Multirate Digital Signal Processing
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Lecture – 07 (Part-2)
Time Scaling-Upsampler and Downsampler - Part 2

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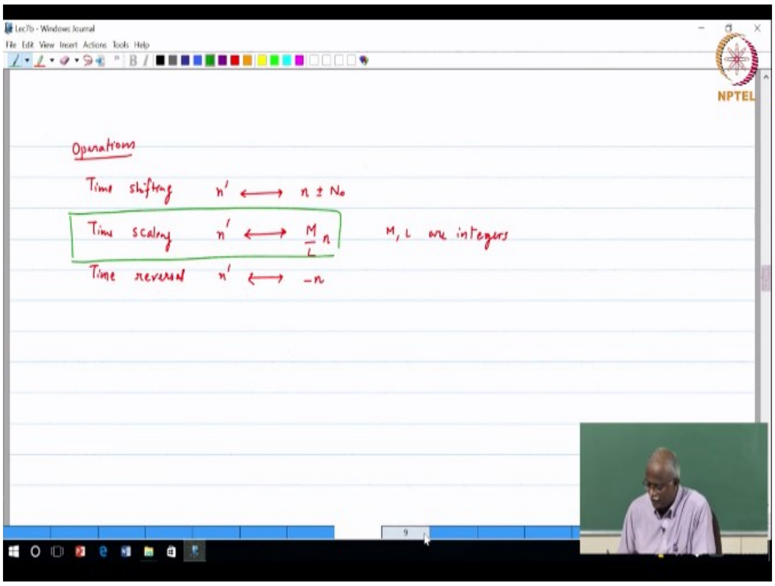

Okay so let us move on to the new material for today, let me just mention one more point. So the various operations that we have looked at which is the reason we are repeating this is because one of the operations is what we are going to be focusing on. One is time shifting, time shifting says that you are going to change your time access to n prime which maps to the original index + or – N naught where N naught is an integer.

Second is time scaling where we take the define a new index n prime, once we in the notation it will actually come as n but think of it as a new index that is there which is n prime. This corresponds to M by L times n, M, L are integers. So I can do a rational transformation of the C index that will be M by L and of course the third one is the time reversal where we said that a new index will map to -n okay that is the time reversal.

And these are the 3 basic operations we are going to be focusing very specifically on that which gives us an integer change in the sampling rate. Again, I am assuming this is material that you would have seen before but nevertheless good for us to make sure that we are comfortable with the, so time scaling.

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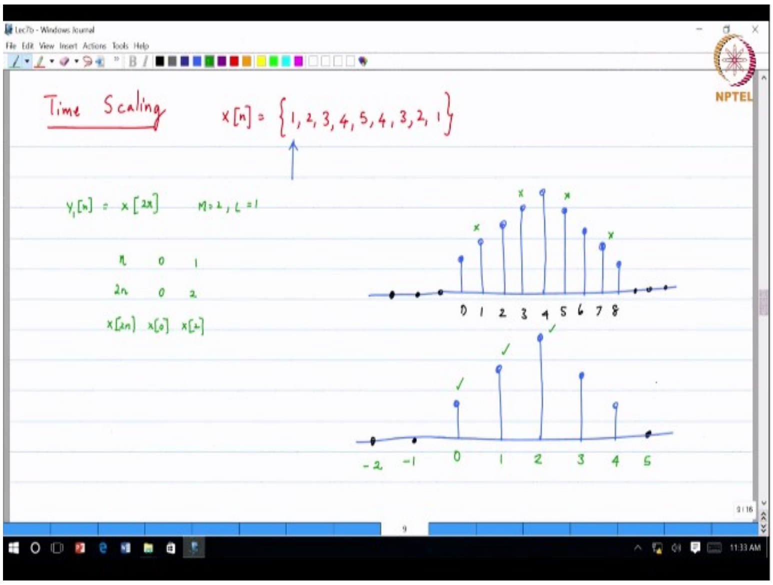

Given a sequence 1 2 3 4 5 4 3 2 1, we are interested in a time scale version y1 of n is equal to x of 2n okay, remember M by L, in this case L is equal to1, so this is a case where M equal to 2, L equal to 1 transformation and we are asked to find out. So the easiest way is to write down your values of n, values of 2n and then correspondingly you find out what is the value of x of 2n, so x of 2n okay. So index 0 corresponds to 0, this would be x of 0, 1 correspond to 2 would be x of 2.

