Probability Foundations for Electrical Engineers
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

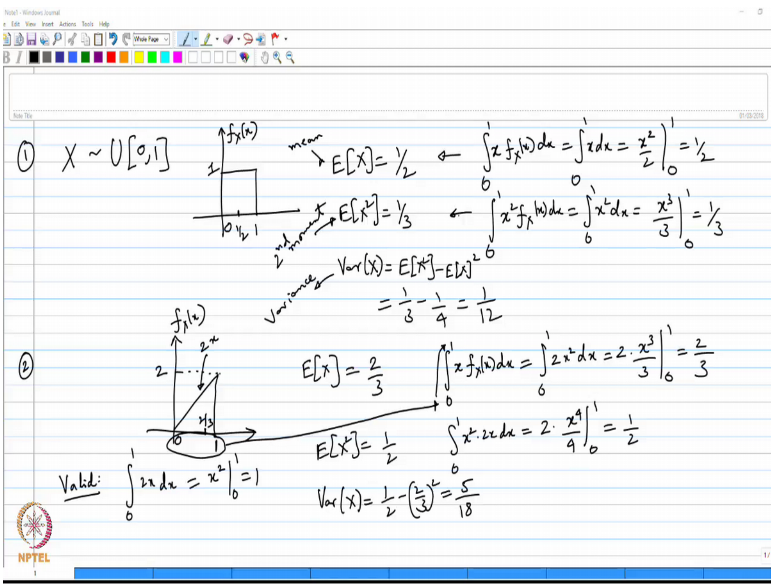
Lecture – 70
Examples: Continuous Distributions

Welcome to this lecture on examples of calculations of expectations; there are two different lectures this week on examples. One example will be with respect to continuous distributions how do expectation calculations look like for continuous distribution.

We will primarily look at expected value of X, expected value of X squared, variance of X those kind of calculations for some standard distributions or also maybe some slightly non standard distributions; just to show you how the calculations will look like. And then maybe I will give some one or two interesting examples in the continuous case.

And also in the discrete case, we will primarily focus on a situation where the distribution might be difficult to compute, but the expectation is easier to compute. And that is a very important theme in this idea of using expectations to deal with random variables and random phenomena ok. So, let me get started.

(Refer Slide Time: 01:13)



Let us take a distribution a very simple distribution to start with let us say X is distributed uniformly in the interval 0 to 1 ok. So, this is an easy distribution if you want to draw the PDF f X of x; PDF it look like this 0 to 1 1. And Prasadhan did this calculation in the lecture you know that expected value of X is half ok.

So, this is easy to do how do you do this calculation? It is basically integral 0 to 1 X times f X of x d x and f X of x is just 1. So, this is just integral 0 to 1 x dx and that is x squared by 2 evaluated between 0 and 1 and that will come out as half.

What will happen to expected value of X squared? This again if you want to do the same calculation, you want to do 0 to 1 x squared f X of x dx and that is just integral 0 to 1 x squared dx. And that is just x cube by 3; 0 to 1 it is one third ok. So, what is variance of X? It is expected value of X squared minus expected value of X whole squared.

So, this 1 by 3 minus 1 by 4 ; which is 1 by 12 ok; so, you might know some standard formulae which will give you these answers; I am not saying there is a way to do it, but this is just a very simple prototypical example of computing mean, second moment and variance ok. So, this is called mean or expected value of x, this is the second moment, this is the second centralized moment or variance. So, these calculations are what people call as bread and butter calculations given any PDF you should be able to do this.

So, let me take one slightly nonstandard PDF and this will also give you a lesson on how to think of PDF’s. So, now if I want a PDF which is say a line from 0 to 1; what should be the height here ok?. So, that is an important question to determine what should be the value here? So, it turns out the correct answer there is 2 and what is this equation? This is two times x ok.

So, that is a straight line which is two times X and this will be a valid pdf; you can you can see that this is a valid PDF by integrating check for validity is this valid you integrate from 0 to 1 2 x; dx the actual PDF this was just x squared evaluated between 0 and 1 and you get 1 and its nonnegative in the interval 0 to 1; so, this is a valid pdf. So, this is a slightly different PDF from the uniform case; higher values are more likely ok. So, this is a simple pdf.

