Probability Foundations for Electrical Engineers
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Lecture - 05
Basic Properties of Probability

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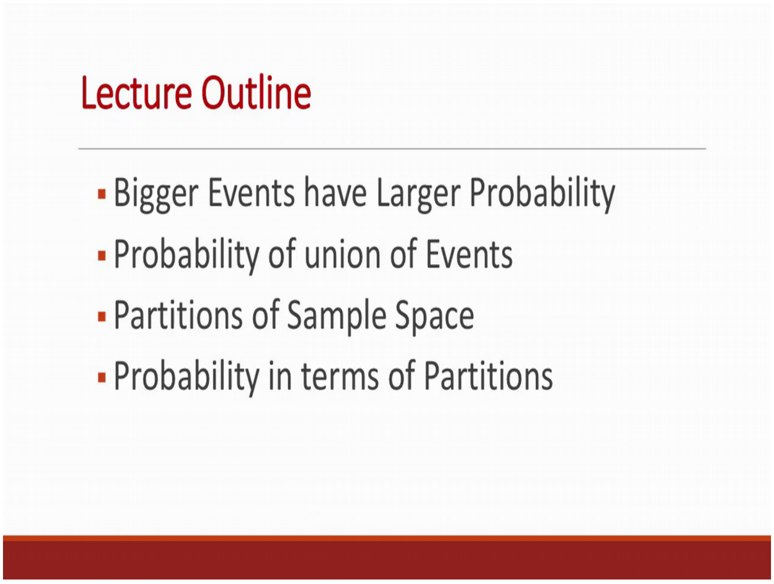

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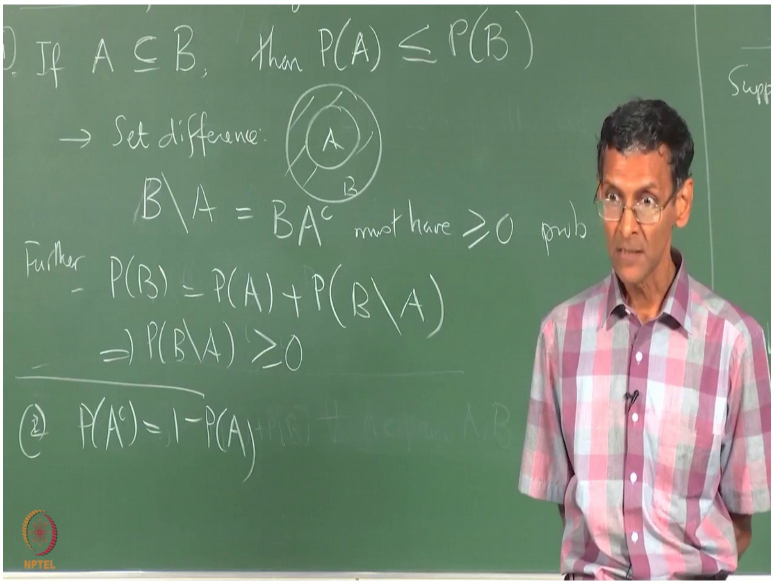

So, this is generally applicable right. Let me say, applicable always these probabilities always hold right, regardless of how you get your omega F P right. The first thing of course, is that if A is a subset of P of B right, then P A must be upper bounded by P B, there is no way, that a bigger set can have a smaller probability right. Now, what do? What is this notation? I think we are saying this for the first time right, this C like elongated C, if you will right this means that, this the set A contain the event A or the set A is contained in the set B or the event A is right. It is contained in the event B right if.

So, this in probability jargon it means that if A happens B always happens, but the reverse need not be true right. A B can have more points or something outside of A right, then A must be the probability that you assigned to A, must be no bigger than the probability assigned to B right, which is; obviously, true, because you have the concept of set difference now right. What is this? The set difference in such cases is clearly set of points in B, but not in A.

