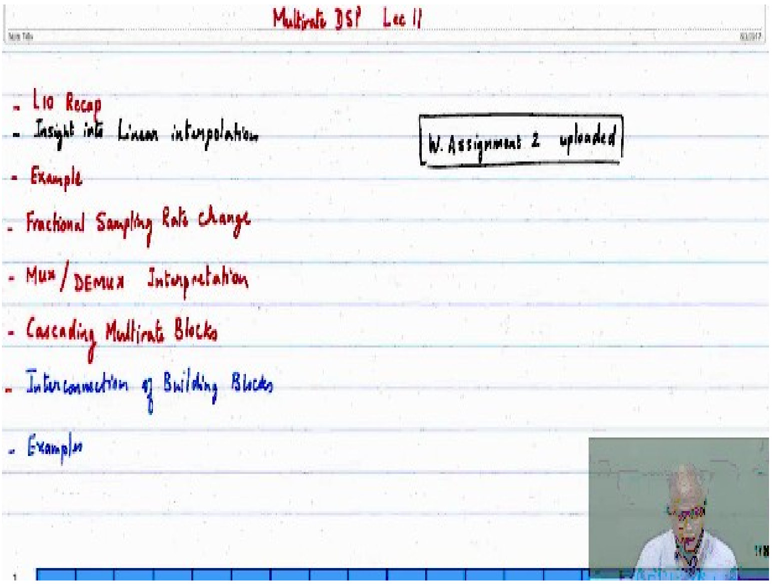
Multirate Digital Signal Processing

Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture – 11 (Part-1)
Fractional Sampling Rate Change – Part 1

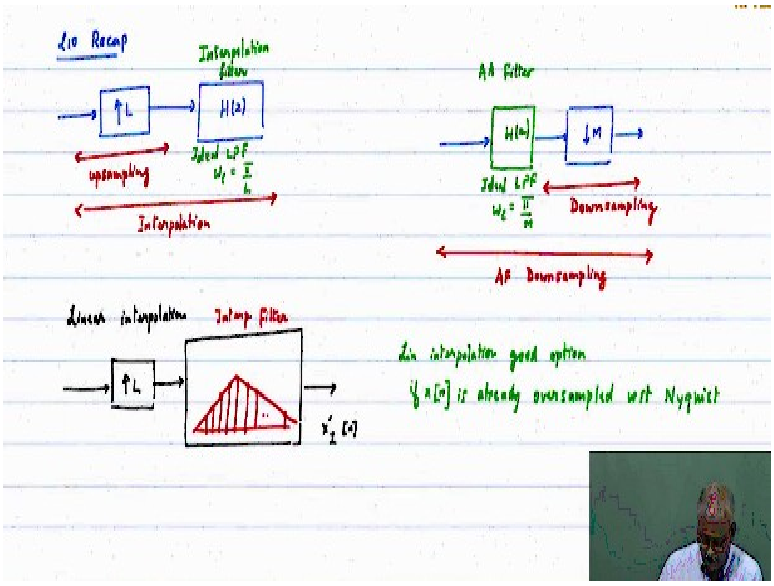
Good morning. We will begin with the summary of lecture 10 and then as you recall we did fair amount of discussion on linear interpolation. We will look at the next aspect of a fractional sampling rate change which will introduce for us all of the elements of the multirate processing that we are interested in doing. The multiplexer and demultiplexer as I mentioned are common blocks in communications.

(Refer Slide Time: 00:42)



Now where do they find a role in DSP, it is in the context of multirate DSP, we will introduced that. Then the complete tool kit of what are all the building blocks that you have. You have the building blocks from DSP, the LTI blocks then you have these 2 linear but not time invariant blocks, the up sampler and the down sampler and then how all of them fit together we will look at it with the series of examples.

(Refer Slide Time: 01:07)



So let us begin with a review of yesterday's lecture, so the basic building block that we have is an upsampler, the 2 basic building blocks, we have an upsampler, we will label this portion of the processing as upsampling. It is insertion of L-1 0s between 2 successive samples; however, because of the number of samples present, you have actually effectively increased the sampling rate of the system.

Then if you cascade it with a low pass filter, H of Z, then the specifications of this filter is that it is an ideal low pass filter with omega c equal to pi over L. It will remove all images outside of –pi over L to pi over L. Such a filter we would referred to as our interpolation filter, okay, ideal interpolation filters. So that means you will know exactly what we are referring to up sample by a factor of L.