So, these kind of problems are also quite standard given a pdf; how do you make sure its valid ok? How do you check that its valid how do you find some constants; so, that it is valid it basically has to integrate to 1 it has to be nonnegative ok. So, if you want to repeat this calculation here; if you want to look at expected value of X how would you do the calculation, you do 0 to 1 x times f X of x dx by the way I am doing 0 to 1 because range here is 0 to 1 ok.

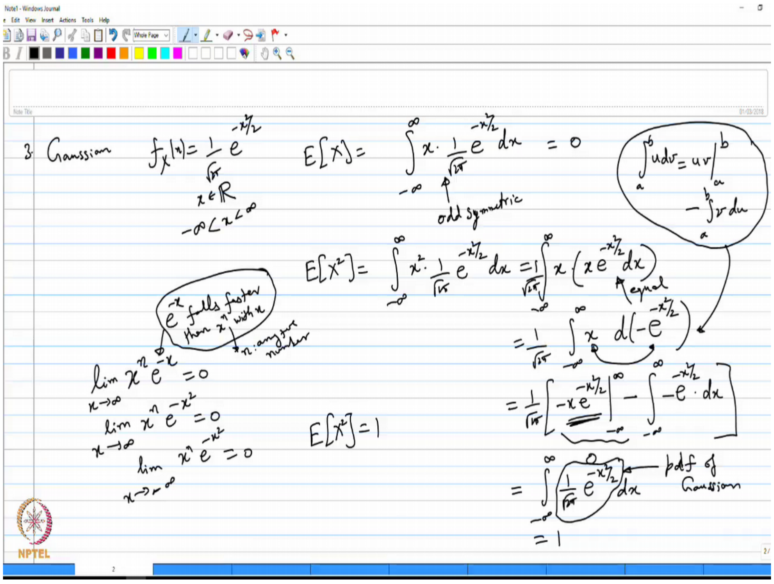
So, this is what gives me the range 0 and 1 right; so, that is enough for my calculation. So, this is integral 0 to 1 two times x squared dx. So, that is just becomes two times x cube by 3 evaluated between 0 and 1; so, that is 2 by 3 ok. So, you see the mean here is larger right; so, uniform case the mean was half 2 by 3 is going to be the right to the right of half, the mean is larger. And that is to be expected because the expected value this distribution, the larger values occur with higher probability.

So, the expected value should be larger; now this kind of inference in this simple case looks very simple I mean what is the big deal you know the distribution itself. So, quite often what will happen in real life complicated problems is you will not be able to figure out the distribution, you will only know the expected value. And you have to make some statements about what will happen to X and those kind of things its intuitive to see what expected value represents and all of those intuition should be very clear to you ok.

So, you can also compute expected value of x squared here little again be a little bit more complicated 0 to 1 x squared times fx which is 2 x; dx. So, that is two times x power 4 by 4 between 0 and half and that is half; half here and what is the variance of X? It is half minus 2 by 3 squared its half minus 4 by 9 right.

So, that is 5 by 18 ok; so, that is the variance you can do check this calculation for yourself ok. So, that is two simple examples of computing expected value and second moment and variance and this kind of calculation this is quite standard.

(Refer Slide Time: 06:44)



So, now if you have distributions involving exponential functions, then the calculation becomes slightly more tricky. So, I want to show you maybe you I mean you have also seen some calculations and in class in many cases; you would have to use integration by parts to simplify your integral. So, I think it is it is important to know integration by parts very well. So, particularly when you deal with integrations involving exponential function that becomes important. Also I think you can identify some symmetries of the function even symmetry, odd symmetry and then quickly do your work sometimes.

So, for instance here is an example in the in the in the Gaussian case; if you take if you look at f X of x is 1 by root 2 pi e power minus x squared by 2 ; this is the standard Gaussian. If you do expected value of x and remember x is going from the entire real line like; minus infinity less than xm less than infinity your whole real line is here.

So, if you do expected value of X, you will get this integration minus infinity infinity x times f X of x; e power minus x squared by 2 dx. The first thing to check is this integral converges and I think that is not too difficult to see, you can see that this integral will converge. And the other thing to check is that some check for some symmetry; if you will if you will notice this is odd symmetric.

So, this integral will actually evaluate to 0 right; so, if you have an odd symmetric function, if you integrate from minus infinity and the integral converges this goes off to 0. So, that is easy enough to do; the other slightly more tricky calculation here is the expected value of x squared ; you are going to have x squared times 1 by root 2 pi e power minus x squared by 2 dx ok.