So, this brings out the, this is A, this is B, this is the set difference. So, B excluding A, this is basically what B intersection A complement right that B intersection A, A complement must have and this is; obviously, in F remember right, because of the rules of F, that says if A and B are in F, all of these things did you derive like, this must be in F again. So, this B intersection A got B A C must have 0 probability and what is P of B P of B also is P of A plus P of B intersection A, sorry B difference A right again this follows because B.

And B in this particular case B and B, B difference a right are exclusive right sorry A and B difference A are exclusive. So, therefore, this additivity has to happen and B equal to P of B right, because the union of A and B intersection A is B. So, therefore, P of must be greater than equal to 0 right. By the way, we will get a lot of bounds in probability theory right. I came up with this phrase, which kind of sounds cued it abounds in bounds right. It is one subject where bounds are rife right from the first second to the last. We will be looking at bounds or I mean no, not I am not, not to say that we do it every class, but it will come, keep coming up periodic very frequently right, more much, more frequently than any other maths class, you may have taken right. So, just keep be prepared for any qualities of this kind right.

So, this may be, we can add, put some numbers here. So, this is the first right, important statement that follows from the axioms. Let us see, if what are the things we can say clearly P of A complement must be right, probability for of the complement must be equal to 1 minus P A, I do not think, we need to spend more time you know this is. So, obvious that right, we do not have to remember, omega is what A union, A complement always right, for any A and they are by definition exclusive and therefore, this follows the most non trivial results.

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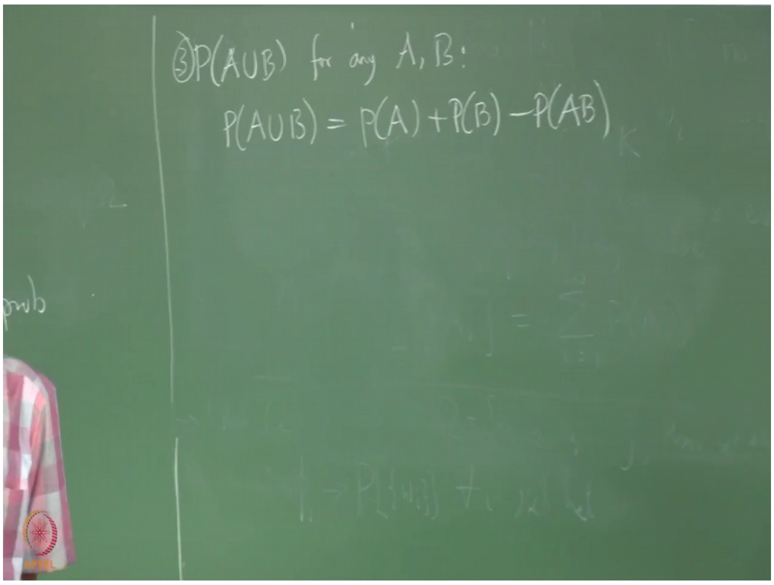

In general for what is this probability of P A union B for any A B not necessarily exclusive. This is a standard result that right is coated in all books, in the first view somewhere in the first few pages right.

How do we get this? How do we write this A, it all depends on writing this A union B write as A union of exclusive events right.

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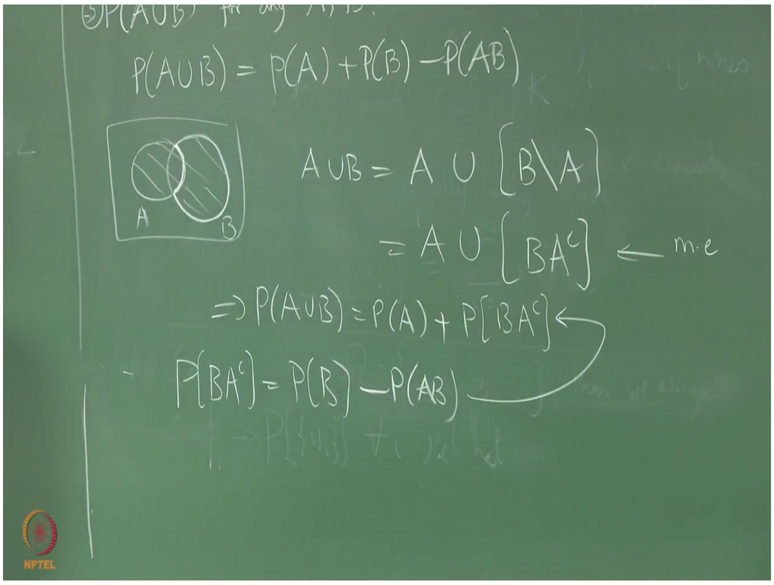

