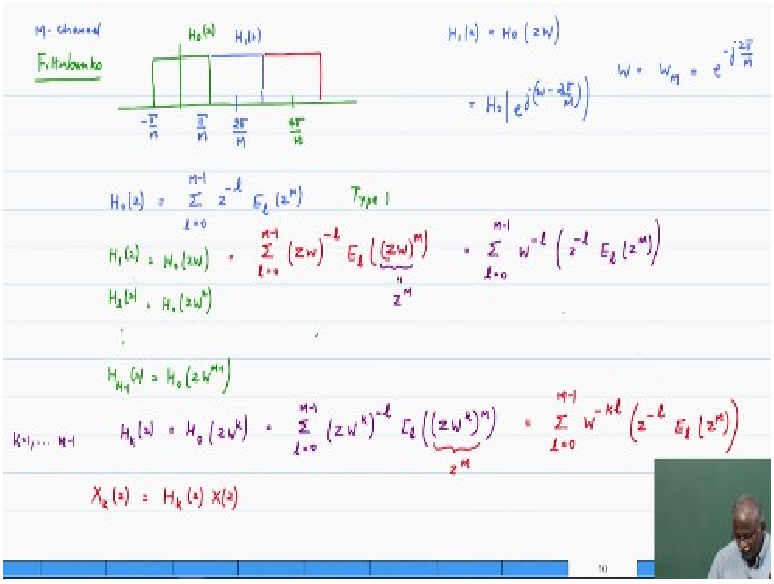

So we are now shifting gears and going to be focusing for the several significant portion of the coming lectures on filter banks, but again we will do it in pieces so that you are able to put it together like a big jigsaw puzzle, but this is the first of those elements. So filter banks we already said that the basic operation that we will get is through a modulation which will shift. If this is H0 of Z that is H1 of Z and we say that H1 of Z is H0 of ZW. This is a M channel filter bank.

So W is actually is equal to WM which is equal to e power –j 2 Pi over M. So this actually H0 e of j omega-2 Pi over M. It is a basic definition of the shifted filters. So if this is –Pi over M to Pi over M the center frequency for the first filter is 2 Pi over M center frequency for the second filter will be 4 Pi over M and so on. So let us quickly get that out of the way this would be 4 Pi over M and so on and the last one will be the 2 PI-2 Pi over M so that is the basic framework.

Now here is where what we have studied with Multirate comes into play in a very elegant way and this is where we will go a little slow, but it is very important that we understand or follow the thought process. So H0 of Z if I were to write it with Type 1 polyphase decomposition. So then this can be written as summation l=0 to M-1 Z power –l El Z M this is Type 1 polyphase decomposition.

We are now quite familiar with this. We are just going to look at this expression. Now from this we will derive H1 of Z we will derive H2 of Z all the way to H M-1 of Z. These are nothing, but H0 of ZW again refers to W subscript M. This one is H0 of ZW square the last filter is H0 of ZW M-1. Now even when we do the first filter itself you start to see some very interesting patterns emerging let us take a closer look.

So basically wherever there is Z replace it with ZW. So this expression would become l=0 to M-1 ZW raise to the power –l El ZW raise to the power M wherever there was Z I have replaced it with ZW to get my first filter. Now please look at what is within the bracket this one if I simplify W raise to the power M will be equal to1. So this is nothing, but Z raise to the power M first observation.

So this one can be written in the following way. This can be written as summation l=0 to M-1 W-l and then I am going to group the terms inside Z-l El z raise to the power M. Now if you have to look at any arbitrary filter HK of Z this would be H0 of ZWK. K is in the range 1 through M-1 and all of this would satisfy the following structure this will be l=0 to M-1 ZWK raise to the power –l El ZWK raise to the power M.