So, in this case it looks a little bit tricky; so, you will have to do actually in this case you will do integration by parts. I am going to show you how I do integration by parts, the way different people know to do integration by parts its different, but this is the way that I like to do it. So, you do integral minus infinity; infinity keep x here and then you will have the 1 by root 2 pi place very little role here ; let it be as it is. And then x times what you will have left is x times e power minus x square by 2 dx ok.

So, now this part one can identify it as follows ; integral minus infinity infinity x times d minus e power minus x square by 2 ok. So, what do I mean by this? If you do this differentiation, you will get that right. So, you will get minus e power minus x square by 2; times the derivative of minus x square by 2 is just minus x; so, that will become the same as this. So, these two are equal they are equal I can check that. So, once you do this you can do integration by parts.

So, this is how I like to do integration with parts this is 1 by root 2 pi will come as it is. And then you multiply these two guys these are the two u and v; u dv right. So, what is integration by parts it says integral udv some a to b is uv evaluated between a and b minus integral a to b vdu right. So, this is integration by parts; you should remember this.

So, I am going to evaluate minus x e power minus x squared by 2 between minus infinity to infinity. And then minus integral minus infinity to infinity minus e power minus x square by 2 times dx ; so, this is my parts ok. So, I have applied this to this guy and got parts ok; so, this is integration by parts. Now this quantity; so, x times e power minus x squared as x tends to infinity actually goes to 0.

So, it turns out this e power minus x falls to 0 much much faster than any polynomial in x. So, these are some things you have to remember; so, x power n; e power minus x limit as extending to infinity is actually 0 ok. So, you can also put it as extending to infinity x power n; e power minus x squared is 0 ok. So, what does this mean? So, this basically says the way e power x; e power minus x falls faster than x power n with x ok.

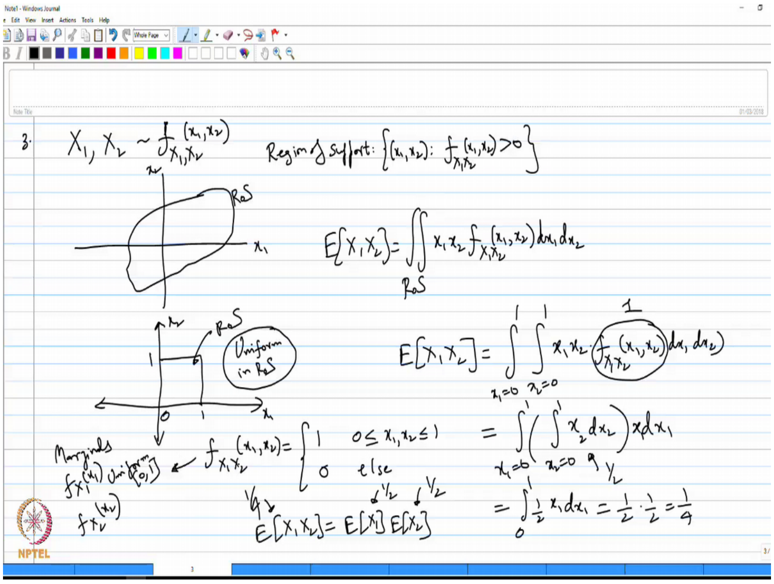
So, this is the important thing to remember for any n could be anything ; any positive number. So, this is an important fact to remember; so, e power minus x is an exponential fall; it is much faster than any polynomial fall. So, this will go through; so, immediately this minus x squared also I mean minus x squared by 2 or whatever you have all of those guys will go much much faster; this is an important result to remember. But particularly when you have e power minus x squared, you can also do extending to minus infinity x power n e power minus x squared that will also go to 0 ok.

So, these kinds of facts about functions are very important to know when you do; when you evaluate this function. So, when you now look at x power; x times e power minus x squared by 2 whether you put infinity or minus infinity; I am going to get 0. So, this first part will just become 0 ok; now only the second part is left and notice what happens to the second part. If I actually simplify this minus infinity to infinity 1 by root 2 pi; e power minus x squared by 2 dx. What is this? This is the PDF of Gaussian right.

Pdf of Gaussian distribution; with mean 0 and variance 1; so, that is something you know. So, this integral is simply be 1 ok; so, after all these calculations we found that expected value of X squared is actually equal to 1 for this; this is example of how to do parts; I think for (Refer Time: 12:57) also mentions how to do parts very very clearly in his lecture.