So the combination of these 2 operations is the interpolation operation. So interpolation for us is not a single step it is the insertion of zeros, replication of spectrum, then filtering to remove the additional copies of the spectrum. So this is one element that we have studied. So up-sampling followed by filtering. Then the corresponding blocks that we have for the down-sampling.

So the down-sampling block denoted by a down arrow and M, you have your input, output, this is referred to as down-sampling, this portion, reduction of the number of samples throwing away M-1 samples, down-sampling process and as you recall we also said that this also can have a filter in fact will require a filter if you want to guarantee that it does not have aliasing.

So H of Z this would be an ideal low pass filter with cut off pi over M with cutoff omega c, cutoff is pi over M, this we would call as the anit-aliasing filter, AA filter and then you have the input to the anti-aliasing filter and this whole process alias free down-sampling or alias free decimation. So that alias free comes on because your and you are exploiting a sufficient condition which says that if I limit my spectrum to pi over M then, so this is alias free down-sampling.

And keep in mind that we say that this, it is a necessary and sufficient condition for baseband signal. So we are not dealing with the cases of bandpass sampling because bandpass sampling can be handled and should be handled differently. Then after we have this inside the observation or the point that we were discussing quite at length and there were several questions after class, very good discussion that we had, was what is the interpretation of linear interpolation.

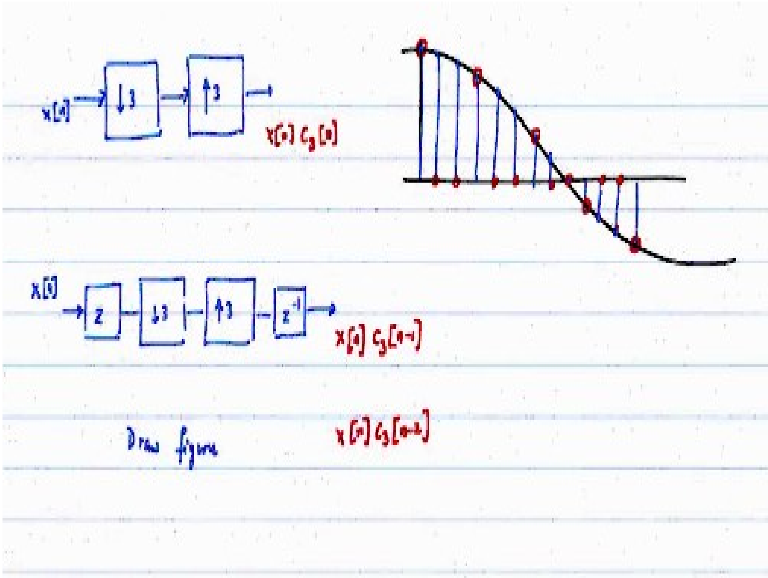
Linear interpolation we said can be understood as an up-sampling by a factor whatever is your requirement up-sampling by a factor of L and then you are using an interpolation filter, but the interpolation filter is not the ideal interpolation filter. So in this particular case we used an interpolation filter which had a triangular response. So basically you had, and depending upon how many samples, the value of L, you will have the size of your interpolation filter.

So basically it is a linear, this is what gives you the convolving the up-sample signal with this. So this is my modified interpolation filter and nothing says that you have to have an ideal low pass filter. You can have different filters, but you have to know what are the consequences of that. So basically this combination is what we called as linear interpolation. Okay so this is xi prime of N.

xi of N you can think of it is ideal interpolative, this is xi prime which is something which is like the ideal interpolative but maybe has some deviations okay. Now the concluding remark was that linear interpolation. If you have critical sampling, that is if you are exactly at Nyquist, then it actually distorts the signal. So linear interpolation is a good option only when you have, input signal is already over sampled.

So it is a good option if x of n is already over sample, is already over sampled with respect to Nyquist. So then what you will find is that you can get rid of the images and whatever distortion occurs is minimal and does not cause a problem, is already oversampled, oversampled with respect to Nyquist, Nyquist sampling frequency okay. So that is the key element okay.

