Multirate Digital Signal Processing
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Lecture - 09
Decimator Properties

Good morning. We begin with a quick recap of the lecture number 8 and the topics that we would like to cover today. We have already discussed about the properties of the upsampler. We have started to discuss about the downsampling process. Today, we will complete the discussion in terms of the properties. We will look at several interesting examples. One particularly interesting example is the linear interpolation.

Very often when you have to find a value between two points, the most natural thing for us to do is to do linear interpolation between those two points. Now we use the word interpolation in that context, we have also used interpolation in our context where we insert zeros and do some filtering.

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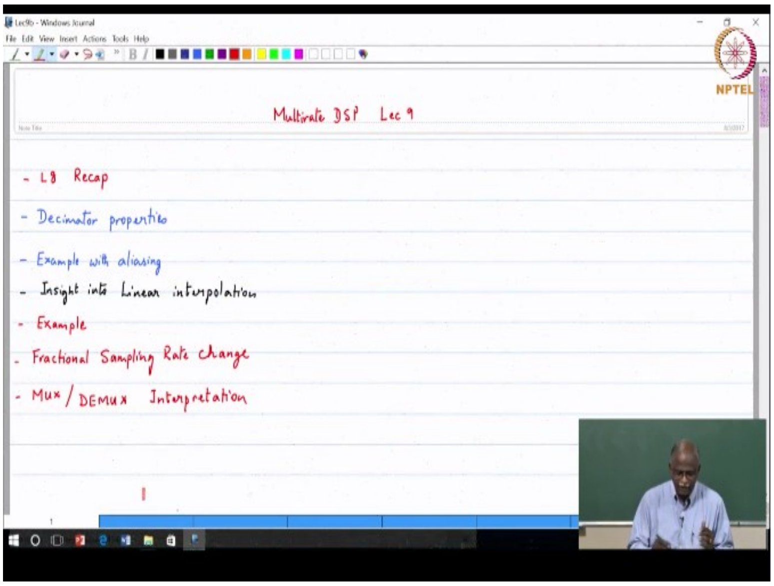

Are they linked? Is there some insight that we can gain into the aspects of linear interpolation? And of course one of the key things is we are not always limited to integer sampling rate change either increase by an integer sampling rate or a decrease, very often we are required to do a fractional sampling rate change.

So that is an aspect that we want to focus upon and then this leads us into some unique structures that are present only in multirate signal processing not in traditional signal processing which is what we refer to as a multiplexer and demultiplexer. You may think well I thought that these things existed only in the communications domain, no they actually exist in the multirate signal processing as well. So that is our goal for today.

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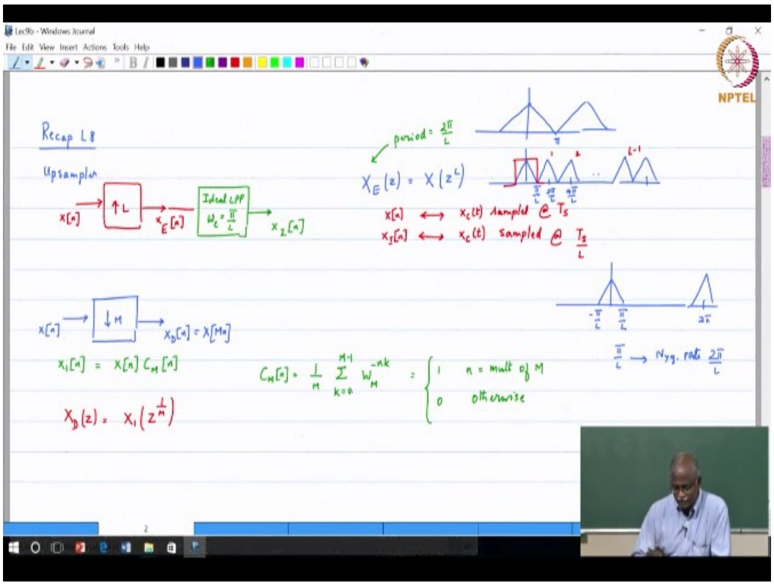

Let me start with a quick summary of what we have covered in the last class and build on that and again if there are questions we will take them as we go along. So the block that we have looked at is the upsampler where we insert L-1 zeros between successive pairs of samples, every set of samples we will every pair of samples we will insert L-1 zeros. Now we showed that this is the spectrum or the z-transform of the expanded signal is related to the spectrum of the input signal by XE of z is equal to X of z power L.