Now looking at what is within the bracket WK raise to the power M is also equal to 1. So this is nothing but Z power M. So we can now write this in the following way summation l=0 to M-1 W -KL Z power –l Elz power M. If there is input X of Z what will be the output. So if the input is X of Z and the output let us say that I call the output of HK of Z as XK of Z. XK of Z is HK of Z times X of Z. So basically it would be this expression multiplied by X of Z.

Now if you go back and review your basic DSP principles of when we write a input output relationship in terms of a transfer function and carefully look at this structure that we have obtained what you will find is that H0 through HM-1. There are M such output. So basically you will have X0 to XM-1 so many outputs are there. Each of these requires you to process X of Z by the same set Z power –l El Z power M.

Once you have processed with that then you are multiplying it by some phase terms and then taking a summation because if you notice X of Z and this set where l-0 to M-1 happens for all of that. So if you were asked to implement this in the form you will get a very, very interesting structure.

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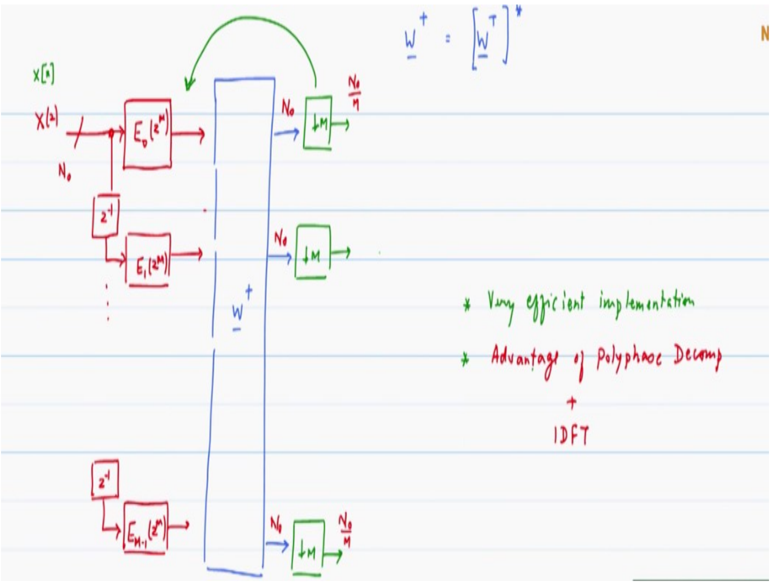

So the structure that implements that if you were to derive it I am sure by spending time you will definitely be able to do it. Let me just suggest that how we want to do that E0 Z power M. Z inverse E1 Z power M…the last one is Z inverse EM-1 Z power M. So X of Z process by each of these. So this is X of Z process by each of these filters is that correct, is it consistent with what we saw in the previous expression let us verify X of Z multiplied by Z power 0 E0 of ZM second one will be Z power-1 E1 of ZM that exactly what we get here.

X of Z E0 of ZM X of Z inverse E1 of ZM X of Z, Z power M-1 E of M-1 of Z. Now what did we do to these outputs? You multiplied it by W -Kl depending upon K supposing it was K=0 what was it just add it all of them together. So if you look at it carefully if you have the powers of W that would have been the DFT matrix this is the IDFT matrix. So basically what you have here is a well known matrix it is the DFT matrix actually just conjugated, but it is a symmetric matrix so therefore transpose conjugate is the same.

By the way anytime we have matrix a dagger symbol means that you do transposition followed by conjugation transpose conjugate. So that is the notation anytime we deal with complex matrices, constant matrices, the dagger symbol represent that. So basically this is like the IDFT matrix. The IDFT matrix will have a scale factor, but other than that this is the structure.

Notice how efficient this structure is to get the output of M Filter banks which are shifted as we have specified where the filters are shifted by multiples of 2 Pi over M that is closely linked to the DFT coefficient. Now if I use polyphase decomposition I will get Z power M that means all of this DFT coefficient kind of get map back to 1. So therefore the expression becomes very simple and then we get this expression.

