Multirate Digital Signal Processing
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Lecture – 07 (Part-1)
Time Scaling-Upsampler and Downsampler - Part 1

Good morning. We begin lecture 7. We will do a quick recap of lecture 6. I would like to do it in the form of some examples and then build on that to complete our discussion on the discrete time processing of continuous-time signals. Last time, we looked at a low-pass filter, we also talked about a differentiator, just want to make a few comments about the differentiator and look at two more examples.

And after that we move into the time scaling where you look at either increasing the sampling rate or decreasing the sampling rate.

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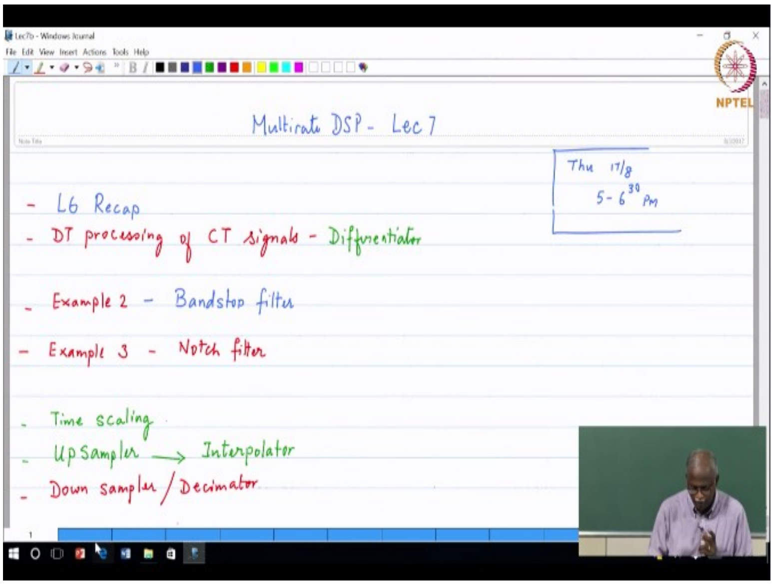

And always keeping in mind that there is an underlying possibility that you might run into aliasing which is a distortion that you cannot remove easily, so that is something that you just want to be careful about okay.

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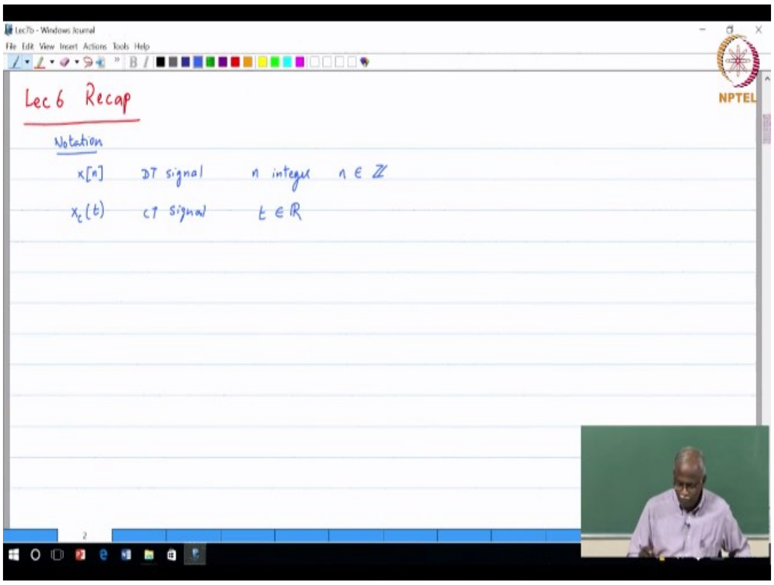

By way of review more than repeating what was done in the last class, let me just mention the notation or the convention that we are following. Again, this is standard in Oppenheim and Schaffer. Anytime we are talking about a discrete-time signal, we use the square brackets. So if you see that assume that it is a discrete-time signal. The argument is an integer okay so n is an element of the integer set.