So that means in the new sequence x1 got dropped, so this sample got dropped, so 0 is present, the sample number 2 is present but their indexing has now become different. So it becomes 0 and 1. So that is the new time index. So basically we are doing a down sampling or a reduction in terms of the time scaling. So basically we have called it as x of 2n which can be then you can see that every other sample has been removed.

And the remaining samples are present okay, so the x of 2n you can think of it as sometimes called compression because you are reducing the number of samples and this is what is called time compression as well, in the time scaling terminology this is called compression. Now the counterpart of compression, again compression basically says I am throwing away some samples from my original sequence.

Now is there a risk with this? The answer is very much yes and the risk is that if there was an underlying waveform that you had sampled by throwing away some samples it is basically like sampling it at a lower rate which means that you may have ended up violating the

Nyquist criterion. So that is something that you want to be a little bit careful about. Again, that is where the downsampling, that is one of the key things that we will want to keep in mind with downsampling.

You should not go down to a rate where you will introduce aliasing into your signal but we would like to keep this as one of the tools that is available to us.

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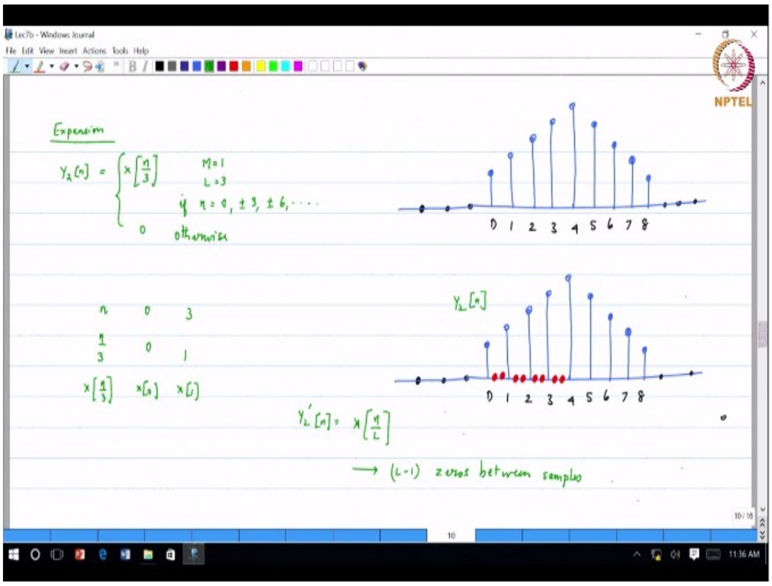

On the other hand, the time expansion, time compression, the other one is time expansion. This is defined as the following y2 of n is equal to let us say as an example x of n by 3. So here we have M equal to 1, L equal to 3 in our scaling of the time and this is equal to this if n equal to 0 +-3, +-6, in the general case it will be 0, +-L, +-2L and equal to 0 otherwise okay. So if you again like the other case where you would say n by 3 x of n by 3.

So this would be 0 0 x of 0, 3 3 1 x of 1 okay, so what happens to those indices, in between they are labeled as 0. So the new sequence this would be y2 of n, this would be the new sequence and this is a sequence where there are 0 valued samples that have entered the expression okay. So between every set of samples you will have L-1 zeros, if L is the sampling, so in the general case if you have a y2 dash of n equal to x of n by L.

So you will this would lead to L-1 zeros between every set of samples okay, again you are familiar with this, just wanted to refresh the notation what we mean by the compression expansion part of it between samples okay. So this more or less sets the stage of saying okay we understand the basic time-domain process of compression and expansion. Now how do we take it from there to become a complete set of tools that we can work with and then.

