Probability Foundations for Electrical Engineers

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Lecture – 08
Examples: Probability Calculation for Equally Likely Outcomes

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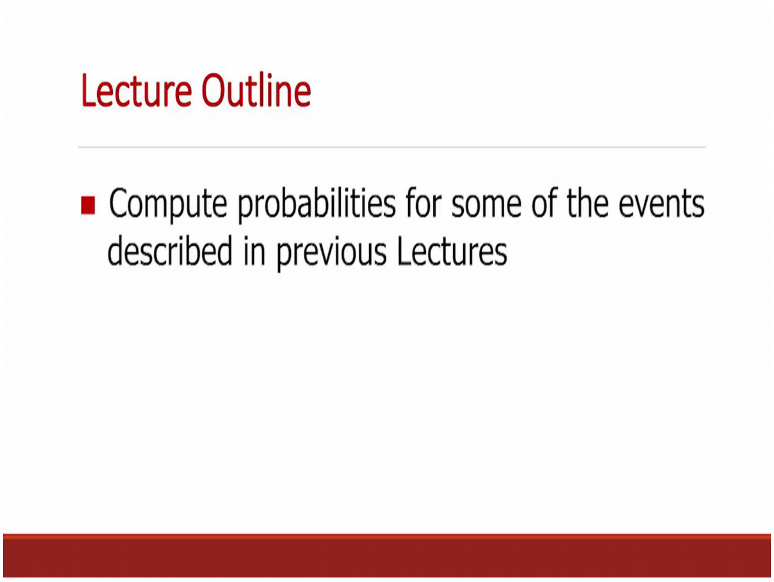

In this lecture we are going to start seeing computations of probabilities for events, once again will restrict our self to simple experiments and define the set of all outcomes, and define some interesting simple events that we can capture. And start thinking of computing probabilities. You must have seen Professor Aravind lectures of how to compute probabilities when every outcome is equally likely in the set of all outcomes in the sample space, all the outcomes are equally likely and how to compute probabilities in such scenarios, it is a very simple calculation you count the number of elements in your event divided by the total number of element elements in your sample space.

We will only see those kinds of examples in this lecture, we will see some simple scenarios and I will also point out how the scenarios can become quickly complicated and you need more advanced machinery to do calculations of probabilities ok so, let us get started.

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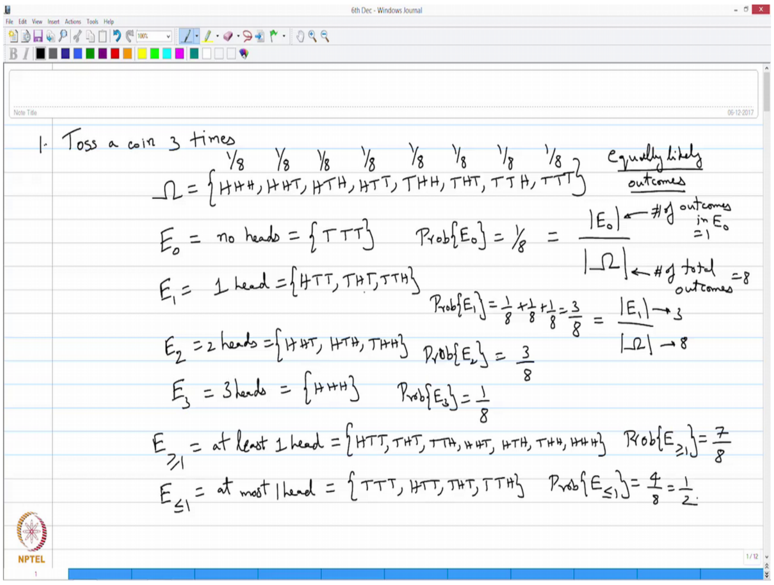

So, the first example we will see will be toss a coin 3 times. Once again we know the sample space for this centricity situation, you can have 8 different possibilities here, let me write it down once again. So, this is your sample space and when you toss a fair coin 3 times very reasonable way to assign probabilities to each of these outcomes is simply to say. All of them have a probability of 1 by 8 each of them has a probability of 1 by 8 ok.

So, that is the most natural way to assign probability. So, this is the kind of scenario that we will do in this lecture mostly, we will look at sample spaces whose outcomes are all equally likely in some sense and then we can go back to our events and start quickly computing probabilities for those events right. So, let us look at the events that I have defined before, I define this event E 0 which is no heads right, this was just simple event which was just T T T, and if you look at what you need to do for the probability of this event it is simply going to be 1 by 8.