Now we also had another results I want you to keep in mind when you have a signal which is band limited to 2 Pi over M anywhere in the 0 to 2 pi range you can down sample by a sample of 2m without aliasing. So which means that in the general case you can also do the following. You can down sample by M each of these branches. Each of these branches can be down sampled by a factor of M without aliasing. Now go back and apply one more principle interconnection of Multirate blocks adders, multipliers.

When you have the decimators they can move to the other side of the linear blocks. So basically the DFT matrix consist of multiplier and adders. So this down sampling by a factor of M can actually move to the left of the. So you can actually move the down samplers to this side now noble identity one more time you do it. So what do you do you actually have x of n coming in.

So now you can replace this with x of n you have the de multiplexer basically the stream input stream is getting de multiplexed into parallel streams each of them being processed by a sub filter. Each of these are polyphase components so their length is much smaller than the original prototype filter and once you do that you do inverse DFT same as DFT you can use the fast transform to implement this.

So you will get your outputs M output with very, very high efficiency. So basically what we can write down here is that this turns out to be a very efficient implementation. We have utilized several of the tools that we have developed. The interconnection of Multirate blocks, the noble identities and of course the shifting of the filters, what happens when you have the DFT coefficient raise to the power M all of those results have kind of come together.

So maybe this is a structure that you must spend some amount of time either redrawing or analyzing and applying. So you get a very efficient implementation, very low computational complexity because you are using the DFT, you are using the polyphase, you are kind of taken advantage all around and the efficiency comes because we have used the advantages of polyphase to the maximum advantage of polyphase decomposition that results from Mutlirate DSP along with the computational advantages we can get using the fast transforms.

IDFT will be implemented using the fast transforms which you have studied in DSP. So therefore you can get very efficient structures for this particular. Now I am glad you asked the question the question what role do these down samplers play? If I did not do this down sampling what would be my input sampling. My input sampling rate was N0 what would be the output sampling rate at this point.

It will still be N0, it would be N0 here, it would be N0 here effectively I would have got M times N0 as my sampling rate after I have done the filtering. Each of the filters produce which is quite obvious because I take input signal I filter it I get an output, input and output are the same rate and I have M such filters at the output. Now if you want to have the most efficient in terms of storage or processing.

Do I need to retain this at N0? the answer turns out to be no because if this was a band limited filter I know I can down sample and recover the original signal. So this one becomes N0 by M all of these become N0 by M and the effective sampling rate at the output if I do the down sampling actually remains at N0. So this is more for storage or for efficient processing. Again you need not do it, but if you do it you get the maximum advantage of the polyphase structure.

And one of the reasons why we would do it is because you actually can do the down sampling by a factor of M retaining the input, output sample rate at the same level without loss of information. Now we need to spend little bit of time on understanding the properties of the DFT matrix. I would like you to spend some effort on that and we will quickly touch upon that.

I am sure you would have probably have looked at it in other context, but let me just make sure that there is an opportunity for us to at least discuss that.

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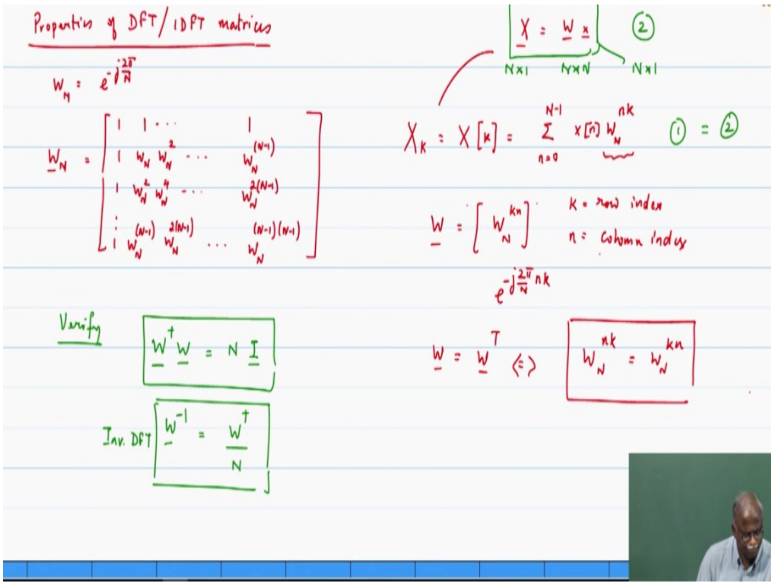