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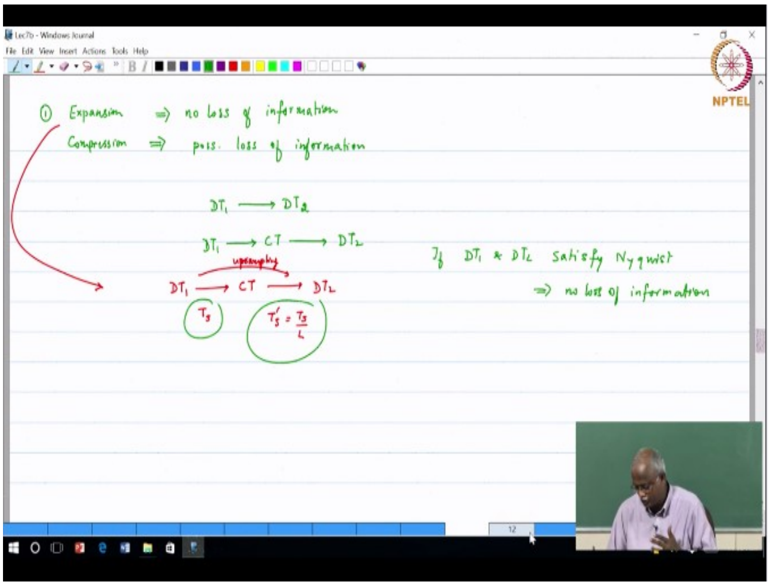

So based on what we have already observed so far okay, so let me just ask a question and then move on to the formal mathematical representation of that. So expansion, sampling rate expansion, I want to ask 2 questions, first is an observation. Expansion basically inserted zeros right inserted zeros. So there is no loss of information, no loss of information, you agree? I cannot lose information in the expansion process.

However, if I do have compression there is a possibility of loss of information because I may or may not be able to recover the lost samples. Am I correct? Possibility of loss of information, possible loss of information. Is it always true that I am losing information? No. When will I not lose information? If the downsample signal is still has got sufficient number of samples, so the information that is a correct observation.

So basically I went from discrete-time with sampling rate 1 to discrete time at sampling rate 2 where the sampling rate 2 was smaller than sampling rate 1 which basically means that why did I not lose information because from discrete-time 1 I could have gone to the continuous time without loss of any information and from there I would go to discrete time 2, you can think of it as a resampling process.

Now of course in this second step if you had not taken care of anti-aliasing, you may have introduced aliasing and therefore you know all bets are off on what you have done to but so compression may lose information, the only time it will not lose information is that if you have retained the Nyquist rate and the second one also retains is still satisfies Nyquist. So both if DT1 and DT2 satisfy Nyquist criterion, then there is no loss of information, otherwise there will be okay, Nyquist then no loss of information okay.

On the other hand, the expansion part of course there is no risk because I have not lost any samples. What is the counterpart of this? I had discrete time at sampling rate 1, also can visualize as moving to continuous time going to discrete time sampling 2 rate 2 and this is through and you can go from here to here through the process of upsampling right upsampling.

Again, we will expand what is the difference between expansion and upsampling or interpolation, we will expand that. So this is effectively like saying I had a sampling period Ts okay which I used to reconstruct the continuous time signal. Now I am going to resample it at Ts prime which is at Ts over L okay assuming that my original sampling rate did not have any aliasing then of course the higher sampling rate will not have aliasing.

Because you are sampling it at a higher rate, so clearly there is no loss of information because you should if you could have reconstructed with the original discrete-time samples, you will definitely be able to reconstruct with the new samples. So the observation is that expansion; no loss of information guaranteed, compression; there is a potential loss of information we just need to be careful that we do not run into that situation.

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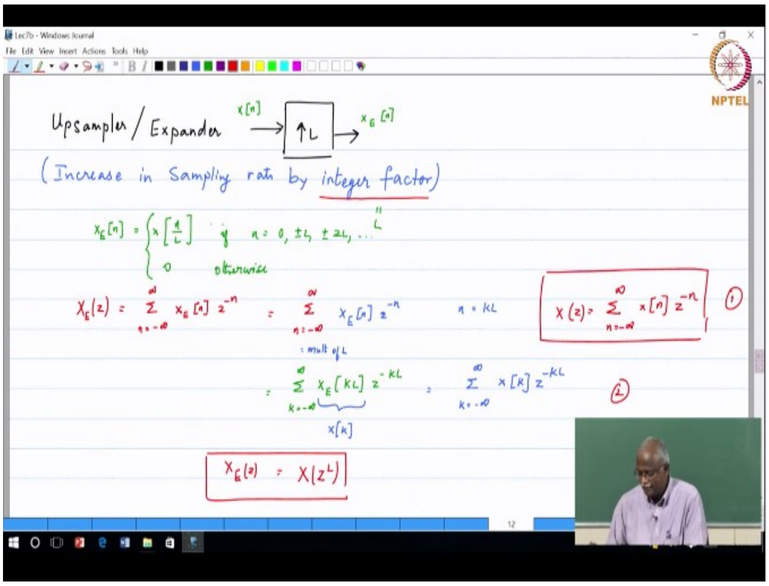

Now a very interesting theory behind multirate signal processing but basically this is our entry point into multirate. So we will take the upsampler or the expansion block indicated by a square box or rectangular box with the up arrow written with an upsampling factor. Now this is an increase in sampling rate by an integer factor and the integer factor is equal to L okay. I am upsampling by this.