So, there are various ways to come to this 1 by 8, like I said for every outcome here I have assigned a probability 1 by 8. So, you see that this has just 1 outcome. So, it is just the probability 1 by 8 there. So, you get that or another way to do it is to think of this as size of E 0 divided by size of the sample space. So, this is also quite important we are going to be looking at equally likely outcomes, this is very important if the outcomes are not equally likely, then you cannot use this formula. If you have an equally likely outcomes set of all equally likely outcomes and you define any event in E naught any event E for that matter probability of that event, is size of that event what is size does this notation means size of the event. So, what is size of this event? It is the number of elements, number of outcomes in E naught number of favorable outcomes in some sense. This is number of total outcomes that are possible field. So, this is a simple formula ok.

So, we also had these other event which is one head. So, we saw that there were 3 possibilities here H T T, T H T and T T H. So, what is going to happen to probability of E 1. You can go and see the probability of 1 by 8 is associated with each of these outcomes. So, you will have to add 1 by 8, plus 1 by 8, plus 1 by 8, and you would get 3 by 8, another way to see this, this is exactly equal to the size of E 1, which is 3 divided by size of omega which is 8. So, this guy is 3 this guy is 8 this one this is 8.

So, you see all these formula work out in the same fashion. So, you can keep going like this you would look at E 2 which is 2 heads, and then that this is the event H H T, H T H and then T H H. Once again probability of E 2, I am going to swallow a few steps here we will work out to 3 by 8 and then we had sorry if I wrote this wrongly, I meant 2 heads and then we have 3 heads which is just the singleton set H H H and once again probability of E 3 is going to go to 1 by 8.

So, this is quite a simple calculation and one can do slightly more sophisticated calculations you remember this event E greater than or equal to 1, which was at least 1 head and this would be everything except the first guy here let me write it down for you. So, this is that and then if you look at probability of probability of E greater than or equal to 1, you get that that is 7 by 8. So, 7 of them here and likewise you can also define E less than or equal to 1, which is at most one head and that is T T T and H T T, T H T, T T H and probability of E less than or equal to 1; so p 4 by 8 which is 1 by 2. So, these are this is a very simple calculation simple situation like this, one can do this kind of counting very very easily, but I want to point out real quick that the counting can become sophisticated very quickly.

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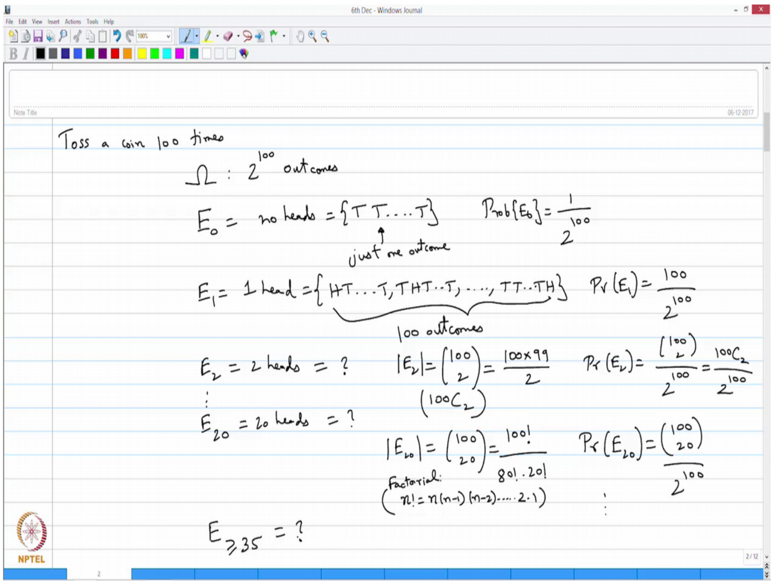

Supposing I say I toss a coin 100 times. So, 100 is just a number you could think of it as sum n or 1000 or something. And then if you start counting enumerating the set of all possible outcomes, there will be 2 to the power 100 possibilities right one could imagine starting to write it out you will never finish writing, it will take a long long time to finish off, but there are 2 power 100 possibilities and you just tossing the coin 100 times it is not all that much right. So, you have so many uncommon unimaginably large number of possibilities in the outcomes. Every possibility every outcome let me let me use the word outcome here just to be very precise. Every outcome in this set omega is equally likely that is true and any event I define I can definitely do a calculation similar to before.

So, for instance if you were to define E 0 as no heads this is just what this is a set with just one outcome all of them being P right just one outcome this is a easy enough calculation just one outcome and if you want to find probability of E naught you are going to write size of E naught which is 1 divided by 2 power 100 that is not too bad it is easy enough to right. E 1 maybe one head is also not too bad how many possibilities would there be you would have H T E H so on right T H T so on T, dot dot T T T H right. So, how many possibilities are there? There are there will be 100 outcomes right the H the single H that you have might be either in the first position or the second position or the third position. So, what will the 100 th position your 100 outcomes.