And again this we said introduces multiple images of the input spectrum and just as a quick sketch of that if this was the input spectrum that you had a spectrum going from up to pi then what you would get is the spectrum compressed. This would be pi over L; the next copy of the spectrum would be 2pi over L, 4pi over L so on all the way to 2pi. So basically we have got additional copies. This is copy number 1, copy number 2, the last one would be copy L-1.

This would be copy L-1 and that is what would be X of z power L and we said that if we followed the expansion of the signal by the zero insertion by a low-pass filtering which basically removed all of these images, removed all of these images then we get a interpolated signal. There are no more zero valued samples, all of the samples have been filled in and the inputs, the resultant spectrum looks as follows.

So you have limited to –pi over L to pi over L all the way to 2pi, so this basically says that if I had a signal which was band-limited to pi over L, think of it as just some frequency. My Nyquist rate would have been 2pi over L. So Nyquist rate would have been 2pi over L but actually I am sampled it at 2pi, so which means I have sampled it at L times higher than Nyquist rate. So that is what the process of interpolation gave you.

You got additional samples to make your signal look like it has been over sampled and whether you interpret it by going into the continuous time domain or by remaining completely in the discrete time domain you can still interpret it as a signal with a higher sampling rate and equivalent representation with in terms of the original signal. Now the other block that we have been discussing is the downsampling block.

And the down sampling block indicated with the down arrow in terms of time is very easy to represent xD of n is x of M n, the corresponding spectral representation or the z transform or the Fourier transform representation not as easy we were in the process of developing it. Basically, we defined an intermediate signal x1 of M which matched x of n at all multiples of M otherwise it was 0.

We said that that sequence is something that can be obtained by multiplying x of n with a comb sequence. Comb sequence is as defined as here. We use the roots of unity, complex roots of unity to just construct a comb sequence and then we were in the process of deriving signal.

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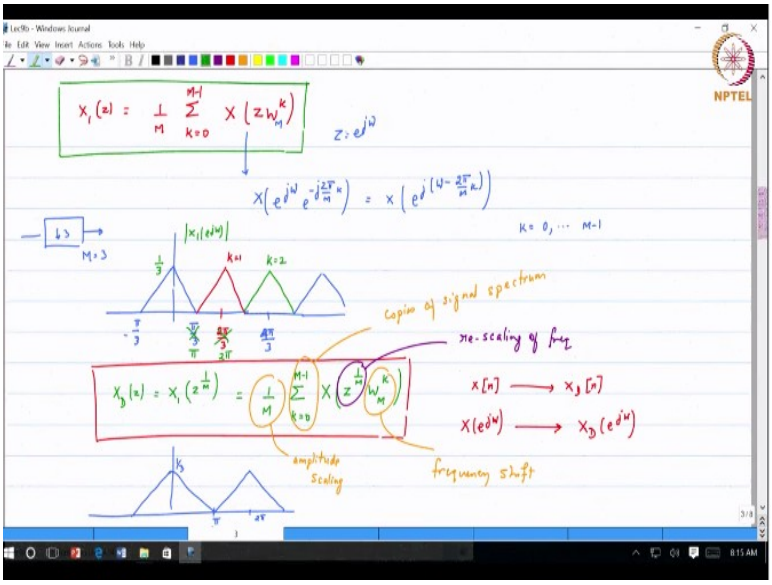

So basically X1 of z, this is the point at which we ended the last lecture. I will pick it up from there. X1 of z, we showed was 1 over M summation K equal to 0 to X of z W K. Now I just wanted to expand this a little bit, so if I write Z if I substitute Z equal to e power j omega then what I get is this term becomes X e power j omega. This if you want to be precise you can write a subscript M so that there is no confusion.