So the notation would be if this is x of n, this is xE of n where E stands for the expanded version, the higher sampling rate and the expression in terms of the time domain already we have described it, let us just write it down. This is equal to x of n by L if n=0, +-L, +-2L and so on equal to 0 otherwise okay. That is my upsampler or my expander of my signal, again no loss of information because the original sequence is preserved okay.

Now very important that I now look at it in the frequency domain, so the next few minutes we would like to obtain the frequency domain representation and please follow along because the notation and the expressions are very, very important for us in our, so we are trying to get the frequency domain or the Z transform, Z transform if I evaluate it on the unit circle gives me the frequency response of frequency transfer via transform.

So I will write it in terms of the Z transform, XE of Z is summation, n going from -infinity to infinity, xE of n Z power –n, that is the standard definition of Z transform. Now I can write this with if the following condition please follow along n equal to -infinity to infinity but I am going to impose the condition that n is equal to a multiple of L because those are the only nonzero samples xE of n all other samples.

So without loss of generality I can impose a constraint on the index of summation that it is a multiple of L, xE of n Z power –n okay, hope you are comfortable with that, I have imposed a condition which does not violate anything in terms of the generality. So anytime you want to impose something that is a multiple of L, it is better to do a change of variables. So n equal to K times L.

Let us say if you want to do that then this summation now gets rewritten as K equal to -infinity to infinity, xE of K times L Z power –KL, wherever there is n I am going to replace it with KL and K will go from -infinity to infinity because that will then capture for me all the multiples of L that are present. Now XE of KL by definition is=X of K. So this is nothing but summation K equal to -infinity to infinity X of K Z power –KL.

And going back to our basic definition X of Z would be summation n equal to -infinity to infinity x of n Z power -n given this we now can say that XE of Z comparing 1 and 2, expressions 1 and 2 we say that this is equal to X of Z raised to the power L okay, very compact expression. It actually gives us a tremendous amount of tools, an insight to work with, that is what we would like to focus on and build up.

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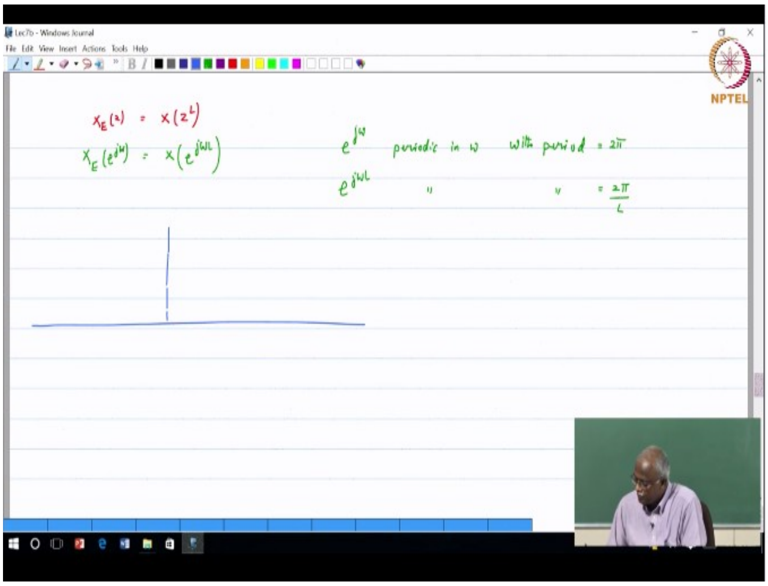

So hope this straightforward derivation I am sure you would have done it previously but the insights are very, very important. So the expression is that XE of Z is equal to X of Z power L. If I were to write it in the Fourier transform, XE e of j omega is equal to X of e of j omega L okay. Now a very important question, e of j omega is that a periodic function? Yes, it is periodic in omega, periodic in omega with period equal to 2pi, e power j omega L by the same token is periodic in omega.