So, probability of E 1 1 can quickly write down is 100 by 2 power 100. So, things will become quickly complicated let us suppose I say 2 heads. So, what happens to the set? So, it turns out there is and you can use some simple counting ideas to come up with the number here the number can be enumerated later on in this course, we look at more more is most such examples of how to do this counting, you must have maybe you familiar with this from your earlier mathematics you have studied it is possible to count such number.

So, so for, you can even count for any number, supposing I put on a number like E 20, it was just 20 heads. So, you can find out you may not be able to I mean if you start writing out all the possibilities it will be too many, but you can count how many possibilities there will be so. In fact, many of you might know the answer to this questions and answer is size of E 2 it is actually 100 choose to right. So, this is the notation we use I pronouncing it as 100 choose 2 you might also it as 100 C 2 this is also possible the actual value is 100 times 99 divided by 2. So, this is the actual number of possibilities that are there with 2 heads, it is a question of selecting the first position for the head second position for the head and then you could have double counting. So, you divide by two. So, this tells you the probability of E 2 is 100 choose 2 divided by 2 power 100 like I said you might also write it as 100 C 2 by 2 power 100 ok.

So, so you see the situation here quite often this will happen in probability you may not be able to exactly write down the entire event as a set of equally likely outcomes, but you will be able to compute how many outcomes are there in the event. So, this is an important thing to know. So, likewise what is it for E 20 size of E 20 it is actually 100 choose 20. So, there is a formula for this, you can write it as 100 factorial by 80 factorial times 20 factorial. So, this factorial is what? N factorial is n into n minus 1 into n minus 2 dot dot dot all the way till 1. So, this is the definition of factorial ok.

So, this is the computation you might be familiar with high school calculations here, for how to compute the number of possible outcomes when you have 20 exactly 20 heads out of 100 tosses. So, probability of E 20 also becomes easy to write down 100 choose 20 divided by 2 power 100 and so on you can do for anything else like this, but the calculation can become quickly more and more complicated, and if you want to define non trivial events like let us say E greater than or equal to 35 let us say. This gets even more complicated you were 35, 36. So, on all the way to 100 and then add them all up and you will get the answer ok.

So, this kind of calculation involves combinatorics and you can quickly see why this is the such sort of combinatorial calculations are very important quite often, these kind of cases will show up in practice and you should be able to compute probabilities in some meaningful way for these things, later on we will see maybe more sophisticated ways of coming up with these numbers. So, hopefully this simple example convinced you that doing probability calculations as number of outcomes divided by total number of outcomes and get a little bit complicated, and you should know some methods of counting to do that in a convincing fashion.

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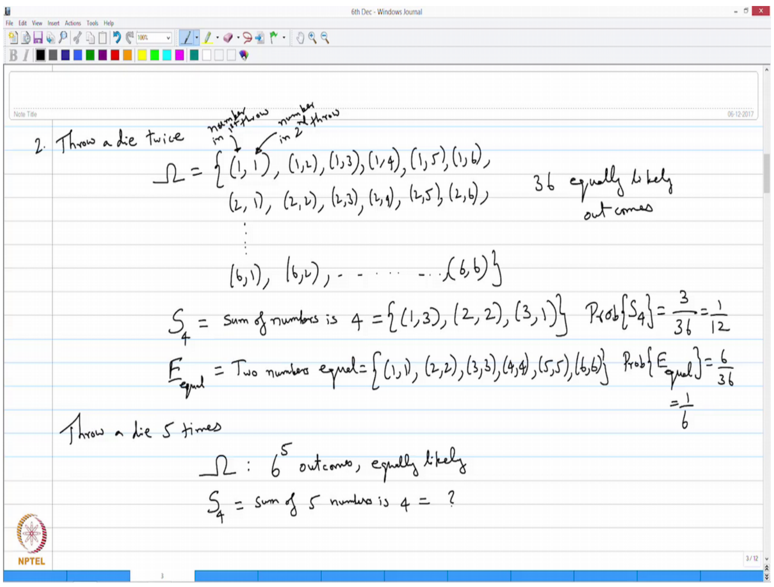

So, let me do the second example, which is throw a die, I will use the slightly nontrivial example where we are going to throw the die twice. So, if you throw a die twice, the set of all possible outcomes you could have a one in the first throw and the one in the second throw. This is the number in first throw; this is the number and second throws. So, you could have 1 comma 2, 1 comma 3, 1 comma 4, 1 comma 5, 1 comma 6, 1 or you could have 2 comma 1, 2 comma 2, 2 comma 3, 2 comma 4, 2 comma 5, 2 comma 6. So, I will put a dot dot dot to go all the way to 6 comma 1, 6 comma 2, dot dot dot 6 comma 6.