(Refer Slide Time: 06:42)



Now let us build on what we have discussed in the last class, several useful result by way of a review let me just ask you to look at this particular block. Down-sample by a factor of 3, up-sample by a factor of 3 okay. So this is x of n and the output is same as x of n multiplied by the comb sequence c3 of n and if you want to have a visualisation of that it is good to sort of keep in mind what exactly we have done.

So if I have a signal, some signal, which I sample. So that is sample number 1, 2, 3, 4, 5, 6, 7, 8, 9 okay, some signal that has been sampled okay. The down-sampling process, down-sampling process says that I am going to retain only a few every third sample. So down-sampling process will retain this, this, this, this and this okay. So every 2 samples in between have been thrown away. So then the up-sampling processes you will restore 0 valued samples in between L-1 zeros.

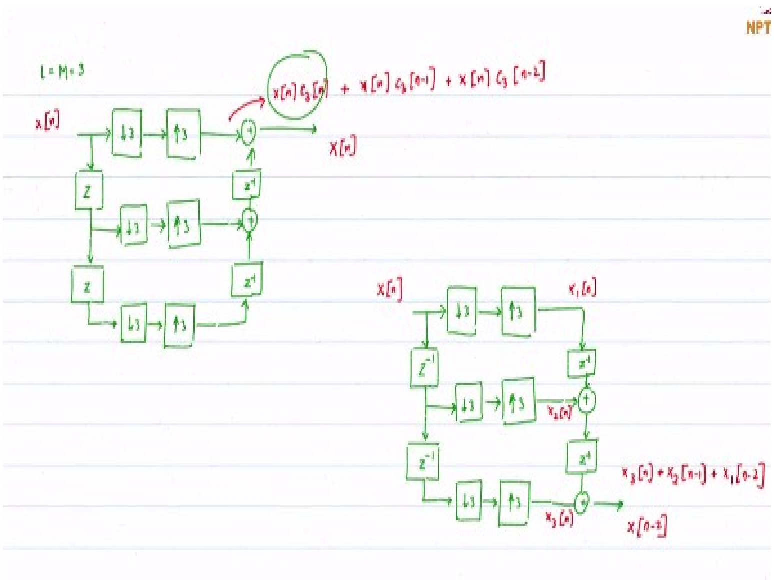
So 0 valid sample is here, these are the samples. So basically the comb sequence keeps the, every, in this case every third sample and the rest of the samples are set to 0 and in yesterday's discussion we had some variants of this. We said that if I can advance this by 1 unit of time then followed by the down sampling by a factor of 3, up-sampling by a factor of

3 and then followed by a delay Z inverse, we said that the output would be if this is x of n, the output would be x of n multiplied by a comb sequence, but shifted by 1.

So it is basically a different subsample sequence c3 of n-1 and you can sketch what that looks like and also you can modify this block to get x of n c3 of n-2, basically it will be Z squared and Z-2 to get the, so please fill in this, fill in this draw the figure, draw the figure, okay, you can try it out, that is just a simple exercise.

Okay now comes a very interesting element of putting all of these pieces together. In the last class we talked about in the context of a comb sequence with keeping every other sample or we split into odd and even sequences, but let us just look at a slightly different case.

(Refer Slide Time: 09:50)



We take M equal to 3 that means up-sampling, down-sampling both are, so L equal to M equal to 3. So the blocks that we have. So the first branch that we have has got down sampling by a factor of 3 up-sampling by a factor of 3 and the output comes out. Now the second branch, I am going to introduce an advanced operator down-sample by a factor of 3, up-sample by a factor of 3 and this will get added to the other signal.

Please note the direction of the arrows very important okay. So the second branch gets added. We need to insert something we need to insert a delay operation here. Z inverse so there is A okay, and then a third branch where we have another advance operator followed by down-sample by a factor of 3, up-sampling by a factor of 3, a delay operation okay. So let us analyse what we have.

So here is a system where there are 3 branches. This is a liner system so I can break it up into 3 parallel things which are getting added. So the first branch is down-sampling by a factor of 3, up-sampling by a factor of 3 that as we looked as we said in the previous case that is x of n c3 of n. The second branch input comes goes through one delay unit down-sampling by a factor of 3, up-sampling by a factor of 3, Z inverse.