And likewise the curved brackets always denote a continuous time, so that is our convention and the argument in that case will be something that belongs to the real valued set. So we will always follow this notation, no deviations. The only time when there is a little bit of confusion possibility is that you go from discrete-time system with sampling rate 1 to another discrete-time system with sampling rate 2.

Both of them will be denoted as n and we have to interpret it in of them but the notation of square brackets, curved brackets will always be, will always be retained okay.

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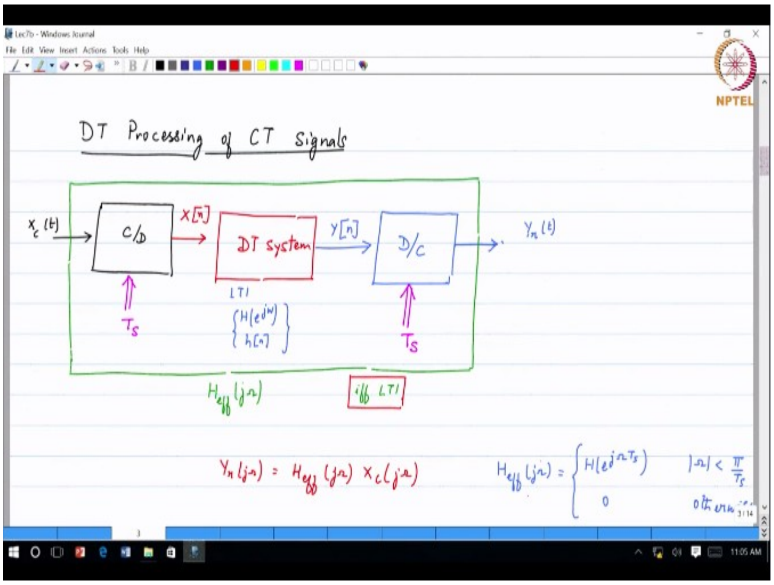

So by way of a quick the motivation for processing a continuous-time signal used in the discrete time domain I think we have spoken about it at length and we are convinced that it is something that will lead to advantages depending upon the application. The key point to note is that the discrete-time system has got a LTI response H e of j omega and that is related to the input output relationship in the continuous time.

And you can think of it as one period of the discrete-time spectrum which is periodic and we are basically removing all of the others and this is provided LTI property is retained.

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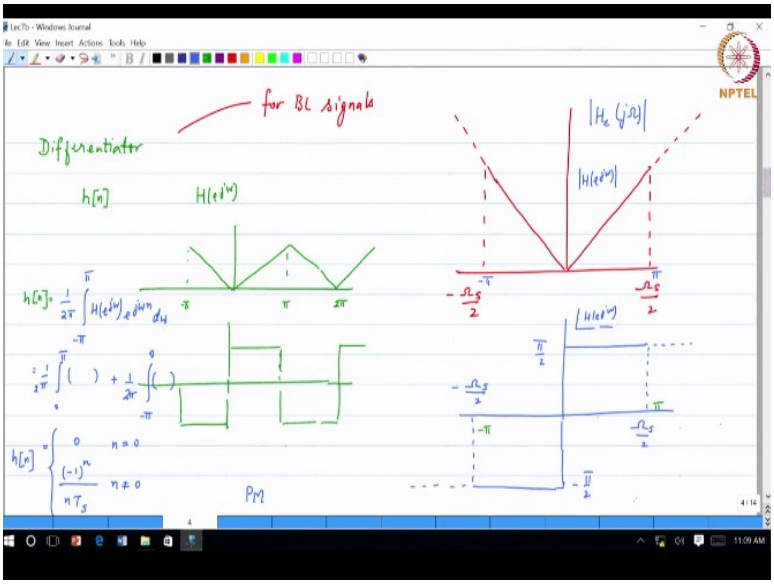

Now in the context of a differentiator, we saw that the magnitude response is something that will increase from 0 going out towards the higher frequencies and the phase is a +pi by 2 for positive frequencies, -pi by 2 for negative frequencies. If we have a band limited signal, then it is perfectly okay to band limit the response of your filter as well because anyway you do not have signal in the outside of that range.