There are 36 equally likely outcomes. When you toss when you throw a die twice and then if you define an event as let us say S 4, which is sum of numbers is 4 you threw the die twice, and the sum that you get is 4. One can now start writing out the outcomes that will show up here, you can have the first throw being one in which case if the sum has to be 4, the second row has to be 3, you can have the first throw being 2, in which case the second throw has to be 2 or you can have the first throw being 3, in which case the second throw has to be 1 this is S 4.

Now, what is probability of S 4? It is the size of S 4 which is 3 divided by 36 and that is just 1 by 12. So, hopefully you agree with me this is the calculation. So, you can do more involved calculation. So, for instance you can say E equal, this is 2 numbers are equal in which case you can have 1 comma 1, 2 comma 2, 3 comma 3, 4 comma 4, 5 comma 5 comma 6 comma 6. So, these results in probability of E equal becoming 6 by 36, which is 1 by 6. So, this is once again sounds like a very simple example, but I can quickly make it more complicated. So, for instance instead of throwing a die twice, suppose I throw a die 5 times ok.

Now, what is going to happen to your omega? In each throw you can have 1 of 6 equally likely possibilities in the second throw again 1 of 6 equally likely possibilities; you are going have 6 power of 5 outcomes. All of them are equally likely yes there is no problem here, but just enumerate them and doing simple calculations of them once again is going to become really really complicated. So, supposing now I again ask the same question of S 4 sum of 5 numbers is 4 ok.

You are going to start going into all sorts of loops here, it is really hard to enumerate all the possibilities very easily and count and do something. So, hopefully once again this example second example at least convinces you, that you know doing calculations in a big sample spaces in this manner of counting the number of events E and dividing by total number of outcomes become bit complicated. But for simple events small events 1 can write it down and this is a nice way of doing calculations.

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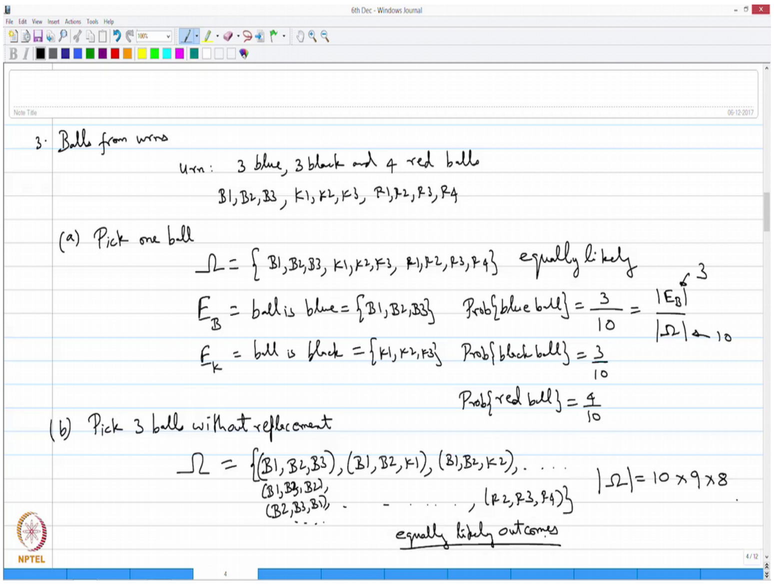

So, the next example I will see is balls from urns, let us once again consider an urn. So, this is a particularly ugly urn. So, let me not write. So, we are going to look at this example of balls from urns. So, let us say we have an urn with 3 blue, 3 black and let us say 4 red balls. So, I am going to name these balls as B 1, B 2, B 3 standing for 3 blue balls and then 3 black balls I will call it K 1, K 2, K 3 and then the 4 red balls I will call them R 1, R 2, R 3, R 4 ok.

So, when you mix up the balls in the urn and throw pick up one at random. So, if you are if you are. So, low the first cases pick one ball at random. So, the set of possible outcomes is it could be B 1, B 2, B 3 or K 1, K 2, K 3 R 1, R 2, R 3, R 4 these are equally likely outcomes. Notice this certain change I made and the way I have written down the sample space if I want to have equally likely outcomes, I have to write down the sample space like this I have to number each of these balls and not treat all the 3 balls as indistinguishable or the same exact type. So, I do that if my outcome set for instance is just B K and R then they will not be equally likely I mean it is a bit difficult to argue that, but if you if you have I f you have an urn with so many balls, any one ball is equally likely to be picked up and this is the basic argument when you pick a ball from the urn. So, you have to have every ball represented as a possible outcome. So, of course, when you pick it up you may not be able to distinguish whether it was B 1 or B 2 or B 3 ok.