So, if you start off with just two events A B right, this A union B is this portion right, which includes all the elements in A and or B right. So, you covered all of these points; obviously, only once in the union. So, what is A union B basically, A union something, if I want to say A union, it is basically this portion. If I mean, if I want to write it as union of exclusive events right, this portion is also written as B difference A right, even when A and B are, B is not superset of A right. So, what is this set difference in general? It is a consists of all those points in B, which are not right. Which are not in A. So, that this statement is right always true.

Where they regard this, whether A is subset to B or A is a superset of B right, in other words this A union B is A union B A compliment right and then which are mutually exclusive. This A is right exclusive of B A C right and therefore, So, I do not have to worry about the intersection when I deal when I just go back to the axiom, I just write this is P of B A C right and then from this it is easy to derive right, that this must be P B minus P A B right. So, in general this P of B A C, whether it is this form or this form P of B A S B A C is P B minus, this is maybe a result worth saying all by itself right, because how do we get this B itself is counted as the union of this and these two mutually exclusive parts. This is B A C and this is B A or A B. So, clearly P P B A C is P B minus P A, this obvious, if you plug this in here, you get this result.

So, this is a very important useful result right, which works for any pair of events A and B and not only that it can easily be generalized right, I can look at A union B union C. How do I generalize it to A union B union C? How do I generalize to A union B union C i?

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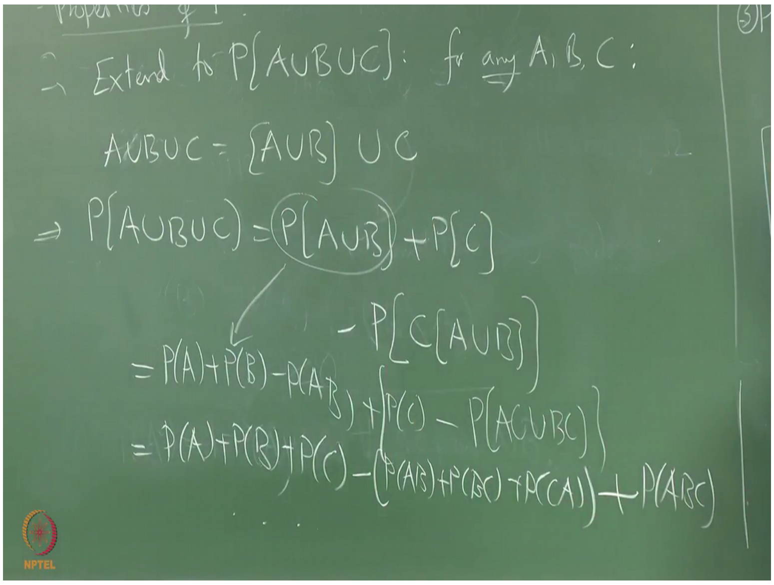

So, as a corollary or whatever extension; so here what we do is you first write this A union B union C as let us say, you first, you were write this A right as A union B union C right. You look at it like this for any A B C not exclusive right to take colon out here then this P of implies this P of A B A union B union C is P of A union B plus P of C minus what by that, what is it? Minus C C intersection A union B right. I will just write it like this, we do not need to put the intersection sign right.