And therefore there is an advantage to doing it in this fashion. So just to complete this problem let me just request you to so this is magnitude H e of j omega, this is argument or angle H e of j omega that is the so basically in order to do that we would have to change it over to discrete-time. So this would become pi, this would be -pi and then we would have magnitude H e of j omega.

And similarly this this would be to I have to change colors here, this would be pi, this would be -pi and that will be the argument. So I would like to obtain the impulse response of this band-limited differentiator, it is a band-limited differentiator because when I map it into the equivalent H effective of j omega that that will show that your differentiator was only differentiating over a finite range of frequencies and not over all frequencies.

So basically the H e of j omega, H e of j omega will have a response that looks like this. It will go from –pi to pi. This is –pi to pi and then it is periodic, so you will see the repetitions with respect to 2pi okay, so that is that and the phase is again defined in the range –pi to pi and of course it will also repeat itself okay. So you can, so given this I would like to obtain the impulse response h of n that would be nothing but the inverse Fourier transform.

So it would be from 1 over 2pi times –pi to pi H e of j omega e power j omega n d omega okay. So again this is just to reinforce the fact that we can represent these once you given the spectrum of the discrete-time filter, we can get its impulse response and basically once you have the spectrum then we can also relate it to the continuous time H effective. So this you would have to split it as two integrals -1 over 2 pi 0 to pi something plus 1 over 2 pi -pi to 0 okay, please verify.

And this you should get to be equal to 0 if n equal to 0and equal to -1 power n by n times Ts, if n not equal to 0**,** this is the expression for h of n. Now you may be wondering how do I design such a filter. It turns out that if you use the Parks-McClellan method and there are 4 types of linear phase filters, type 1, type 2, type 3, type 4. If you pick the appropriate type, specify the magnitude response, you will get a filter that pretty much looks satisfies that.

And it should have impulse response given as follows. So the what are we leveraging, leveraging the fact that we can design discrete-time systems with a lot more flexibility, a lot more ease then we can do analog systems and always making sure that if the input signal is band limited then I am always permitted to sample it, take it into the sample domain, use the discrete-time toolbox that is available to us and then bring it back into the continuous time domain okay, so let us move on.

So the next application or just so that you start thinking along the lines of what we are trying to do.

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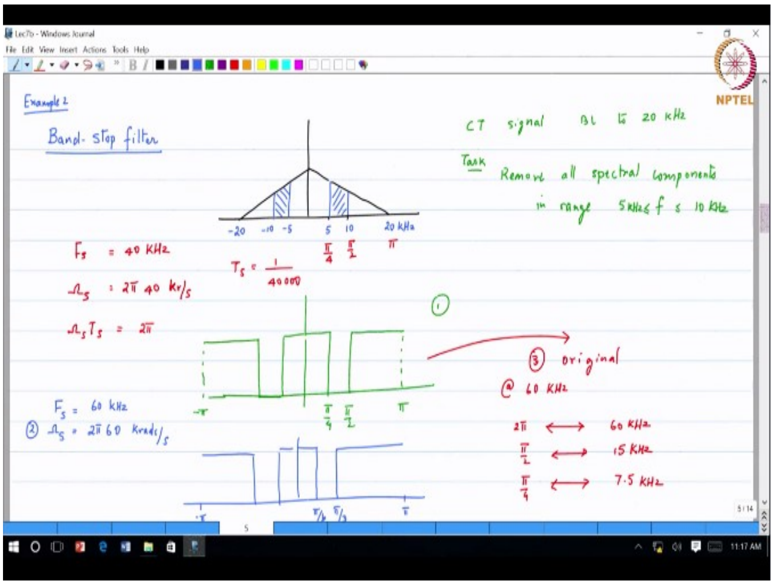

So we are given a continuous-time signal, so here are the data that is given. It is a continuous-time signal with spectrum which is band limited. So band limited to 20 kilohertz, so I am assuming that you will be able to interpret that there is a spectrum from -20 to 20 that is the input spectrum. So the task is as follows.