But usually when we talk about it in the context of down sampling and if the down sampling is by a factor of M then we assume that the W term is defined with, so this would be e power –j 2pi over M into K. So this can be also written as e power j omega -2pi over M into K, where K goes from of course 0 all the way to M-1 okay. So here is where we see the shifted versions and in today's lecture we will see this graph a few times.

But let me just draw it, so supposing I had a signal that was band-limited to –pi over 3 to pi over 3 okay and I was basically doing M equal to3 downsampling by a factor of 3. That is the case that we are looking at. Now what we are saying is the spectrum of X1 of e of j omega, so if I were looking at the spectrum of X1 e of j omega will first of all introduce a scale factor 1 by 3. The spectrum for K equal to 0 is the original spectrum.

Then K equal to 1, we get a copy of the spectrum shifted and centered around 2pi over 3 and then for the K equal to 2 we get another copy of the spectrum shifted to 4pi over 3 okay. So two copies of the spectrum K equal to 0 1 2 will be the number of copies and then you would have the original signal again. So this is what X1 e of j omega, it has multiple copies of the input spectrum shifted by multiples of 2pi over M okay multiples of they are centered around these figures.

And this here is a case where we saw this corresponds to K equal to 1, this corresponds to K equal to 2 okay. So this is easy to visualize, so having visualized this then it is just a very, very simple step to go from there to the down sample signal because we already know the following relation XD of z spectrum of the down sample signal is X1 of z power 1 over M okay. This we derived in the beginning of the last lecture.

So it is nothing but a scaled version or we call it a frequency stretched version of the input of the signal X1 of z. So this can be then be written as 1 over M. Please pay careful attention to the substitution of the variable z power 1 over M. So wherever there is z I am going to replace it with z power 1 over M, so which means that there is a scale factor 1 over M summation K equal to 0 through M-1, X is the first place where I am encountering z and the only place where I will encounter z.

Z replaced by z power 1 by M and whatever is the argument multiplied by omega WM raised to the power K. So nothing else changed, only this happened okay. So this is my expression for the downsampler. Now very often when we go directly from X of z, so basically if you go from X of n all the way to XD of n which will then mean that you are looking at X e of j omega the spectrum of X e of j omega from that down sample version XD e of j omega.

There is all sorts of confusion okay because you do not know where the scaling is occurring, where the shifts are occurring but what we have shown is that if you define an intermediate signal X1 of z, then it is very easy for us to construct what X1 e of j omega looks like, X1 e of j omega is the original signal with shifted copies at multiples of 2pi over M, easy to construct. Now the last step is when I want to construct the down sample signal, we know that this relationship XD of z equal to X1 of z power 1 over M is nothing but rescaling of the frequency.

So which means that whatever was the original frequency, I strike it off and then multiply it by the factor M. So this if I can cancel this and call it as pi because in this case M is=3, this one I will cancel and call it 2pi okay that is it. Equal to 2pi after that the spectrum will repeat, so therefore as far as XD of z is concerned for this particular example, it is as follows. Height of 1 over 3, this is pi and I have this as my copy and this is at 2pi okay.

So that is it, that is the down sample signal, the spectrum absolutely no confusion, the 4 you keep in mind when we write down the down sample signal first of all amplitude scaling, 3 of these were also observed when we were writing down the spectrum of X1 of z, so I will just repeat that, 1 is the amplitude scaling, this was already present in X1 of z, the copies, copies of the inputs of the signal spectrum.