We have use that convention from all you know, everywhere we will be using that convention right. So, now, this we expand using that. So, this is P A B P A plus P B minus P A B, this guy comes here then of course, the P C comes here, minus this I am just putting another bracket for this portion right. This is now P of what C P of A C union B C. Now, note that yesterday, we talked about distributive law right. C intersection A union B is A C intersection A C union B C. This is the easiest form that you should be a lot to remember in some sense is multiplication over addition which all real numbers and complex numbers right. They obey the other one, is little harder right, that union distributes or intersection is not. So, intuitive, but intersection disputing over union as I think very, you know we made latch on to that right. So, if you take this thing inside C inside you, get exactly this.

And now, this once again can be expanded using right that probabilities expanded using that. So, finally, what do you get I am not going to write all the steps. So, you tell me do it and tell me what do you get, you get P A plus P B plus P C minus P i B plus P B C plus P C A all of those with A negative sign and then you get plus P A B C, whether the plus P A B C comes from that comes from here AC intersection B C is just A B C right, if you draw a Venn diagram that will become very clear A C intersection B C is A B C. So, therefore, to cut you know one problem that I have with this board is that right. They I do not know, how they recording, I want to say this on camera does not matter, but it is right, it this the bottom portion is not that usable, anyway I will go, stop, write, one more line and hopefully, it will P A B P A plus B B plus P C minus P A B plus P B C plus P C A minus plus P A B.

So, be you can see a pattern emerging right. From this, if you want to extend this to you the union of 4 events, what happens something very similar to this right. It will be an extension, the individual probabilities will come first right, but the danger is right, I mean if you know in the exclusive case all of these joint probabilities will go to 0. So, these guys right will are the only guys who will survive, when you have a collection of exclusive events right, but when you have norm exclusive events right, then all these are the terms, start to come into the picture right and the next step would be this portion would be the same, but instead of P A B C, you will get A B C A B B C D C D A and. So, on right that all those will come on the plus sign and you will get a minus of P A B C D. So, it will keep, you know the extra terms will come with A plus minus alternative plus minus sign right. It is called the inclusion exclusion principle.

I do not want to write, the general version of it right. You can go, look it up right, inclusion exclusion principle right for the union of N events right, in general. So, it can be extended and whether it becomes quickly somewhat right, unwieldy shall we say. So, once again; now, let us appeal to a bound to help us out. Right now, these bounds are helpful of course, only as we will see right, only if the number that we come up with is a meaning for a useful number right, up the upper bound or lower bound. So, in this case let me write the bound out and let me write let us see, how useful it can be the union bound.

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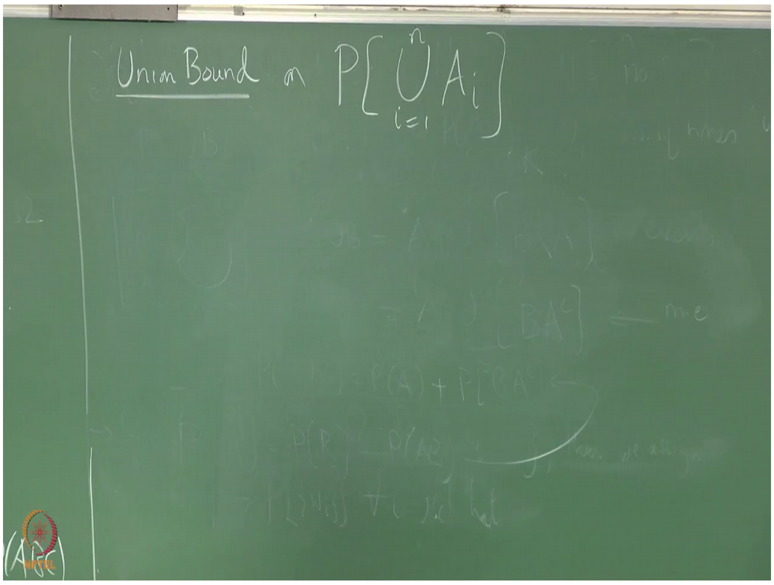

Union bound on the union of some N events A I, which need not or the probability of the union of any events A I. They need not be exclusive right any events right. Exact calculation is messy as we just now said right.