Remove all spectral components, all spectral components in the range 5 kilohertz to 10 kilohertz, components in the range of f greater than or equal to 5 kilohertz to 10 kilohertz which is what we have shaded in blue to show that that is the portion that we want to remove and of course we would like to use a discrete time filter to satisfy that. So here is the sampling rate, decided to do it exactly at Nyquist just for illustration 40 kilohertz okay.

So now I need to design a digital filter that will satisfy the corresponding conditions. So if my sampling frequency is 40 kilohertz okay, so this basically well I should strictly speaking should write it as 2pi times 40 kilo radians per second okay, so that strictly that is omega s but when I convert it into the discrete time I would have to multiply omega s by Ts and my Ts is 40 kilohertz okay.

So my sampling frequency is 40 kilohertz, so maybe just to be consistent let us do the following. So let me define sampling frequency and this is the continuous time frequency okay. So sampling frequency is 40 kilohertz, the corresponding sampling frequency in radians per second is 2pi times 40 kilo radians per second okay. So this tells me that the sampling period is 1 over 40,000 okay.

So if I convert omega s times Ts that will map to 2pi right. So if this is 2pi basically 40 kilohertz is 2pi, 20 kilohertz would be pi, 10 kilohertz would be pi/2 and 5 kilohertz would be pi/4. So the corresponding discrete time filter would be a band stop filter, I would pass all frequencies from 0 to pi/4, cutoff the frequencies from pi/4 to pi/2 and then allow the remaining frequencies to pass through okay.

So this would be pi/4 pi/2 to pi, so if it is a real-valued, it will be symmetric. So I get another stop band on the other side and this is at –pi, -pi so basically this would be the low-pass filter that we are to design okay. So this is observation number 1, so this is how you would design a filter given the constraint. Now just so that we quickly make a couple of more observations. Second observation, if I had changed my sampling frequency to 2pi times 60 kilohertz 60 kilo radians per second okay.

So which means that I have actually sampled it higher than Nyquist which is at 60 kilohertz sampling rate, sampling period would be closer to each other. Then, we would have to relook at what would be the frequencies that we would have to design. It is still a low-pass filter sorry it is still a band stop filter. The pass band edge goes from 0 to 5 kilohertz, now 2pi corresponds to 60 kilohertz.

So 5 kilohertz corresponds to pi/6 okay and 10 kilohertz corresponds to pi/3 and of course you will have a much longer pass band all the way to pi and you can draw it on the other side as well. So again depending on how I have chosen my sampling rate, I would have to design so there is a link between the design of the discrete-time system and the sampling rate that you have chosen that is the second observation that we make.

Now if I had kept the original filter, this is observation number 3, so if I had kept the original filter that means I had retained this filter, so keep the original filter which is cutoff from pi/4 to pi/2 that is the band stop portion but I had sampled my signal at 60 kilohertz okay, so which portion of the spectrum did I remove? You would have to recalculate, so 2pi corresponds to 2pi in the discrete-time domain corresponds to 60 kilohertz right.

I am cutting off from pi/2, pi/2 corresponds to 15 kilohertz, pi/4 will be one half of that 7.5 kilohertz. So this if I had taken the original filter with the pass band up to pi/4 and then another pass band from pi/2 to pi, I would have sample it at 60 kilohertz, I would have actually done a removal of a portion of the spectrum but I removed a different portion of the spectrum okay.

So what this example is to illustrate is that your sampling period is important, your design of the digital filter is important and there is a whole lot of flexibility that you have once you go into the digital domain okay. Any questions on this example should be yeah. **"Professor - student conversation starts."** No, no, this is a design of a digital filter. So when you specify your digital filter if you remember in DSP you will always specify from –pi to pi or from 0 to 2pi, you would have to specify your digital filter in that range.

This is the specification of the digital filter, so your question is very valid because the spectrum of the signal does not go all the way to pi but it goes to some portion less than pi because you are sampling but the digital filter result is designed so that it has a frequency response from –pi to pi right. So the digital filter design and specification does not strictly depend on even if I change the signal for just for discussion purposes, the digital filter remains specified from –pi to pi.