Signal spectrum, how many copies M-1 additional copies and the frequency shifts, so all of these were present in X1 e of j omega. The scaling multiple copies and the frequency shifts which is what we saw. Now the last step of writing the spectrum of the down sample signal basically boiled down to doing a scaling of the frequency, rescaling of the frequency that is all, rescaling of the frequency okay.

Once you have this picture then it is very easy for us to work with the down sampler and not have any confusion at all. So the down sampling process always my recommendation would be is think in terms of an intermediate signal X1 of z which is the input signal multiplied by a comb sequence which will give you this spectrum, shifted scaled, shifted multiple copies. Then, the last step of getting this spectrum would be just the requirement that you have to rescale the frequencies and make sure that you are looking at a period from 0 to 2pi okay.

Any doubts on the notation that that we have used? Okay now we will spend a few minutes on an example which may seem like it is a little bit repetitive but again keep lookout for those things which will help you make sure that you will never have a doubt or a confusion whenever it comes to downsampling. So here is the first a series of figures probably easier for you to draw on your notebooks but let me do the best that I can.

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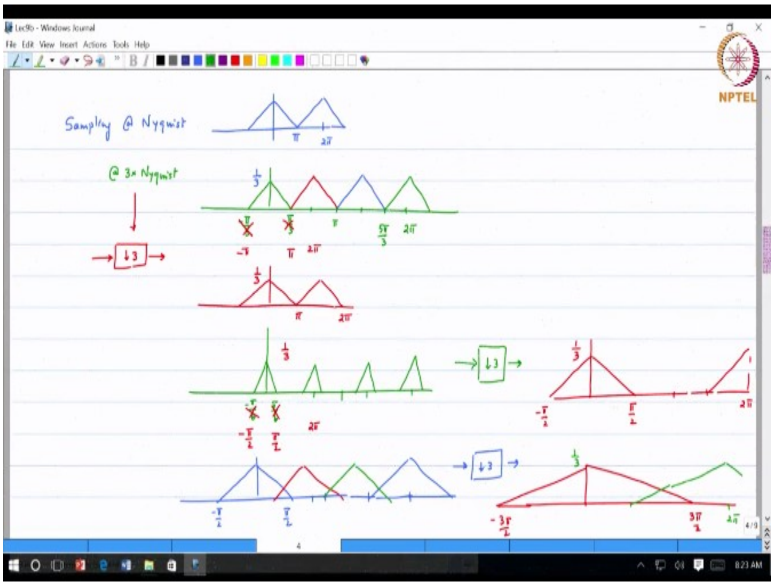

So the first one is I am going to draw the spectrum of sampling at Nyquist rate okay. Again, keep in mind that this is a series of figures preferably or draw it all on the same page, so then you get the, so Nyquist sampling rate the spectrum looks like this. This is 2pi okay, so that is this is pi Nyquist sampling rate. So there is no gap in the spectrum. Basically, you are sampling it exactly at the twice the highest frequency content

And therefore basically the two copies of the spectra just touch each other. Now of course reconstruction or removal or you know anything else you want to do with this is known. So next one which I want to look at is sampling at 3 times Nyquist. Again, let me just quickly draw this picture okay. If this is pi by 3, -pi by 3, this is pi, this is 2pi and this is the copy of the spectrum going from 5pi by 3 to 7pi by 3 okay.

So basically this is a case where you have over sample. If you had over sampled at 3 times the Nyquist rate, this is the spectrum that you would have observed okay. Now take this signal one that is being sampled at Nyquist rate 3 times Nyquist. So which means that if I go down by a factor of 3 take this signal, pass it through a downsampling sorry it is downsampling by a factor of 3.

Then, I should not have any aliasing because I actually did over sampling by a factor of three to begin with and so and we should verify that whatever we have done, so downsampling by a factor of 3 first of all says get me the shifts of the spectrum right in the intermediate signal. So first thing is to draw the shifts of the spectrum. Again, this is a repetition of the figure that we drew in the last previous page.