Exact calculation need means that, we need the intersection probabilities for not just the pair wise right, terms we need intersection probabilities of take. It events take 3 at a time, 4 time and so on right. So, how do we simply, you know write a simple upper bound on this, it turns out that this union can be no bigger than I mean what than? What you would get if all of them were; in fact, mutually exclusive.

Students: Summation.

Yes. So, the summation of probability is you say is A, is going to be an upper bound right that this portion out here is an upper bound to the union probability of the union.

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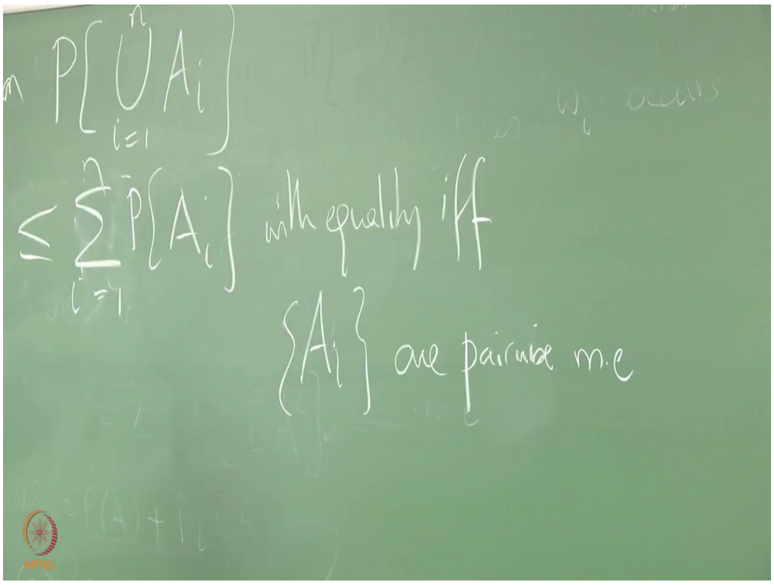

Sorry, with equality if and only if what the A i are pair wise exclusive, there is any two events in this collection, must be mutually exclusive, every pair of events that you pick up from the A i must be exclusive.

If you want to actually formally prove this; you have to start with the right, with the first pair itself. Let us say A 1 and A 2 P of A union A 1 union A 2 S are clearly upper bounded by P of A 1 plus P of A 2. A 2, because P of A 1 A 2 is going to be non negative right. Then you extend that by induction you get this. So, this is the union bound very useful and many examples especially in bounding, some error probabilities of in communication. So, on and. So, on right, but the utility of this union bound, what is when does it become totally useless, in the right hand side is bigger than 1 or even equal to 1 right, then the union bound is useless. So, it is useful

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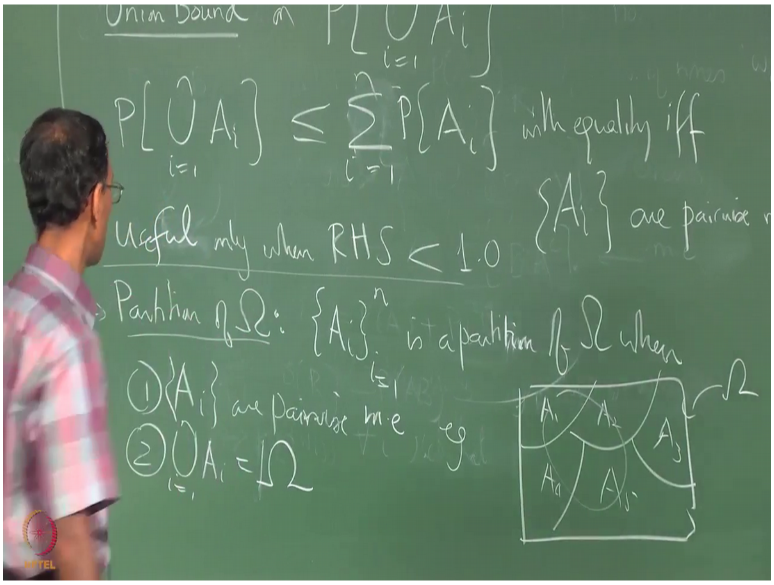

Only when the R H S is upper strictly smaller than 1 you know that any probability has to be right smaller, you know at most one right.