So I would have to for completeness show this portion of the spectrum as well okay. So this would be –pi to pi okay good. **"Professor - student conversation ends."** Any other questions? Okay.

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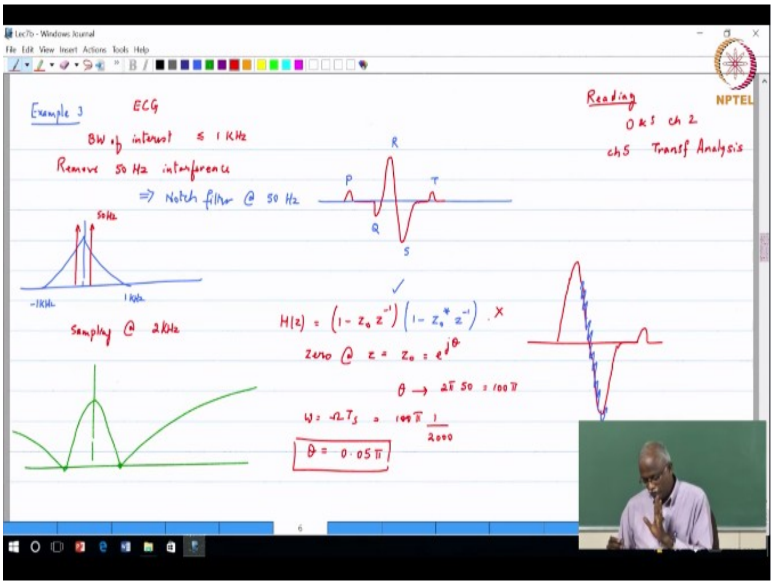

Just as we said that one of the things that I am we are assuming that you are familiar with in terms of the reading assignment, so let me just mention it here reading assignment we said Oppenheim and Schaffer chapter 2 where we talked about all the different types of sequences and their manipulations or transformations of sequences. Similarly, there is another chapter which is chapter 5, which is a chapter which we call as transform analysis.

So this is where we talked a lot about the design of filters, the response of LTI systems, where to put the poles and zeros, again we will not be revisiting that but except to use that as a tool in our multirate discussions. So here is an example which probably gives you a chance to review or recollect or refresh yourself regarding some of the contents that you would have covered in chapter 5.

Again, I would request you to brush up on that in case you are not very familiar with that okay. So here is a very practical example something which we encounter all the time. We are looking at the ECG signal okay. ECG signal you probably have not had any need to take it but if you have seen an ECG signal, basically it has some pattern that looks like this. The doctors are looking for 5 points, p, q, r, s, t and they look at spacing frequency, the heights and other things.

And then they are able to interpret that into saying a normal heart, something that has got irregular behavior, so lot of information that is obtained from there. So the bandwidth of interest, bandwidth of interest in the ECG signal let us say is in the range of 1 kilohertz, 0 to 1 kilohertz. So it is basically the information content is in the 1 kilohertz range. So this is the signal that you would easily sample at 2 kilohertz probably a little bit higher if you want to be safe. Now there are impairments that that do happen, one is noise present.

So basically you are interested in signal that is in the 0 to 1 kilohertz range, there is noise outside of 1 kilohertz. So you would have an anti-aliasing filter that would remove all the unwanted noise. So anti-aliasing filter takes care of the noise that is outside but there is one in band impairment which is very, very common in ECG signals that is what is shown in the blue on the right hand side.

Anyone guess what is this? 50 hertz interference, so basically there is a superposition of 50 hertz which is your AC supply and occasionally there could be harmonics of the 50 hertz, you may have 100 hertz, 150 hertz as well. So again let us assume that we have a 50 hertz interference and the task is to design a filter that will remove the 50 hertz but will not mess with the rest of the spectrum.