So, I think they gave you a different way of in which I write it down maybe I gave you more steps on how to do it. So, some things like this you should remember typically when e power minus X appears in the pdf; you will have to use integration by parts to simplify the expression. So, hopefully this is clear enough to you; given any other kind of problem involving these kind of PDF's, you should do it ok.

(Refer Slide Time: 13:32)



So, the next important thing to remember in calculations is quite often; you love calculations involving multiple random variables I just want to give one example of how to do that. So, let us say you have X 1 and X 2 having some joint PMF 1 x 2 ok.

So, the way I like to picture this PMF usually is I draw a two dimensional plot x 1, x 2 ok. And then identify draw the region of support ok; so, what is the region of support? This is something very important to understand in any 2 D PDF; the region of support is the set of all x 1, x 2 for which f X 1 of f 1 x 1 this guy is greater than 0 ok.

So, we will assume simple continuous distributions and this all of this is easy to define. So, the set of all values for which this guy is greater than 0. So, this will be some region here ok; so, some region like that ok. So, in this region you should have a good handle on ok; so, you should be able to describe the boundaries of this region, you should be able to do a 2 dimensional integral over this region all of these things you should be able to do.

So, for instance if you were to find expected value of X 1, X 2 what would you do you will have to do integral over this region. So, I call it RoS integral over the RoS maybe I will write on a double integral for your comfort; you have to do x 1, x 2 f X 1, X 2 of x 1, x 2 dx 1, dx 2 ok. So, how do you evaluate integrals like this; it just depends on the integral like this is evaluated depending on the region; it is not very easy to give a general formula.

So, maybe I will take a very simple example to show you how this is done. So, let us take a very very easy example first; so in the first example my region is just 0 to 1 1 here ok. So, this is my region of support remember this is x 1 and x 2, this is my region of support ok. So, I have not told you what the PMF's; so, I will say uniform ok. So, this is a very standard assumption; so you put a uniform distribution on this region of support So, you can see what the actual description is f X 1, X 2 of x 1, x 2 is actually 1 if x 1 and x 2 are between 0 and 1 this 0 else.

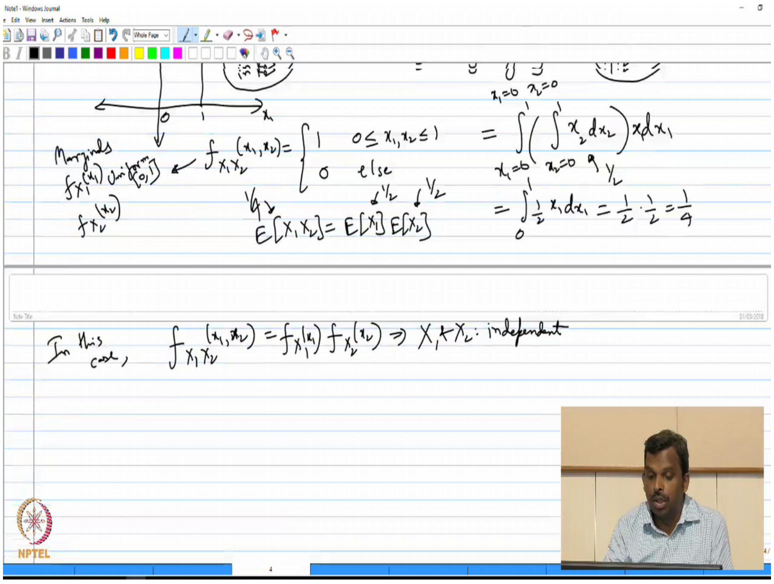
So, this is x 1, x 2 ok; so, now, if you want to compute expected value of X 1, X 2 simply going to do an integral x 1 goes from 0 to 1, x 2 goes from 0 to 1 x 1, x 2 right times f X 1 X 2 of x 1, x 2 dx 1, dx 2; the same formula, but I know this guy its now 1 right. So, it is just the integral 0 to 1 x 1 and then integral 0 to 1; x 2 see this integral will actually separate out right.

So, you do x 1 equals 0 to 1 you can first do the x 2 integral and then do the x 1 integral. This integral we have seen before value is just half right and then you have x 1; I am sorry function forget x half. So, this becomes integral 0 to 1 ; half x 1 dx 1 and the half will come out and then you will have an integral 0 to 1 x 1 dx 1; that is another half.