So, what there is no point in saying my upper bound is 2 or 3, upper bound is 1 period by definition by construction right, sorry it should be useful. All these mistakes that I make right, fine let them be on camera, because it you know, it shows up the problems of talking and thing you know write and writing at the same time right, you start a sentence in some form right and then you write the opposite of what you know your intended right and then turns out that if you go and make a correction does not matter right, we want this, these lectures to be as realistic as possible right, not some kind of A anyway. So, I think maybe, I was pretty much one more point maybe I have a couple of minutes.

So, maybe I will just conclude that right. Supposing, you have a partition of omega. So, this is anotherr property which I stop numbering anyway, but or anyway, let me not write a number. What is the partition of omega, clearly this in the discrete case right, the elementary events P of omega, I are, I am sorry the omega element remains omega i are a partition, a partition is where the, you have a collection of mutually exclusive events right, which the union of, which make up what you are trying to partition. So, this A i let us say i equal to 1 to N is a partition of omega when, what two things have to hold right. When 1 A i are pair wise exclusive right, there is any pair of events out of this collection have to be mutually exclusive, by the way this notation, i equal to 1 to N i suppose, right you by.

Now, you are going to use to it right, it means just the collect or entire collection A 1 A 2 up to A N that is the short form of writing, you can use this freely right. Whenever you have such things to talk about and I wish they would use this more often in the D S P course as well, but they do not use it right, for some whatever reason right, but this is exactly the same kind of thing that you been studying since you looked at discrete time signals and so on right. So, they are pair wise mutually exclusive and. Secondly, the union of all of these A i s is omega itself. Now note, that right A i s can be arbitrary events in this, as long as they satisfy these two conditions right, they need not be elementary events in this, in the discrete case; obviously right you cannot right. You taken it down, all the way to the elementary out events right and those will; obviously, be a partition.

But what about the continuous case right omega being having an uncountable infinity of events right, which is somewhat sometimes called continuous right, there also you can partition with that discrete with a countable number of A i s and sometimes this N need not be finite. It came in B right, infinity countable, infinity right. So, in this case then any event B, which is constructed using this omega must be expressible right as a union, sorry right as A union of B A i s of the right. So, basically what we have is, we have I hope this is. So, I have A 1 A 2 A 3 some A 4. Let me stop with this, I have A 1 A 2 A 3 A 4. So, I have some let us say right, something like this the omega. Now is some, this outer rectangle right, it is been partitioned using these 5 A A i s, then if I take some B any B must be like this right, it must sit inside omega maximally right.

So, what can you say about B B is basically the union of the B A i s, i equal to 1 to N he need not involve all N, but he cannot involve any more right it mean B, you may not get right B may not need all of the A i s to make it up does not matter some of those B A i s may be 5, maybe empty right maybe the right which is, but that is it is right. You can always take A union with phi. So, and it is important to note that this B all these B A i s are exclusive right and therefore, this P of B is what it is the sum of P B A i s right. So, we will pick up, I mean just last thing maybe I will say it again tomorrow, in tomorrows class in.