So, some events may not be interesting to you, you may not be able to have an event it says the chosen ball was B 1 that event may not make sense, it may not be measurable in some sense, it is not outcome that you can distinguish at the end. So, maybe you are only the smallest event you can define is B 1, B 2, B 3 you can have some such restrictions like that that is not wrong. But if you want to have equally likely outcomes, you have to have each ball numbered like this at least in the sample space. So, this will let you do the calculations more easily.

So, for instance if I define an event E B as the ball is blue, this is a feasible event remembers the balls are distinguishable this 1 2 3 it is just a number that I have internally to calculate probabilities, not really visible to the outside world. So, ball is blue this is going to be the event B 1, B 2, B 3 and now you can calculate probabilities; probability of E B or else a blue ball, picking a blue ball is 3 by 10, 3 is the size of E B right. So, this is the size of E B by the size of omega, this guy is 3, this guy is 10, I get 3 by 2 ok.

So, you notice you can even when you think that the fundamental outcome is not equally likely you can kind of break it up into smaller fundamental outcomes, even though they are not very I mean you cannot really define all sorts of events here you can do calculations like this just based on this. So, this is an important thing to know as well. So, if you can also define an event each K which is ball is black that would be again K 1 K 2 K 3 and that probability of black ball is once again going to be 3 by 10. So, likewise you can define an event where you have a red ball and in that case we will have 4 by 10. So, that is the different events this is the simple example.

Now, if you slightly complicate this example and I say I pick 3 balls without replacement right. So, immediately the equally likely outcomes now are going to become much more complicated here. So, what will be the outcomes here? I mean if you think about it I can have let us say B 1, B 2, B 3, one possibility or I could have B 1, B 2, K 1 or I could have B 1, B 2, K 2 so on and so, on I mean so many different possibilities are there you will end all the way at R 2, R 3, R 4 right. So, this is the set of all possible outcomes actually I have not written down all the possibilities here so. In fact, you can also have B 1, B 3, B 2 or you can have you know B 2, B 3, B 1 all sorts of combinations of this is also possible. So, once they start writing out things like this, I have to be very careful about how I account no that is quite important. So, if you were to count the number of possible things here. So, the correct answer would be 10 the first ball that I pick can be any one of the 10 possibilities, and the next ball that I pick can be any one of the remaining 9 possibilities, and the next ball that I pick can be one of the remaining 8 possibility. So, this is the total number of possibilities in my outcome. So, even though I cannot write it down in a very clear and convincing way, if I actually count the number of outcomes I will have 10 times 9 times 8 and what is important is these are all equally likely outcomes. So, this is very important.

So, now when you start defining events, you can once again do probability as number of outcomes in the event divided by total number of outcomes in this omega. So, let us define one event.

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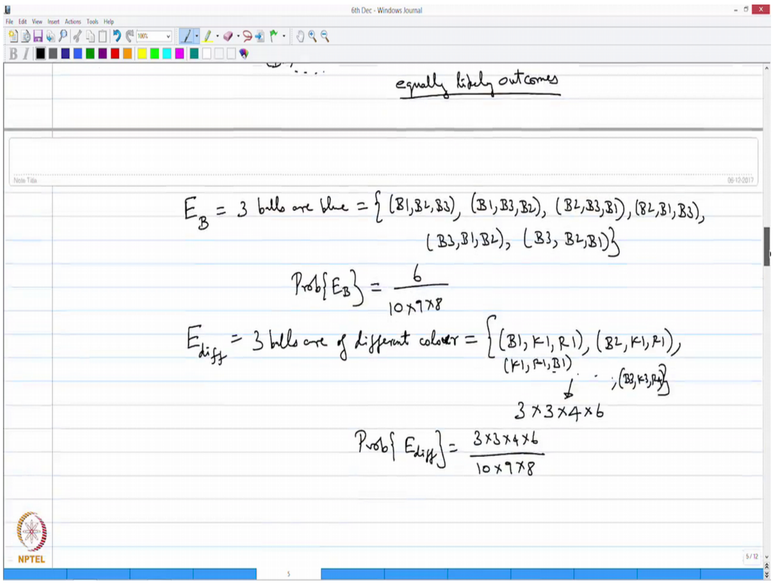

Let us say EB once again here this is 3 balls are blue. So, this is going to be an interesting sort of set there is going to be B 1, B 2, B 3 or B 1, B 3, B 2 or you can have B 2, B 3 bone. So, it can just come in any order B 2, B 1, B 3 or it can be B 3, B 1, B 2, B 3, B 2, B 1. So, that is 6 possibilities and that is it right the 3 balls can come in any order B 1 can come first, B 2 can come next, B 3 can come next or you know any other order B 2 can come first B 3 come like that. So, any order in which they can come and that is the only possibilities how many possibilities are there are 6 here ok.