So, you simply get 1 by 4 ok. So, this is very simple case and you see in this case; it is actually that expected value of X 1, X 2 is actually equal to expected value of X 1 times the expected value of X 2; why is that? Because this gives you the marginal pmf; what are the marginals? f X 1 of x 1 and f X 2 of x 2 they are both uniform and in 0, 1 ok.

So, you have to check as you have to check the marginals and once you check the marginals you see that this guy is half, this guy is half, this guy is 1 by 4 and we are verified that the expected value of X 1, X 2 is this. In fact, this is actually a very trivial case in this case X 1 and X 2 are independent. So, if they are independent the p PMF to PDF itself factors ok; so, notice what happens here.

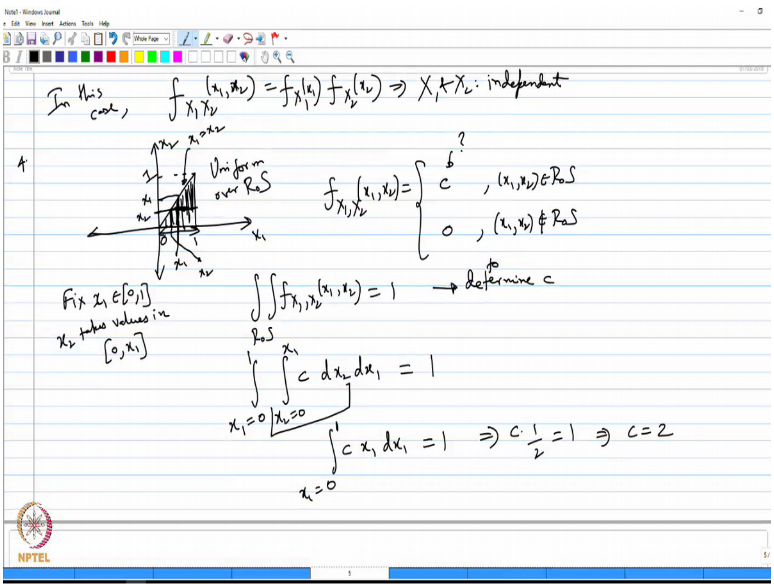
(Refer Slide Time: 18:06)



So, since X 1 and X 2 are independent; you have f X 1 X 2 of x 1, x 2 is actually equal to f X of x 1 times f X of x 2 X 1 of X 2; x 2 x 2.

So, in this case this implies X 1 and X 2 are independent ok. So, if they are independent then; obviously, this needs to be true it is very easy to see ok; so, that is the kind of problem.

(Refer Slide Time: 18:37)



So, I am going to complicate the problem a little bit to give you the next example where we will look at again a joint PDF with the region of support ok, but I will take the region of support as a triangle ok. So, just like I did in the previous case in the 1 dimensional case; I will take the region of support as a triangle, but now this is a very different plot right.

So, this is x 1, x 2 and I will say uniform over region of support ok. So, I am going to consider a joint distribution which looks like this ok; think about it once again x 1 and x 2 take values within this region and they are uniform ok. So, for instance; so, I can take a value here x 1 and x 2 can belong here, but outside of this there will be no x 1, x 2 value ok

So, in this part on so, so only the shaded part in the triangle x 1 and x 2 can lie in ok. So, its slightly restricted its very different from the previous picture, where you had the entire square and x 1 and x 2 could take values anywhere there this is within this region ok. So, here you will see the calculations will change a little bit the first thing to calculate is what is the joint distribution? What is the joint distribution and how do you describe this?

So, it is uniform; so, it is going to take a constant value, but what will be that constant value ok? So, so; so, there you have to do some careful description. So, it will take some value for x 1, x 2 in the region of support and 0 for x 1, x 2 outside of region of support ok; so, this I know. Now what is what should be this constant? How do you determine this constant? You have to integrate right over the region of support this guy and equate it to 1 ok.