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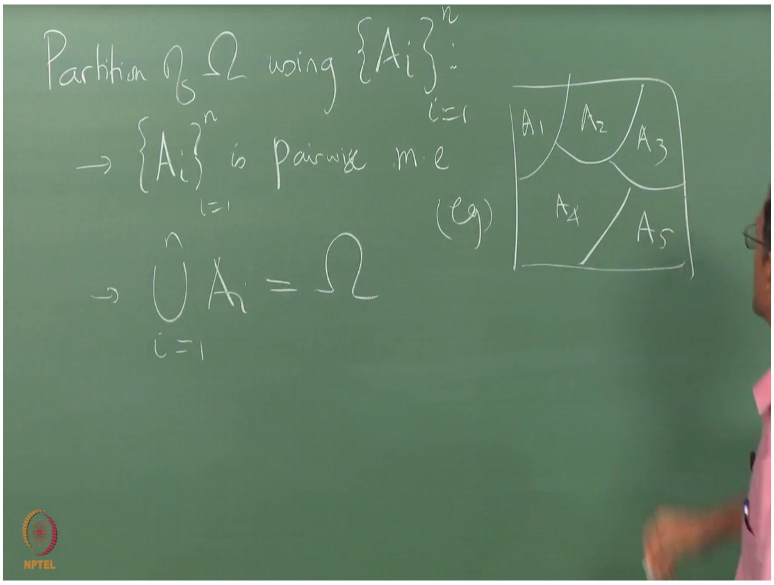

So, I have N, N divisions of omega basically, right N, non overlapping divisions of omega and union of course, this is.

So, whenever we have a situation like this right, physic in pictorially it is drawn like this right, A 1, A 2, A 2 A drawn like this then if I have any B, any event B right; obviously, the event B has to live again in omega as A subset of omega right and therefore, it has to intersect the A i s. So,

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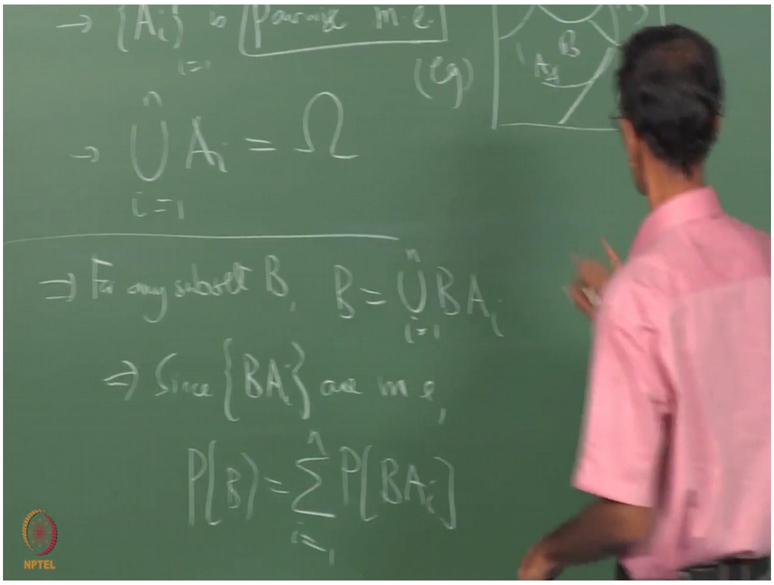

For any B any, say any subset B, which is in the Borel field of course, sorry which is in the sigma algebra F right for which we want to you, know assign a probability and that. So, we have B will be the union of P A i right. This may sound trivial, but it is actually not right. It is actually right, a very useful thing, a way to construct this B right, using these intersections right, this idea of making up, this B using intersection, using the parts of B which intersect each of the A i s is a very useful idea right. Especially, in more complicated experiments right as we will see, as we go along.

So, here of course, we flattened out everything we have right. A single flat representation of omega, the A i s and B and so on right it, but later on when you look at two stage experiments and. So, on right, one experiment and then feeding another one, then this kind of idea will be pretty much, the only way you can approach right, this problem right. So, you construct this B, the overall event B as A union of B A i and because the A i is a pair wise, mutually exclusive right B A i s will also be exclusive, you can clearly see from the Venn diagram, that B B A 1 is no intersection with B A 2 and so on right and therefore. So, there since B A i S are mutually exclusive, the probability of B is now the summation of B A i right using our additivity formula.

So, this rule which I am stating formally right, just using A i s and using this part of the course will, I am not going to do any examples right now, but you should be able to relate the later examples to this result right. So, as right now, we just concentrating on writing down some intuitive and useful results right, which we will later on expand use using. You know examples right. So, for the time being I am just running through some results right.