Basically, you want to preserve as much of the signal both in the time domain and in the frequency domain as possible. So the frequency domain interpretation of this problem is that we want to remove the 50 hertz component, 50 hertz interference and in the process you will remove the 50 hertz content in the signal as well but you are saying make it as narrow as possible so design as tight a band stop filter or a notch filter.

So this would actually then say that you have to design a notch filter, notch filter act with a notch at 50 hertz and make it as sharp as we can okay. So let us do two things, one is let us just draw a spectral interpretation of the problem, we have some information signal up to 1 kilohertz. So that is +-1 kilohertz okay and we have a 50 hertz component which is very low frequency compared to the bandwidth of the signal.

There is a 50 hertz component, so basically this is 50 hertz component and so that will show up, since it is a sinusoid it will show up as an impulse or Dirac Delta and in case harmonics are present you will have to draw that. Let us only take the case of the 50 hertz okay, so we will assume that we will sample at 2 kilohertz, sampling at 2 kilohertz, we would have to design a notch filter for this.

So transform analysis says that if I have to remove a sinusoid completely, I have to put a 0, my transfer function must have a 0 at that frequency. So my H of z, my discrete-time filter's transfer function must have whatever else it has it must have a 0 on the unit circle at that frequency. So basically it must have a transfer function which is 0 at z equal to z naught okay and z naught is a 0 on the unit circle.

So we will call it as e power j theta and what is theta? Theta corresponds to the 0 location must correspond to 2pi times 50 hertz, so that that is 100pi if you would look at it in the continuous time. Now this is actually getting sampled into the discrete time domain, so omega is equal to omega times Ts. So that would be 100pi times the sampling frequency which is 1 by 2000.

So this in the discrete-time basically says that theta must be at 0.05 pi, 0.05 pi so pi by 20 whatever that you just can locate. So basically it is very close to the so yes I have designed this and this I have located the 0, I do not want a complex impulse response. So what do I do? 1-z naught conjugate times z inverse right, that will give you a real impulse response, very good. So now I say can you please sketch for me the spectrum of your notch filter okay.

So the 0 is located here and here, filter does a reasonably good job but it starts to droop right from there and then it does this right, that is the notch filter because that is the response if you so the notch filter took out the corresponding component but it also took out with it lot of the other portions of the signal you may actually have ended up doing a lot of damage to the rest of the spectrum as well, so clearly this is not the right option okay, that is not the right option.

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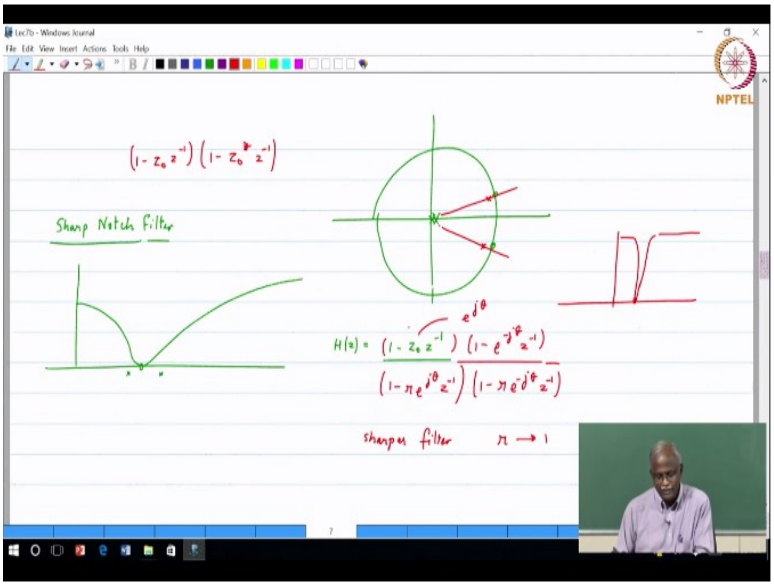

And before we even go further let me just also ask you a couple of related points connected to that. So if I have 1-z naught z inverse 1-z naught conjugate z inverse okay, these are the 0s which we had looked at. So can you tell you what the pole-zero plot looks like just for completeness. So this is the unit circle okay. I have one 0 at z naught, one 0 at z naught conjugate, they will be on two sides, you can locate them okay.