So basically at the output I have x of n c3 of n from the top branch that is what is coming from this side. Now coming from this portion, the second branch is a + x of n c3 of n – 1 and just for a moment follow along on the second branch. If I follow from the input there are 2 delay elements that is Z squared. Down-sampling by a factor 3, up-sampling by a factor of 3 and then passing through 2 delay elements that turns out to be x of n c3 of n-2 okay.

So take x of n common, c3 of n, c3 of n-1, c3 of n-2 basically comb sequences interlaced you will get back exam x of n okay, so if this is x of n, the output will turn out to be x of n and I think you can either draw it or you know visualize it whichever way you want to justify okay. Now just want to make sure that you are comfortable with this. So let me give you a task, which I am sure you will find interesting and copy okay.

So I am going to modify this figure a little bit and then ask you to work with this and then let me know what is the answer, okay. So delete a few things okay, so I am going to modify this figure. So the upper branch is down-sample by a factor of 3, up-sample by a factor of 3 and is going to now, that is not going to be the one that is producing the output. I am going to feed it downwards to another delay element Z-1.

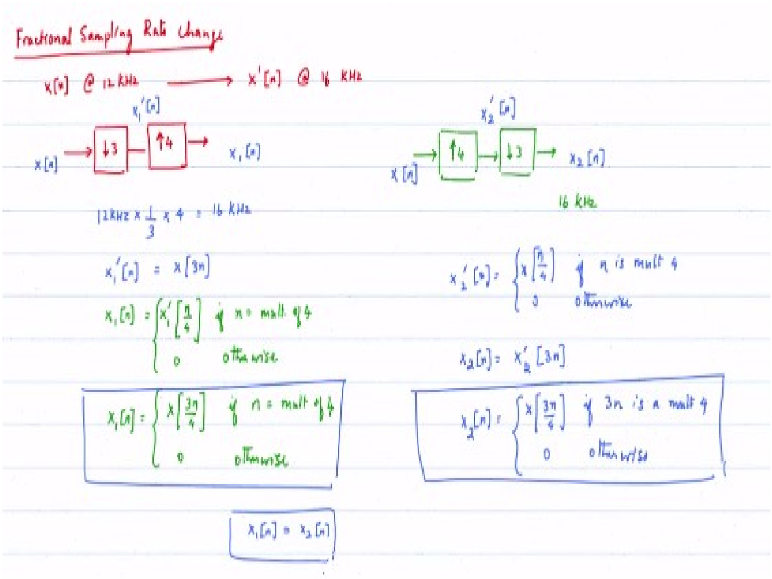
Okay, let me quickly clean up my diagram, both sides left and right I now have delay elements. Okay so this is a plus sign and okay. Now the second branch turns out to be Z inverse down-sample by a factor of 3, up-sample by a factor of 3 comes to the output, goes downward, that means to another delay element and then comes out. The third branch goes through 2 delay elements, down-sample by a factor of 3, up-sample by a factor of 3 goes out okay.

Can you tell me what the lowest branch will be? What will be the effect of the lowest branch? or let us take the other one. The upper branch is very easy to see, it is x of n, down-sample by a factor of 3, up-sample by a factor 3, that means it is this first sequence okay, which has then been delayed by 2 units of time okay. Now go through the second branch, second branch effectively had one advance operator and one delay operator, but now has got 2 delay elements okay.

So if you look at it carefully and then analyse you will find that this one also turns out to be that you have to do it systematically, basically verify that if this input is x of n, you still get the interlacing property, all the 3 down-sample sequences get properly interlaced, the output turns out to be x of n-2. Please do verify probably it is helpful if you define a sequence x1 of n, define another sequence x2 of n and a third sequence x3 of n.

So the output is x3 of n + x2 of n - 1 + x1 of n - 2 okay. So you can write down the expressions and make sure that you are comfortable with this result. Then let me proceed on to the fractional sampling rate because this will also teach us and give us a lot of interesting insights.

(Refer Slide Time: 16:40)



So fractional sampling rate change, it is an example, but it also gives us a lot of the insights into what we are trying to develop as far as the multirate framework. Fractional sampling rate change. So I have a signal x of n which is sampled at 12 kilohertz okay, want to transform it into x prime of n which is basically x of n, but at a different sampling rate at 16 kilohertz.

So yesterday we did say that I can visualise this as a system where I do a down-sampling by a factor of 3 and then up-sampling by a factor of 4 which means effectively I will get 12 kilohertz, down-sampling by a factor of 3 this gives you a 1 over 3 multiplication factor of the sampling rate. Up sampling by a factor of 4 gives you a multiplication by 4, this gives you 16 kilohertz that gives you the target that you are looking for right.