So, probability of E B all 3 balls being blue, so that is just 6 divided by 10 into 9 into 8. So, this is slightly more complicated calculation than the previous case, but nevertheless it is possible to do this it is not very hard. So, here is a interesting situation. So, E different is 3 balls or of different color. Now, one can start thinking about enumerating this is again little bit complicated. So, you could have say B 1, K 1, R 1 or you could have B 2, K 1, R 1 so on and so forth right.

So, how many possibilities would there be. So, maybe the last one you can say here is B 3, K 3, R 4 right. So, how many different possibilities here, it looks like quite a few if you if you look at the size here. So, you the first blue ball can be one of 3 possibilities the next black ball can be one of the 3 possibilities, the next red ball can be one of 4 possibilities and then also I mean the sequence does not matter right. So, you can also have K 1 R 1 V 1 right. So, the it can go any which way.

So, I can have the black ball firs, the red ball next and the blue ball last and all of those things are there. So, it turns out you know you can see the calculation becomes quite murky here and you can actually multiply this by 6. So, you pick any one of the 3 red balls blue balls any one of the 3 black balls any one of the 4 red blue red balls. So, that is 3 into 3 into 4 possibilities and for every possibility you pick you have 6 different permutations within them and all of them are possible outcomes and they are all different outcomes here. So, that is the kind of calculation one needs to do. So, if you do that probability of E def ends up being 3 into 3 into 4 into 6 divided by 10 into 9 into 8 ok.

So, you can see this, this kind of calculation can get bit dicey you know I mean how to get all these numbers right maybe you missed something it can happen. So, this method of calculating probabilities by enumerated equally likely outcomes, may not be the simplest method, but it is useful it has it is utility in various places. So, same thing I can repeat in my fourth example which is draw cards from a pack.

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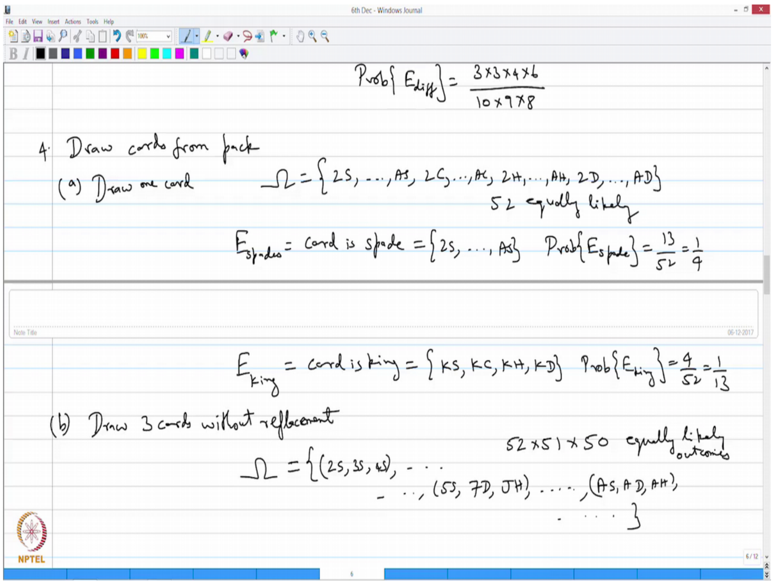

Once again my set of all possibilities is a 52 in number, if you remember we start with 2 spades, go all the way to a spades and then 2 hearts, 2 clubs go all the way to ace clubs then 2 hearts go all the way to ace of hearts, and then go for 2 diamonds all the way to ace of diamond. So, this is 52 outcomes these are equally likely right when I pick one card ok.

So, first examples once again pick one card draw one card sorry when you draw one card these 52 outcomes are equally likely and then we have this questions like a E spades which is card spade once again this is very easy card being spade it is just these 13 outcomes and the probability will simply be 13 divided by 52 which is 1 by 4, it is not too bad to calculate same thing with the E king which is card as a king.

In this case I can exactly write down what it to be it could be king of spades, king of clubs, king of hearts, king of diamond, and probability of E king this once again 4 by 52 which is 1 by 13. So, easy calculations to do in subsets, calculations once again will become a bit murky if I draw 3 cards without replacement. So, what will happen to omega here? I am drawing 3 cards without replacement and the probably the outcomes will all be triples right.