Now is this complete? The 0s are there, map the 0s, poles, are there any poles in the system? z equal to 0 how many poles? 2 poles. So I should not forget that. So these are two poles okay. So this is the transfer function of the system that has the following response. Now obviously it was not satisfactory. So if I want a sharp notch filter. So the intuition from chapter 5 Oppenheim and Schaefer says I have got a 0 so the response is like this.

Now how do I make it sharp? I have to add a pole very close to where I want this, so basically I have to add a pole somewhere you know around that point okay. So yes that is the correct thinking, so I will now design a transfer function which is of the form 1-z naught z inverse divided by, so maybe we will call this as e power j theta okay and you would have to also have another term 1-e power –j theta z inverse okay.

And the corresponding pole 1-r e power j theta, let us say I put it at the same location okay. Now would this work? 1-r e power -j theta z inverse, does this work? So basically what we are saying is we removed these poles, we have if you look at this I have put a pole here and a pole here. Will that work? Does that give you a sharp notch filter? The answer is yes because when you come close to the zero, the zero will dominate it.

It will force the function, anywhere you go further away the pole and zero more or less will match each other in terms of the magnitude response. So we will make the plot, so the frequency response that you will get is you will have a notch at zero but pretty much everywhere else it will have a flat response.

So you will have a fairly sharp notch okay and if you want the notch to be sharper than what it is what you would have to do is to make it sharper filter what you would have to do, you have to send r closer to the unit circle. So it is almost doing a pole zero cancellation. Everywhere else it looks like a pole zero cancellation except at zero. In this form, you can assume that the poles and zeros are matching.

That means there are 2 poles, 2 zeros and the only thing is this is a system that could become unstable because you know if due to quantization effects or something if this pole looks like it has gone on to the unit circle or for some reason outside that behaves like a pole, outside the unit circle the stability would be the issue but by and large this would.

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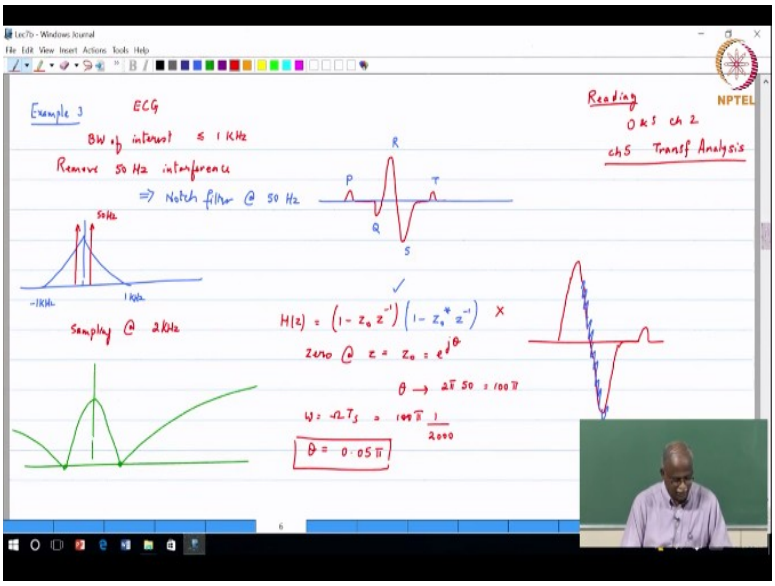

So this would be a system that you would easily design for a removal of that 50 hertz notch and design it very nicely, so that your discrete continuous time signal ECG signal looks clean. Now the advantage of going to the discrete-time domain is that if you found out that the harmonic let us say the third harmonic 150 hertz also is present to some degree to a smaller level but it still has to be removed, you just have to add another set of poles and zeros so that you can remove that notch as well.

So chapter 5 very important for us because that tells us how to design LTI systems to the requirements that we have and then once we are able to do that there are lots of interesting advantages.