So that is corresponding to K equal to 1, this corresponds to K equal to 2, the 2 shifts, the scale factor of 1 over 3. Then, the last step will be the rescaling of the frequency. This becomes –pi, this becomes pi, this becomes 2pi. So the new spectrum for the down sample signal draw that in red, down sample signal basically goes back to the original spectrum goes all the way to pi, 2pi but there is a scale factor of 1/3 which came in through the process of the downsampling that we have done, we have incurred okay.

Now for the benefit of clarity would like to look at the same downsampling by a factor of 3; however, the input spectrum is not at Nyquist rate but slightly is less than Nyquist rate. So basically look at this particular scenario, so I am going to redraw a spectrum which I would like you to look at and study for the downsampling. So this is from –pi over 6 to pi over 6. Now what would this signal look like if I sent it through a down sampler?

Of course, there will be a repetition at 2pi right. What would happen to this signal if it were down sampled by a factor of 3? So if this signal down sample by a factor of 3 what would it look like? I would have to create multiple copies at 2pi by 3 and 4pi by 3. Please go ahead and do that. So basically 2pi by 3, 4pi by 3 copies of this spectrum okay and then you would do a scaling 1 over 3, scaling of 1 over 3.

And then you would do a rescaling of the frequencies. So this would become multiplied by 3 will become –pi by 2, this will become pi by 2 and this will become 2pi. So the spectrum of this signal after the downsampling process looks like this. It goes from –pi by 2 to pi by 2, -pi by 2 to pi by 2 there is pi, 3pi by 2 and 2pi and of course the downsampling did not of course has brought the images closer to each other because previously they were further apart but.

And it looks like there is a stretching in frequency and we know exactly where the stretching comes from. Now all of this was done to answer the following question. Now if I gave you a spectrum that had the following form where the signal spectrum went from 0 to pi by 2, –pi by 2 to pi by 2 okay and of course that means the other copy of the signal would be here okay. Now if I took this signal and then down sampled it by a factor of 3 what happens okay?

So first and foremost, I have to have to look at the shifted versions. Once we look at the shifted versions, then you do the rescaling. So the shifted version says there is a copy at 2pi by 3, so 2pi by 3 there is a copy, at 4pi by 3 there is a copy and of course there will be a rescaling process okay. So if I were to after the rescaling if I were to draw the new figure then it would look like this.

It will go all the way to 3pi by 2 because pi by 2 will go to 3pi by 2 and go to -3pi by 2 on the other side, -3pi by 2 and the copy of the signal or the other portion of the spectrum will start at pi by 2 and then go up to 2pi, so starts at pi by 2 goes all the way to 2pi okay. So the spectrum itself I have been able to reconstruct 1 by 3.

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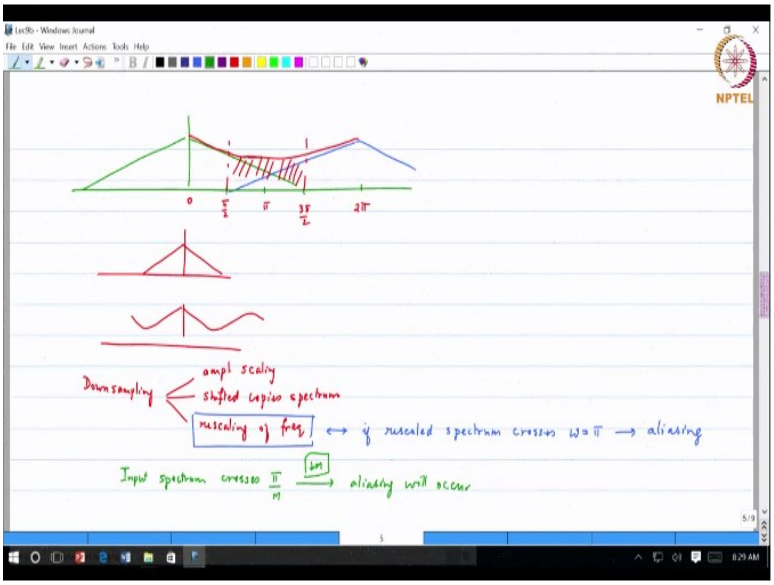