So properties of the DFT matrix. So properties of the DFT/IDFT matrices. So in case you are not very familiar with this I would say just request you to brush up on linear algebra and some of the basic definitions that we would like to discuss here. So W subscript N is e power – j 2 Pi over N. This multiplicative factor would come when we do an N point DFT or IDFT. So N point DFT would basically have the following matrix.

First row will be all once second row will be 1 WN, WN squared …WN N-1. First row, second row, third row would be WN squared WN 4 …W N 2 times N-1. And if you come down the list the last row would be WN raise to the power N-1. WN raise to the power 2 times N-1 and the last term would be WN raise to the power of N-1 into N-1. Again exactly what you would have studied in DFT, IDFT relationship.

So this is what we refer to as the DFT equation X subscript K or X within bracket K this is the same as X of K. This can be written as summation this is not a matrix this is a scalar. This is summation n=0 to N-1 it is end point DFT. x of n WN n K or K n. Now if you remember this was what we observed to say that the Filter bank actually comes from a DFT matrix because this is how you compute the DFT coefficient.

You take the inputs and you multiply it by these phase terms and then you sum them up to get the expression that we are interested in. So the W matrix consists of entries which are Wn raise to the power Kn where Wn is the e power –j 2 Pi over N. So the element of the nth column and the kth row. K represents the row index. Row index n represents the column index and basically what we have is this will become e power-j 2 pi over N times nk.

And I think it is easy to see that this is a symmetric matrix because if you do W is equal to W transpose the row index and the column index will interchange and basically it just says that WN nK is the same as WN Kn so these are basically it is a symmetric matrix. So but it is more than a symmetric matrix for the reason if you have if this equation can be written in the following way.

X is equal to W times lower case X, lower case X representing let us say time domain. So this is the N*N matrix let me write the dimensions in a different color. This would be N* 1 this would be N * N and the input vector is again an N * 1. So basically that is the DFT equation in matrix form. So this and this are identical, so equation 1 and 2 are identical. One is writing it in matrix form the other one is expressing it as a summation.

So given this result we can also verify I am sure you can verify the following result W dagger that means transpose conjugate. Transpose will be the same thing basically multiply the conjugate if you multiply W dagger times W what you will get is N times the identity matrix and you can use basically the properties of the roots of unity to verify that. And I am sure you would have done it already in the context of the DFT.

So this result basically tells us that the inverse DFT. So this is the inverse DFT. Inverse DFT is equal to W dagger divided by 1 over N and which is what we have in the IDFT equation you will have 1 over N term and why did that 1 over N term come because W dagger is a scale factor you basically are taking care of the scale factor.

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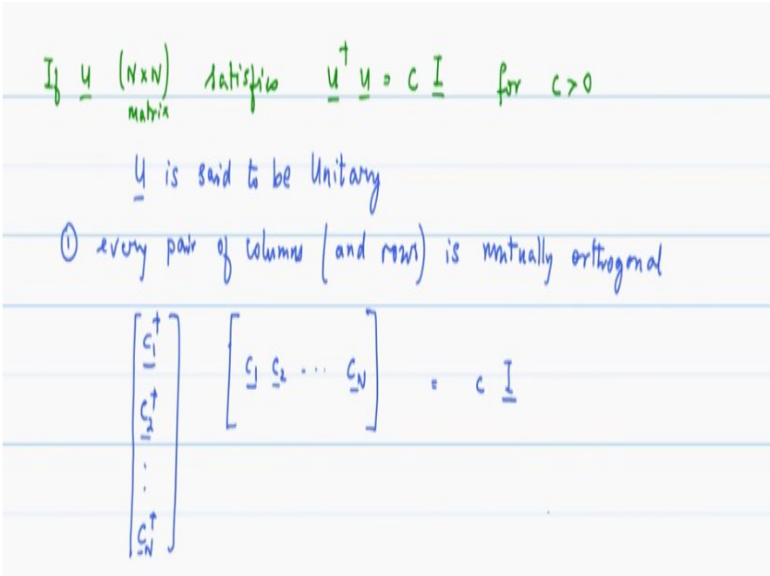