And of course we can do the up-sampling first, you may say well you know I heard somewhere that up-sampling should be done first so do the up-sampling by a factor of 4, down-sampling by a factor of 3, now look at the overall sampling rate, this is also at 16 kilohertz both are, what I would like to do first is analyse these 2 structures because it introduces for us a very important concept in multirate signal processing especially if we try to interchange the blocks.

One of the things that we want to focus today is when am I allowed to interchange these 2 blocks? Obviously down-sampling and up-sampling are both time variant blocks and therefore interchanging of time variant blocks is generally not permitted. So is there a possibility of these 2. So let us analyse this a little bit and it will give us a lot of insight. So let me label the signals.

This is x1 of n, x of n. this is x1 of n, this is x1 prime of n. So can you write for me the time domain expression, x1 prime of n is x of 3n that is the down-sampling okay. That is equation 1. The second equation says that x1 of n is equal to x1 prime of n by 4 if n is equal to a multiple of 4 and equal to 0 otherwise okay now can you rewrite for me the expression for x1 prime of n. So x1 prime of n is equal to x of 3n.

So can you help me write this equation? So x1 prime of n by 4 will be x of 3 n by 4 whatever is the argument multiplied by 4 okay, what is the condition, x1 of n, x1 of n is x of 3n okay. So basically what we are saying is this condition will hold if n is a multiple of 4 which is the same as n is the multiple of 4 right, but okay, so equal to 0 otherwise. Actually I just have to be little bit careful here.

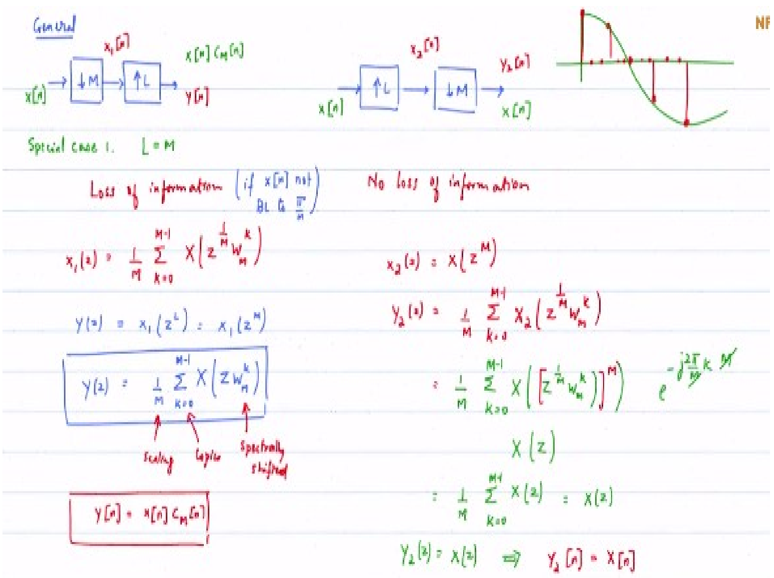
x1 of n is x prime of n by 4 that is the down-sampling operation. I now have x1 prime of n which is x1 prime of n is x of 3n. So x prime of n by 4 if n is the multiple of 4 that is x of 3n if n is the multiple of 4. I have not changed the condition. So in this particular case, just be careful okay. So basically x prime of n is, x of n is x prime of n by 4 is another multiple of 4.

And x prime of n by 4 is nothing but x of 3n by 4 okay. So this is the final expression for x1 of n. Now let us look at this sequence, same input x of n, let us the output be called as x2 of n and the intermediate signal x2 prime of n. Now x2 prime of n is x of n/4 if n is the multiple of 4, 0 otherwise and x2 of n equal to x2 prime of 3n down-sampled. Now if I substitute x2 prime of 3n give me the expressions.

It will be x of 3n by 4, the argument of x2 prime has to be multiple of 4, so if 3n is the multiple of 4. If 3n is a multiple of 4 and equal to 0 otherwise, okay. So x2 of n. Now please compare the 2 expressions that we have obtained given the fact that 3 and 4 are relatively prime. So therefore n being a multiple of 4 and 3n being a multiple of 4 are exactly the same condition. So there is nothing different between the 2 terms.