Because you have just multiplied omega by an integer factor did not affect the periodicity properties, this is valid with period equal to 2pi by L. So the period of the function has shrunk or in other words there is a very interesting change in terms of the property of the underlying the function e power j omega okay. So what does this mean and how do we understand it? So we have understood the time domain, what happens with the expansion. Now the frequency domain interpretation equally important okay.

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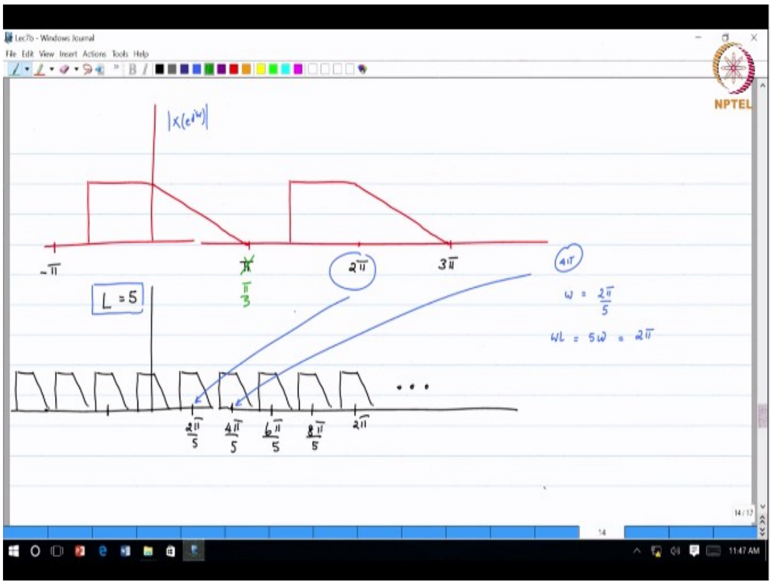

So let me show you a figure that will help us. Supposing this is my expression for X e of j omega magnitude, it is from –pi to pi and it has got a spectrum which is not symmetric, so which means that x of n is complex. Nevertheless, it is a well-defined discrete time sequence, complex sequence for which the spectrum looks like this. So I have now done the upsampling or the expansion by a factor of 5, L equal to 5, what does it do to my spectrum?

So basically what I would need to do is if there was a frequency or omega equal to 2pi by 5 okay, omega times L, omega times L would be equal to 5 times omega that would be 2pi. So whatever was at 2pi will migrate to 2pi by 5 right, that is for my expanded signal because of the scaling by a factor of 5 this is where it is going to go, whatever happened at 4pi, whatever was there at 4pi that comes to 4pi by 5.

And eventually you will find that there is a so what has happened as far as the spectral interpretation is concerned is that you can think of the spectrum of X e of j omega, all you did was changed the frequency axis. So you just said okay this is no longer pi, it is going to be pi by 5, this is no longer 2pi, it is going to be 2pi by 5, so basically what you did was just a rescaling of the frequency axis okay.

So the upsampling part is very, very straightforward, it just says all you did was rescaling of the frequency axis which again is not very surprising because we did say that we did not make the observation that the periodicity is going to now fold by a factor of L, is that true? Yes, now the spectrum looks periodic with period 2pi by 5 and other than the scaling of the frequency nothing else has happened.

Basically, you got all of the information, information is contained you just have multiple copies of the original signal okay.

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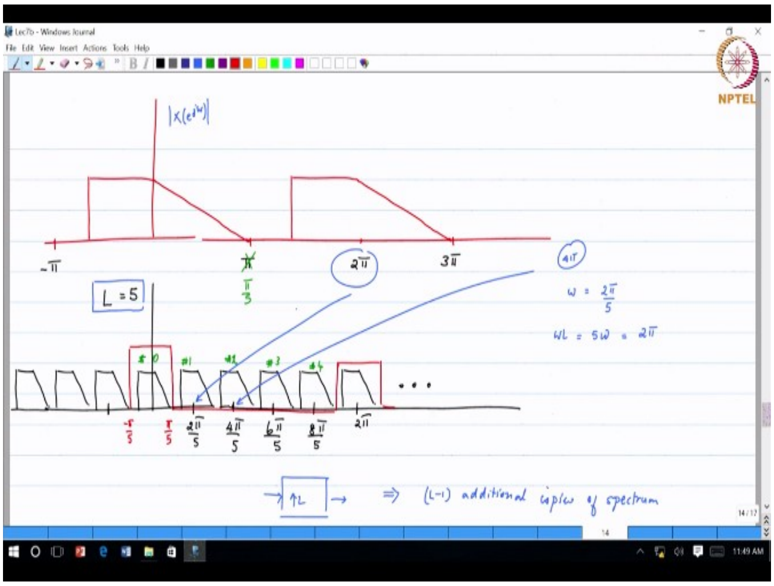