Now if you have come this far then the next step would be okay, so I will just zoom in and draw this figure. So this is pi by 2, pi, 3pi by 2, 2pi, spectrum, copy, the other copy of the signal and basically if I look at a composite signal that is I have just drawn it, so the signal itself will look like this. It has a scale factor like this and then in this portion, it is some combination of the green and the blue and then it will repeat itself okay.

So this is 0, this is pi by 2, this is pi, 3pi by 2, I think I have to redraw the red line and the blue line 0 to 3pi by 2 it goes to 2pi. This is 2pi okay, so the other copy of the signal is from pi by 2 to 2pi and then repeat. So the copy of the resultant signal looks something like this. So it has some resemblance to the original signal but in the part where there is overlap, you know really we cannot say anything.

Because the two the copies have overlapped and therefore to interpret this signal we can say that yes the original signal had a very clear band limited nature but now after this process the one that I have has something like this. It has got some distortion that especially towards the high frequencies. Now what exactly happened there this is where this is the region where aliasing is occurring.

And the spectrum where there is the portion of the spectrum where there is overlap is a place where because of the aliasing we will we may not be able to recover the original signal. If we are to now look at these 3 examples okay. So I had a case where the signal was limited to pi over 3, I did that downsampling, it was clean. Over sample the signal, down sampled of course there was no issue.

When I exceeded pi by 3 and I went all the way to pi by 2, down sampled it; I saw that there was an aliasing that happened. So here is a layman's way of describing the signal. So when you do the downsampling, the following 4 steps happen. One there is amplitude scaling. There is shifted copies of the spectrum. So that is two and three, shifted multiple copies of the spectrum, multiple copies of the spectrum shifted by multiples of 2pi by L and the last one is rescaling of the frequency.

Rescaling of the frequency and the rescaling of the frequency we described in layman's terms as a stretching of the frequency. Now basically what we can say about the occurrence of aliasing is that rescaling will happen when I do the downsampling, so if my rescaled spectrum, if the rescaled spectrum crosses omega equal to pi then I will be having the effects of aliasing. Why?

Because the next copy is going to be centered around 2pi, this initial spectrum has gone beyond pi, so when I shift it there is going to be overlap between these two copies of the signal. In other words, rather than talk about the rescale spectrum, you can also make the statement in terms of the original spectrum. So if the input spectrum, input spectrum before rescaling means that if the input spectrum crosses pi over M then when I do downsampling by a factor of M aliasing will occur, aliasing will occur okay.

So we have talked about it completely in the discrete domain. We remember not gone into why aliasing has occurred. Basically, we have shown that the input signal is periodic. When I do the downsampling, there is amplitude scaling, shifted copies of the spectrum and a rescaling of the frequency and we have shown that if the input spectrum has content beyond pi over M then the downsampling process will introduce overlapping of images.

This overlapping of images is very similar in fact is exactly the same as what happens when you under sample a band limited signal there will be overlap of the copies of the spectrum but this is not overlap of the continuous spectrum but in the discrete domain itself we have shown that there will be overlap of spectrum, I cannot I may not be able to recover the original signal which is what is typically the description of aliasing.

So just to summarize, the down samples signal XD of z can be written as 1 over M summation K equal to 0 to M-1 X of z power 1 by M, that is where the frequency scaling occurs. Now if you go directly from X of z to this expression there is always this confusion, you know do I scale by M, do I stretch by M, what happens to the shift, all of those doubts will occur but if you go through this intermediate step can avoid that confusion.

And will be very clear in your mind what exactly is the spectral interpretation of the downsampling process okay. So now let us build on this for a few more minutes while we have that. So let me make the following statements.