Now once you study these properties can just note down that the following observations if a square matrix if any square matrix satisfies the following properties N x N matrix satisfies the following conditions which is u dagger u is equal to constant times the identity matrix. For some C greater than 0 some scale factor C then u is said to be unitary. Unitary matrices have got many interesting properties.

I will just mention the one that is more important to us. The property that every pair of rows or pair of columns. So this is the property that it is visible right from the basic equations every pair of columns or and rows are mutually orthogonal. This is a powerful result mutually orthogonal. Now how does it manifest itself? If you write down the columns as C1, C2 up to CN these are column matrices.

And what does the basic result say u and u transpose conjugate so what you will get is C1 column you do transpose conjugate it becomes a row vector C2 dagger, CN dagger. Now C1 dagger time C1 becomes identity or some scale factor. C1 times C2 is 0. So basically you can see how this will manifest itself as some constant times the identity matrix and similarly you can show it for the rows as well.

So the DFT matrix is very rich in symmetry. It has got lots of very interesting properties, but as far as we are concerned it is a matrix that we would like to leverage for our Multirate structures, but I want you to maybe do the following exercise.

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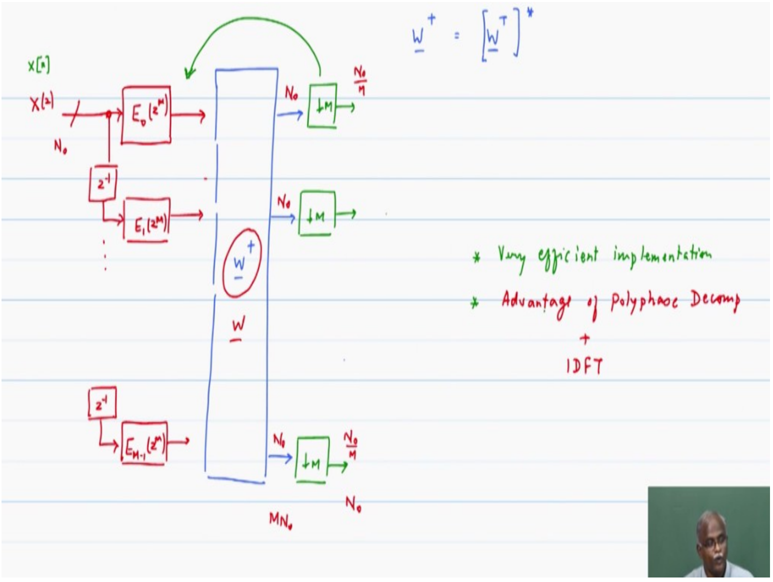

Can you take a look at this structure and replace this with the DFT matrix and then think about what is happening, what change has happened? So it is not a very difficult task but something for you to think about because this is now going to give us a lot of insight into the structures that we are going to be developing and this is also going to tell us and what you will find is that these outputs just get cyclically shifted that is all.

There is nothing profound happening, but this is what it is going to give us a lot of insights the use of Multirate or the DFT as a spectral analysis too, but this is very important that you see the relationships between the shifted filters when use the IDFT they have shifted in this direction. Now when you use the DFT you will find that the shift happens on the negative side, but in case if you do look at we will pick it up from here in the next class and develop. Thank you.