So, it could be 2 spades, 3 spades, 4 spades or you know anything else you know it could you could have like 5 spades 7 diamond jack of hearts, I mean all over the place maybe all the way till I do not know ace of spades ace of diamonds ace of hearts like that I mean every single triple and it can occur and it can occur in any order right all of those things are equally likely. So, how many possibilities are there here the first card could be any one of the 52 possibilities, and then second card could be any one of the remaining 51 possibility the third card could be any one of the remaining 50 possibilities so many equally likely outcomes ok.

Now, you can start defining events you can say- what is the probability that, all 3 cards are spades. So, I could ask the same question again right.

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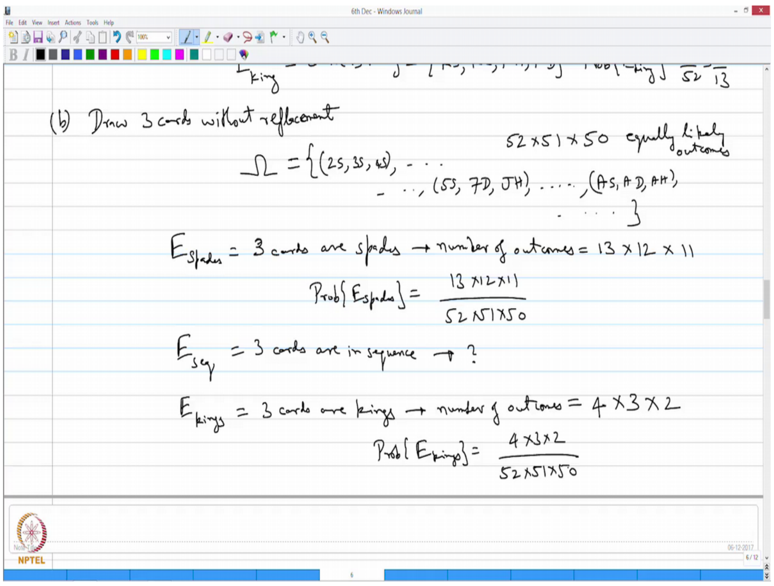

So, I could define an event here I will say E spades, it is just 3 cards are spades what will happen here? You should start now counting what are the possible sequence of 3 cards I can get in spades. So, this once again if you want if you start enumerating you will have so many possibilities, but then number of outcomes here is in some cases easy to calculate. It the first card you pick if you want all 3 to be spades the first card you pick can be any one of 13 possibilities, following which is the second card has to be any one of the remaining 12 possibilities in spades and the third card has to be remaining 11 possibilities in spades.

So, if you actually count the probability of E spades here as outcomes divided by this you will have 13 into 12 into 11 by 52 into 51 into 50. So, this is this is not too bad, but you know if you complicate your event a little bit you remember we defined an event earlier which is called E sequence. So, 3 cards are in sequence, this is a little bit more complicated. to compute enumerating this number is a bit complicated you have to go through things more carefully, but maybe this is easier.

So, what about 3 kings 3 cards are kings, how to how many number of outcomes you have? this is again not too bad the first card you pick can be any one of the 4 kings, the next card could be any one of the remaining 3 kings, the next card could be any one of the remaining 2 kings therefore, 4 into 3 into 2. So, probability of E kings it is just 4 into 3 into 2 divided by 52 into 51 into 50. So, this is the kind of calculation one can do if you stick to equally likely outcomes ok.

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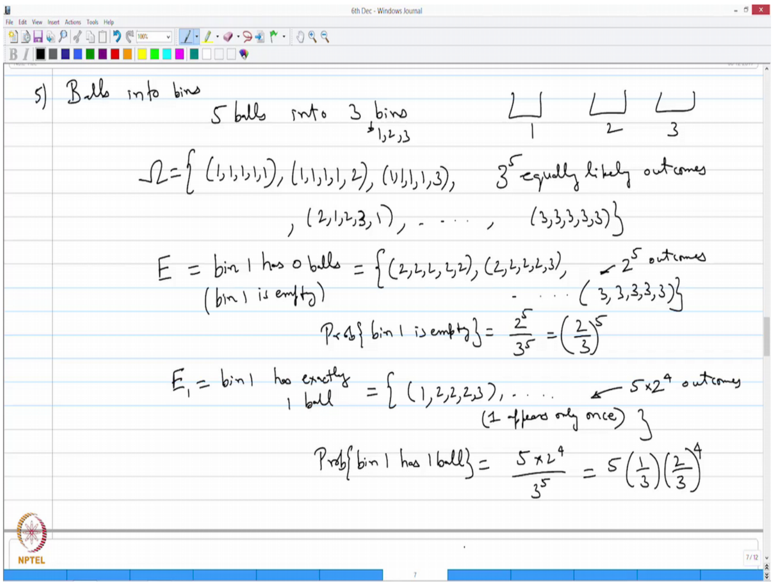

So, the last example that we have been carrying along is throwing balls into bins. So, let us say we throw I mean I think I picked out this example. So, let us consider the case of 5 balls being thrown into 3 bins. So, to get equally likely outcomes you have to figure out each ball went into what bin. So, you have to have to have that, and that could be any one of 3 possibilities each ball one first ball could go into any one of the 3 bins following which the second ball could go into any one of the 3 bins again following which the third ball could go into any one of the 3 bins again. So, the bin in which each ball ended up in can be equally likely to be any one of the 3 following which the second ball can also do that etcetera.