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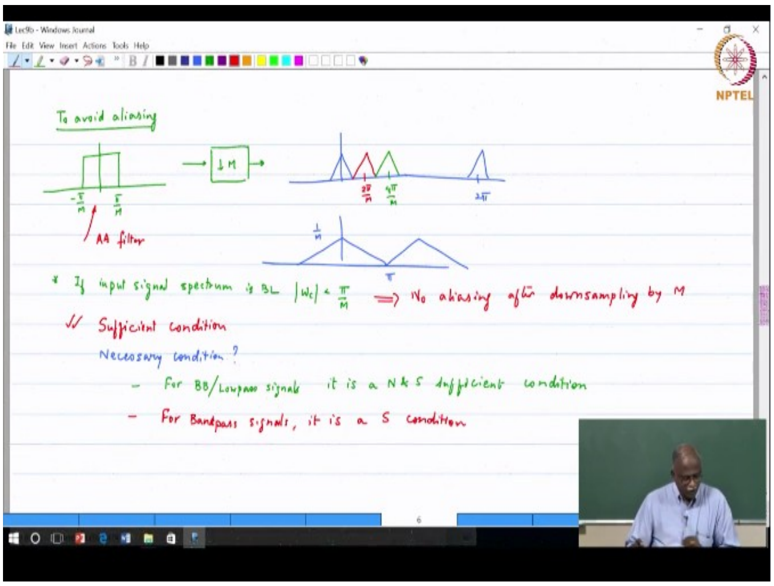

So to avoid aliasing that would be one of the important things that we would want to do in multirate signal processing because aliasing is a nonlinear distortion removing it is not easy. So therefore we want to be careful. So one of the things that we would do for to avoid aliasing is to pass my input signal through an ideal low-pass filter –pi over M to pi over M. In that case, I am guaranteed that there will not be any signal beyond pi over M.

And then when I pass this signal through a downsampling by a factor of M then the output spectrum before we do the rescaling because I have only pi over M, then remember there will be shifted copies at 2pi over M, at 4pi over M and so on all the way to 2pi okay. Now after doing the rescaling then what we have is something that goes from 0 to pi, this is pi, 1 over M okay. So that is the spectrum.

If the input spectrum signal spectrum is band limited to pi over M that is an observation. If input signal spectrum as we have shown is band limited to pi over M that is omega c is less than pi over M that is the cutoff frequency that you have used or whatever is your band limit of your signal either through the band limiting process of the filter, then there is no there is no aliasing okay, a very important result.

And just like we used an anti-aliasing filter in the process of sampling, what did we do? We used an anti-aliasing filter to make sure Nyquist property would be satisfied. This is exactly the digital version of the anti-aliasing filter. So this is the anti-aliasing filter. So it is intuitively very satisfying to see that whatever we understood from the aliasing occurrence of aliasing in continuous time and we sample the signal that same thing is also present when we do sampling rate conversion in the digital domain okay.

So which basically no aliasing after downsampling by M, one very important observation at this point is that first of all this is a sufficient condition right because we are saying that any time you give me a signal that is band-limited to pi over M I down sample. So clearly this is a sufficient condition because we have not put any conditions in terms of satisfying. The minute you tell me spectrum is the band limited to pi over M this result holds.

So it is a sufficient condition. Question is is it a necessary condition as well? Are there cases where you are not band limited to pi over M and you still do not incur aliasing in the process of downsampling and the answer is yes that occurs with bandpass sampling okay. So basically it is not a necessary condition; however, you can qualify it with the following statement that if you are dealing with not if for baseband low-pass signals okay.

So for BB stands for baseband or low pass signals okay signals of the type that we have shown low pass signals, it is a necessary and sufficient condition. If it is a low pass signal, it has to be band limited to pi over M otherwise you will run into aliasing. So it is a necessary and sufficient condition okay. On the other hand, if you are dealing with bandpass signals, for bandpass signals that means you do not have a signal content all the way to the low frequencies.