So effectively x1 of n is equal to x2 of n, okay. A very important observation that needs to come along with this one is this true in general that I can interchange the word. What this is telling me is I can do the down-sampling followed by up-sampling or the other way. So let us quickly make sure that we have this very clear in our minds and they will not be any confusion as far as this thing is concerned.

(Refer Slide Time: 24:00)



So let us take this particular case, the general case, down-sample by a factor of M, up-sample by a factor of L. Up-sample by a factor of L, down-sample by a factor of M okay. So special case number 1, special case number 1 is the case where L equal to M and of course I will take special case number 2 which is L not equal to M okay. So right off the bat so that means this down-sampling by a factor of M, up-sampling by a factor of M, what is the output? x of n, cM of n multiplied by a comb sequence okay.

Now if I up-sample by a factor of M what does it do. If I have inputs, if I have a sequence where I have the following samples, okay, if I have these as my samples. Now up-sampling by let us take L equal to M equal to 2, it does the following, it inserts 2 zeros between every set of samples and then down-sampling by a factor of 2, removes those 2 samples. So what I get at this point is x of n. Input x of n, output x of n.

So clearly when L and M are same, they cannot be interchanged because one of them here there is no loss of information, no loss of information, on the other hand the case where L equal to M and I do the down sampling first, there is loss of information. So I have to be very careful with this. So obviously the first observation is that if L and M are the same they cannot be interchanged without the penalty that you will have to pay.

Obviously this is the case where L and M are not the same. Now can I make the general assumption that I can always interchange when they are not the same and we will work out in a very careful manner how such a question is supposed to be answered, but maybe just let me start by giving you the framework. So the way we would analyse is that let us say that this is x1 of n and this is y of n. x1 of n, y of N.

So X1 of Z would be 1 over M summation K equal to 0 through M - 1 X of Z power 1 by M W M K, correct, that is the down-sampling operation yes. If x of n not band limited to pi over M that is a good point, yeah. Because if you have not band limited, if you are band limited to pi over M then the first step does not produce for you any loss of information and yes the signals are different, but you can still hopefully recover the information that is of interest, that is a good point.

So X1 of Z is given by this, Y of Z is X1 Z power L that is the up-sampling process okay, and in this case let me write it as X1 Z power M. So I have to go back to my equation number 1 and replace Z with Z power M, okay, please be very careful, I am replacing Z, wherever Z occurs I replace it with Z power M, so if this is equal to 1 over M summation K equal to 0 to M-1 X, there is Z power 1 by M, I am going to replace Z with Z power M.

So that means Z power M by M which is Z power 1 and then I have W M K, this is Y of Z, okay. Now notice that there is scaling, amplitude scaling is there multiple copies are there, copies they are spectrally shifted. The only thing different from a down-sampling process is that there is no scaling of frequency. So spectrally shifted. So this combination of scaling copies and spectrally shifted is by the comb sequence.

So it is straightforward or we can go back to the earlier lectures and you can confirm that this is nothing but x of n cM of n. x of n multiplied by the comb sequence. If I call this as let us call it as x2 of n and this output as y2 of n, X2 of Z is X of Z power M. I am choosing L equal to M, X of Z power M. Y2 of Z is the down-sampling process, 1 over M summation K =equal to 0 to M-1, X2 of Z power 1 by M WM raise to the power K. Basically I have written down the equation for expression for a down-sampler.

Wherever there is in X2 of Z I am going to have to replace X of Z power M okay. That means the argument of X 2 of Z is going to be raise to the power M as an argument of X, 1 over M summation K equal to 0 to M-1 X of Z power 1 by M WM raise to the power K this whole thing raise to the power, let me use a different colour may be easier to see that, this whole expression okay. Now Z power 1 by M raise to the power M gives me X of Z. W M raise to the power K is E power –j 2 pi over M multiplied by K.

Now if I raise it to the power M, M and M will cancel, this becomes coefficient equal to 1. So this effectively becomes 1 over M summation K equal to 0 to M - 1 of X of Z. You add it M times and then divide by M trivially equal to X of Z. So Y2 of Z = X of Z which implies that y2 of n = x of n, which is exactly the intuition or basically when through the sketching process we already knew that they were different and this is a more of a formal structure when we say that yes they are indeed different.