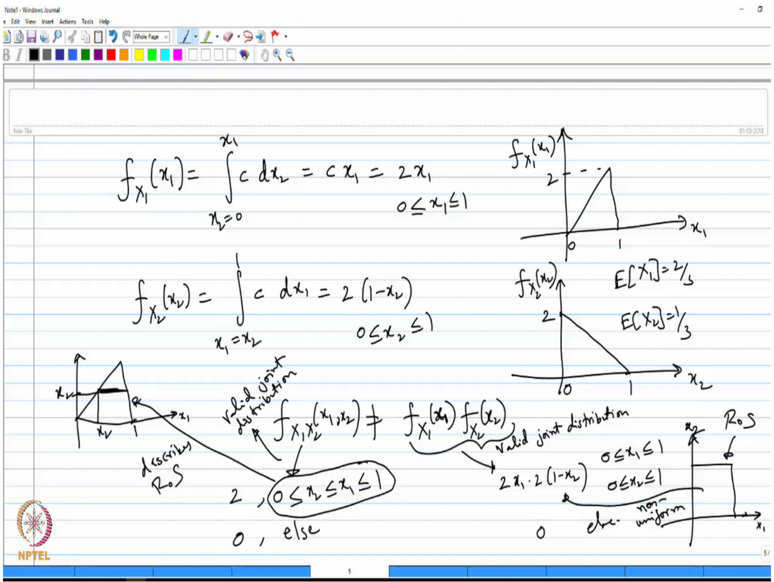
So, this will determine c. So, how do you do this integration? I need a description of the of the region of support ok. So, they describe the region of support I will first take x 1 from 0 to 1 ok. So, if you notice in the region of support x 1 takes values from 0 to 1. Now for every particular x 1 ok; so, what is this line? You should have a description of this line this is x 1 equals x 2 ; so, for every x 1; x 2 takes values from 0 to x 1.

So, if we fix x 1 in the range 0 to 1 x 2 takes values in 0 x 1 ok. Once I fix x 1; x 2 can take values only from 0 to x 1 it cannot go outside of that; that is the constraint of this reason as opposed to a square we have only a triangle. So, once they fix an x 1; x 2 can go only from 0 to x 1 ok. Now I can do my familiar integration this needs to be; so, this needs to be equal to 1.

So, let us simplify the lhs this one is an easy enough integral integral c dx 2. So, you will get integral 0 to 1; x 1 equals 0 to 1 that integral is simply c times x 1 dx 1, this needs to be equal to 1 which implies. Now this integral is also easy c times integral 0 to 1 x 1; d 1 we saw that before that is half is equal to 1 which implies c equals 2 ok.

So, this is how you determine the constant. So, if you want uniform over a certain region of support; you will have a constant which is different from 1 ok; so, this is 2 alright. So, now let us try find the marginals also; if you find the marginals you will get a very interesting sort of picture here. So, what is going to be the marginal?.

(Refer Slide Time: 22:32)



The marginal ; if you want to really find f let us say X 1 of x 1; I have to integrate out x 2 from the equation. So, it particular x 1; I have to integrate f X 1 x 2 of x 1 over 0 to the actual range right.

So, this will be x 2 will go from 0 to x 1 right for particle x 1; x 2 goes from 0 to x 1; c is the value to PDF I am going to integrate over dx 2 ok. So, this simply works out as c times x 1; c we saw as 2. So, it is two times x 1 is it ok; what is the marginal distribution? 2 times x 1 and x 1 takes values from 0 to 1 ok. So, the marginal distribution of x 1 will look like this ; it is the same distribution that I considered before the triangle distribution; what do you think will be the marginal of x 2 right?

So, for x 2 I have to integrate over x 1 right. So, let us look at that f X 2 of x 2 I have to integrate the same guy dx 1, but what is the range of x 1? If you look at the range of x 1 right, if you fix an x 2 this is also x 2 this point is also I am sorry this point is also x 2. So, if you fix an x 2; x 1 goes from x 2 to 1 ok.

So, the range is different slightly different x 1 goes from x 2 to 1 think about it because notice that this is the triangle the region of support is a triangle like this ok. If I fix my x 2 here right this will also be x 2. So, for a particular x 2; x 1 can take only these values it goes from x 2 to 1. So, this guy will actually be c times c times 1 minus x 2. So, c is 2; so, 2 times 1 minus x 2.

So, how does f X 2 look. So, its 1 times 1 minus x 2; if x 2 is one it goes to 0; if x 2 is 0 it goes to 2. So, it will look like this; so, x 1 is much more likely to take larger values, x 2 is much more likely to take smaller values and this is the distribution ok, but our x 1 and x 2 independent not really right. So, if you do f X 1 times f X 2. So, you notice f X 1, X 2 of x 1, x 2 is not equal to f X 1 of x 1 times f X 2 of x 2 ok.