For bandpass signals, it is a sufficient condition. It is not a necessary condition. For bandpass signals, it is only a sufficient condition; it is a sufficient condition okay. So very, very important maybe another way of visualizing it is as follows.

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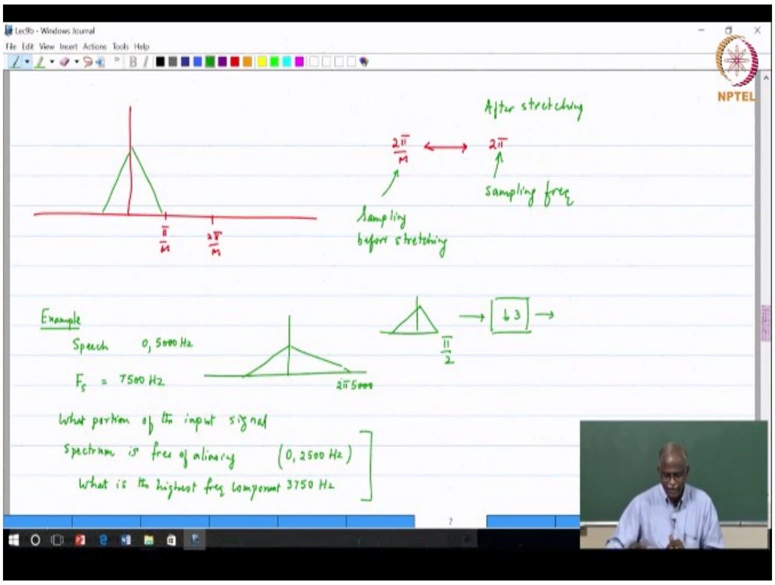

The way to look at it is look at just on the frequency axis; on the frequency axis here I have pi over M, 2pi over M and so on right. Now the before stretching if this is the spectrum after stretching 2pi over M will go to 2pi right, this is after stretching and if this is what we refer to as the sampling frequency after the down conversion process okay. Now the same frequency before stretching is this one.

So this is the sampling frequency or what corresponds to the sampling frequency before we have done the stretching okay and if 2pi by M is my sampling frequency then my input signal better be limited to pi over M for Nyquist rate to satisfy Nyquist rate okay. So again you can visualize it in multiple ways, whatever is the way that you are comfortable with that I would recommend you to do that okay.

Very quick question, very quick example of what often of something that translates into the exact same scenario that we have looked at, so consider that you are looking at speech data which has spectrum in the range 0 to 5000 hertz okay. So spectrum frequency continuous time frequency is 2pi times 5000. Now the sampling frequency Fs is equal to 7500 hertz. Now clearly this would be a case where you are not satisfying the Nyquist criterion.

Now what portion of my spectrum, what portion of the input signal spectrum is free from aliasing? Aliasing will occur, it is going to affect some portions of my spectrum which portions of my input signal spectrum is free of aliasing okay and maybe it is you are seeing it right away but please do draw it and make sure that you are comfortable with it, is free of aliasing and you should find that the answer comes out to be zero to 2500 hertz is the portion of the spectrum that is free of aliasing.

After the process of sampling, what is the highest frequency content in your signal? Now you may say why you are asking me the question, if there is a frequency content all the way to 5000 hertz wait, wait that was before you sample, after you sampled if 7500 was your sampling frequency, it is 3750. What happened to those frequencies above 3750? They got aliased.

And you know which portion of the aliasing, which portion of the spectrum they affected? They happen to have affected from 2500 upwards of the spectrum. So what is the highest frequency component okay? Please make sure that you are comfortable with these two results and basically this corresponds to the case where you have a spectrum digital spectrum all the way to pi by 2 downsample by a factor of 3.

And I think this is an example we have looked at in the process of today's lecture where we said that yes there will be aliasing and what portions will be affected and this exactly maps to that if you work out the details okay, so let me stop here.