So observation if I am doing an upsampling by a factor of L, expansion by a factor of L, this means that there are the 2pi mapped on to 2pi by L, so until you get to 2pi so effectively you will get L-1 additional copies of the spectrum, additional copies of the spectrum okay. So we can maybe even indicate it here, this is copy number 0, that is the original it has been shrunk in terms of the spectral, this is copy number 1, additional copy number 1, additional copy number 2, number 3, number 4 and after that the periodicity kicks in okay.

So the important question is if I were to look at it in the context of information in the signal, do I need the copies number 1 2 3 4, the answer is no because they are exact replicas of signal number 1, so what you would permit the process to do is to remove and how will I remove those, I will apply a low-pass filter which will go from –pi by 5 to pi by 5 and basically remove the rest of the spectra until I get to 2pi okay.

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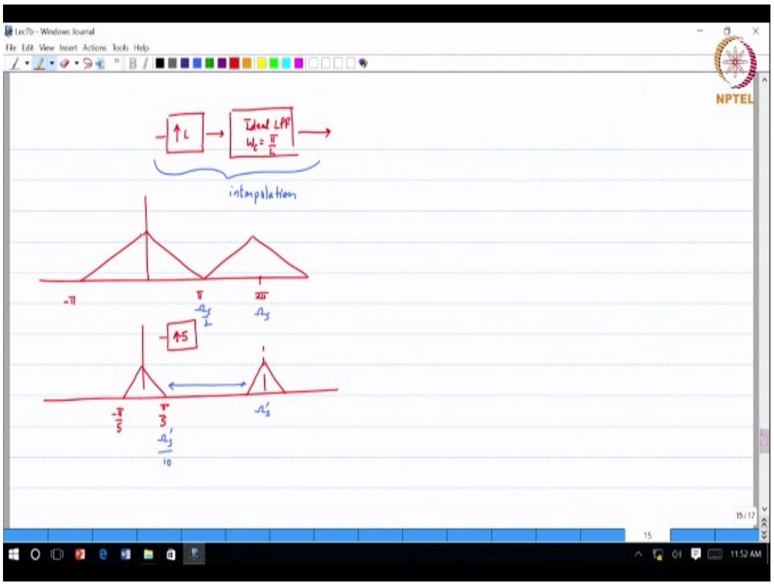

If I removed through the process of filtering, the unwanted spectra then what do I get in terms of the visualization, I now have upsampling by a factor of L followed by a low-pass filter. Let me call it as an ideal low-pass filter that means it has got a brick wall response, the cutoff frequency is pi over L. If I do this then what I will, what I would have achieved is if the original spectrum had a representation of this kind from –pi to pi, the period of 2pi, basically it was less as Nyquist sample signal.

What I would have done is the following, if I had done upsampling by a factor of 5, pi and -pi would have shrunk to pi by 5 to –pi by 5, there would have been 4 additional copies but those got removed and what I am left with is the repetition with the period of 2pi okay. So there is I have created a gap between the copies of the spectrum. Now this is exactly the same thing that would have happened if you had taken the original band-limited signal and sampled it at 5 times higher than Nyquist.

Because what happens when you have sampled it at Nyquist you get the spectrum exactly from –pi to pi, so this because this is if this is omega s, this will be omega s by 2 now but if this is omega s is your new frequency then this is because it is 5 times higher this will be omega s by 10 okay because 5 times higher, it should and this this would be omega s by 10.

So over sampling comes in very naturally in from our intuitive understanding of over sampling comes only if you have the low-pass filter included in the expression. So the combination of these two is what we call as interpolation. Interpolation is the same as sampling at a higher rate, you now have full now no zero valued samples, all have nonzero samples.

And they are at a higher sampling rate than the Nyquist rate and therefore you get this. So we will stop here but again look ahead to the representation of the compression operation. The corresponding, the other, the reduction in the sampling rate and we will then look at the frequency representation and derive the tools by which we can do upsampling and downsampling. Notice that the filtering will come as a very important part of upsampling.

It will also come as an important part of the downsampling process which we will see in the next class. Thank you.