So, think about how it will look. So, the left hand side right; so, the left hand side is what? This guy is we have this description; then it is 2 for 2 in the region of support and what is the region of 0 outside of it? What is the region of support you have to describe it carefully here. So, one way to write at this is one way to write it think about why this is true. So, for every; so, you fix an x 1; x 2 lies from 0 to x 1 right. So, one way to write this thing is to say 0 less than or equal to x 2 less than or equal to x 1 less than or equal to 1 ok.

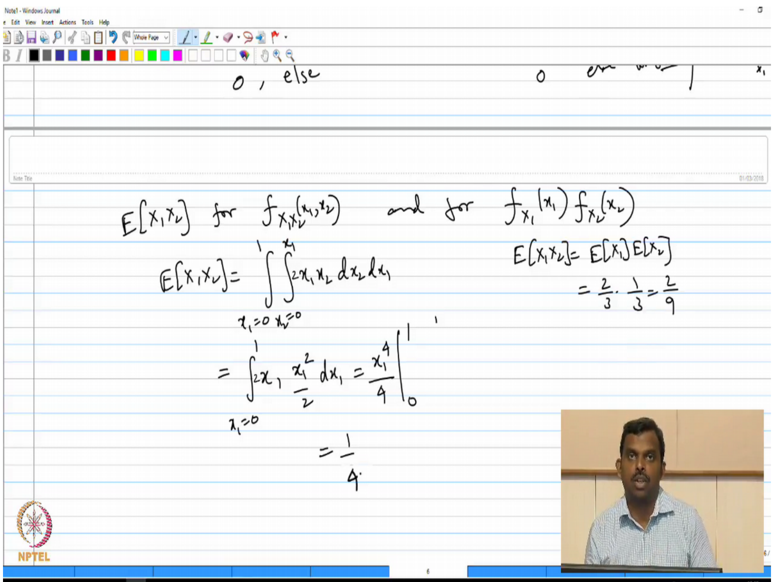
So, 0 else ok; so, this is a succinct description of the region of support ; it describes this region of support ; this triangular region of support is described by 0 less than or equal to x 2 less than or equal to x 1 less than or equal to 1 ok. For a particular x 1. x 2 takes values from 0 to x 1 and think about why this is true.

So, this is in x 1; x 2 this is true please clear to you. Now what about this product? This product on the right hand side what will it look like ok? So, this will look very different this will be 2 times x 1; times 1 times x 1 minus x 2 for what values? For any x 1 between 0 and 1 and any x 2 between 0 and 1 and 0 else.

So, this will actually have a region of support what will be the region of support for the right hand side? The region of support will actually be a square ; this will be the region of support x 1, x 2 and it will be non uniform ok. So, it will be non uniform in that region of support ; it will take some 2 dimensional function ok. So, it will be non uniform ok; so, we see that there are two different joint distributions here. This is also a valid joint distribution, this is also a valid joint distribution both of them have the same marginals right; you can calculate marginal for both same marginal, but the joint distribution is different ok.

So, this is a very nice example to look at how this can happen and you can also check that the expected value of X 1; X 2 if you calculate ok. So, let us do the calculation next that is an interesting the calculation also ok; it will be different from the expected value of X 1 into expected value of X 2. So, we see here expected value of X 1 will be 2 by 3 and you can imagine expected value of X 2 will end up being 1 by 3 and if you actually calculate expected value of X 1; X 2 ok.

(Refer Slide Time: 28:46)



We can calculate the expected value of X 1, X 2 for two joint distributions for f X 1, X 2 of x 1, x 2. This uniform in the triangle and for the joint distribution f X 1 of x 1 times f X 2 of x 2 ok.

So, this distribution the answer will be what? Expected value of X 1, X 2 will actually be the product of expected value of X 1 into expected value of X 2; you can check this. So, you will get 2 by 3 into 1 by 3; which will just be 2 by 9. For this guy it will not be the same it will be something else. So, here if you do expected value of X 1, X 2 you actually have to do that double integral ; you take x 1 from 0 to 1, you take x 2 from 0 to x 1 2 and then you do x 1; x 2 dx 2, dx 1 ok.

So, this will be a different integral. So, let us do x 1; 0 to 1 2 x 1 will be outside and x 2 dx two integral from 0 to x 1 is going to be x 1 square by 2 right you can do that. So, you have x 1 power 4 by 4 0 to 1 ; so, this is 1 by 4 ok. So, we see clearly these two have different correlations expected value of X 1 into expected value of X 2.

So, this also was meant to give you an example of how to do calculations with joint PDF's and expectations etcetera.