So, if you want to write down equally likely outcomes, you have to capture that information in a vector. So, you should have for instance maybe you number the bins as

1 2 3; bin 1, bin 2, bin 3 maybe I should write that down here 1, 2, 3 and these balls are being thrown and you want to track whether each ball ended up. You could have that all 5 balls ended up in the first bin or you could have the first 4 balls ended up in first bin last ball and one ended up in the second bit or you could have the first 4 balls ended up in 1 in 3. So, on it will go. So, you could have something like you know 2 1 2 3 1.

So, all sorts of possibilities are there may be the last 1 is all 5 of them ended up in bin 5. So, if you think about it these guys are equally likely outcomes right the actual place in which each of these balls ended up is an equally likely outcome. So, how many such equally likely outcomes are there for each coordinate in this vector of 5, I can have 3 possibilities. So, you have 3 to the power of 5 equally likely outcomes ok.

So, now you can see already this has become complicated, now if I start asking questions like if I define E let us say E an event E as that bin 1 has 0 balls. So our, another way to think of it has been 1 is empty. So, if you want to define an event which has bin 1 is empty, ns you should have a vector a vector of length 5 which should either be you know 2 or 3 right is that you could have 2, 2, 2, 2 3 likewise. So, each coordinate should either be 2 or 3 you cannot have 1 anywhere ok.

So, how many possibilities are there here. So, this has 2 power 5 outcomes do you agree is that hopefully that is clear to you. So, if I say bin 1 has to be empty, what are the possible events here? In these vectors I should never have a 1 if I have a 1 then some ball ended up there. So, clearly that is wrong right. So, that cannot happen. So, only 2 or 3 must be there. So, each entry in this vector has to have only 2 possibilities. So, there are 2 power 5 possible outcomes here.

So, what is the probability of bin 1 being empty it is 2 by 3, 2 power 5 by 3 power 5 that is not too bad a calculation 2 by 3 power 5, if you know more advanced probability you probably could have figured out this 2 by 3 power 5 quiet by yourself, but what about this next question let us say we have might want to say let us say event E 1 maybe I call it bin 1 has exactly 1 ball. So, it has exactly 1. So, which means what? 1 appeared ok.

So, you should have vectors here and one should appear only once like this in this vector 1 appears only once. So, it is not too bad to count this as one could appear in the first place or in the second place or in the third place or in the fourth place or in the fifth place there is 1 5 possibilities for one to occur. And then once one has occurred what should happen to the remaining things it could be either 2 or 3 right. So, there are 2 to the power of 4 possibilities is that. So, think about that you had one possibility where one occurred that could be in any one of 5 places and the remaining 4 places you could have 2 or 3 and all these things you can add up. So, the number of possible outcomes here this has (Refer Time : 37:52) put an arrow here this has 5 times 2 power 4 outcomes think about how I did that.

So, if you want to find probability that bin 1 has 1 ball exactly 1 ball that would be 5 times 2 power 4 by 3 power 5, there is another way to write this, this some of you might like 5 into 1 by 3 times 2 by 3 power 4. So, these are ways in which you can do calculation. So, you can define more sophisticated events here you might want to say you know bin 1 is empty, bin 2 and bin 3 are not empty. So, that becomes little bit more sophisticated in writing down events in this fashion, later on maybe we will see more interesting ways of writing down these probabilities will give names to these probabilities and come up with maybe more simpler ways of figuring out answers, but for now this is a kind of calculation we have to do ok.

So, once again let me summarize this lecture; one way to compute probabilities is to break down your sample space and write it in terms of equally likely outcomes. So, that might be very easy in some cases, in some cases you have to think about it very carefully and write it like in the balls from urn situation particularly when you have indistinguishable situations, you have to actually distinguish them when you write it in the sample space. So, that you get equally likely outcomes, but then you have to define your events which do not rely on the distinguishing. So, then you can do the calculation very nicely. So, once you do that you get these kinds of answers, but once the sample space becomes very large and the event is a little bit more trickily defined, it is difficult to count the number of outcomes and we need better methods to intuitively argue for these probabilities. And as we go along in this course you will study more advanced and sophisticated methods for such calculations.

Thank you very much.