**Probability Foundations for Electrical Engineers**
**Prof. Aravind R**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**

**Lecture - 55**
**Independence to Two Continuous Random Variables**

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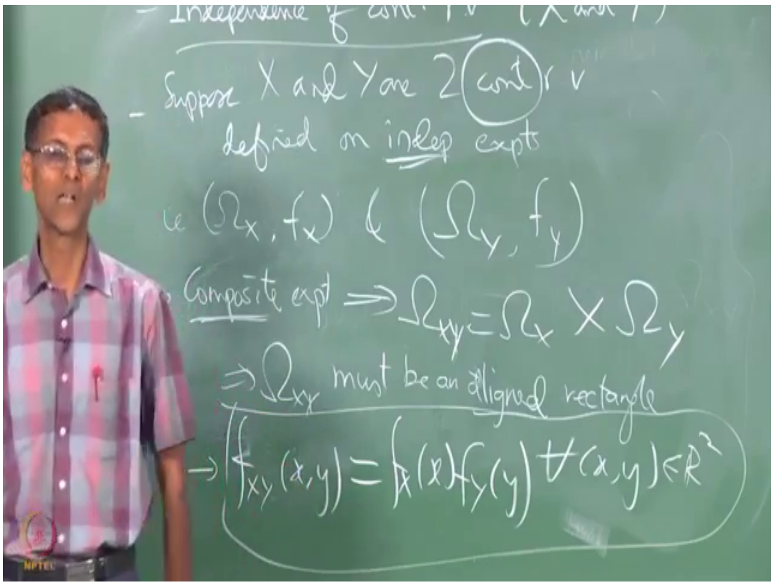


So, let us start with this independence of continuous random variables say X and Y. I am first going to take the situation, yes you have a question please ask. There is no question maybe we can limit these conversations is there a question again or any comment you want to make it. Please let everybody get the wisdom of.

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So, rather than going into the mathematics straightaway let us assume that you have two independent experiments with two continuous random variables rather than discrete random variables is not it. Then say how do we combine the two we will see, that we will give us enough mathematical insight into the situation is not it. Suppose x and y are two continuous random variables defined only.

Independent experiments ideas they have to be continuous random variables right. So, this is how we did in the discrete case we combined using a how multiplication similarly we are going to do the same multiplication here also; ie in otherwise you have omega x and f x for x omega y f y for y, these two let us say are given. I mean in general they can obviously, be different there is no reason why I said this f x and f y have to be the same I think nothing of the sort. How to combine them in a composite experiment? So, composite experiment with mean what, you are you are performing both of these of course, in two different corners of the room, but you are plotting the result in one, on one plane if x takes.

A value out here as minus three and y it takes a value plus 5 you are going to put that point three minus 5 on the plane say this is the this is the value I got in their joint experiment. Just like in the discrete case no difference except that if you look at all the possible values of omega x I mean omega y taken to here, what is omega xy it has to be the Cartesian product of omega x and omega y, is not it. So, it automatically implies that for the composite exponent omega x y is always omega x cross omega y the Cartesian product, every possible combination will feature an omega x y.

So, what can you say about this region omega xy now? It has to be rectangular in shape or it has to be that maybe a whole plane is also considered rectangular in the sense right, but essentially it has to be rectangular not only any rectangular it has been aligned rectangular it has been aligned with the two axis cannot be some crazy to write some rotated rectangle and so on. So, this must be aligned with the axis when I say aligned I am talking about aligned with that x and y axis fine that this give you intuition about omega xy. What about f xy? I will obtain f xy by taking the product of f x and fy. What does that give me? So, this is all important equation of this composite join marginal I claim in this case is exactly the product of this for all xy actually turns out in R square.

Not just omega xy because outside that 0s will anyway taken, this is 0 outside, omega x is 0 outside omega y and this product will be 0 whenever either x is not outside the omega y is outside omega xy. So, any point outside omega xy this will give you 0. So, this is I am here saying this is I am not what I am trying to say is I am constructing supposing I construct this function like this, I claim that this function constructed like this is the valid marginal sorry you are valid joint pdf. Think about this.

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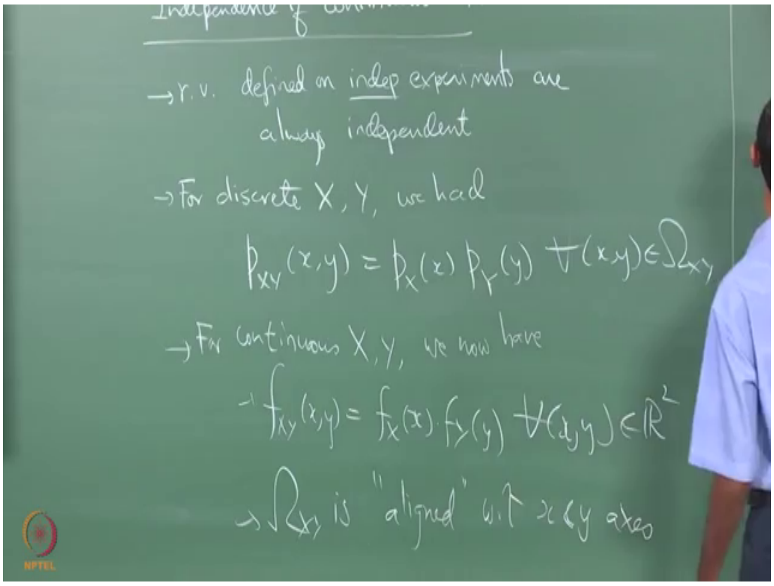


So, let us get going. With this is not the first time you saying in the independent random variables we saw them earlier in the discrete case and we had this property where the joint pmf was given by the product of the marginal pmfs for each and every point where you had nonzero joint probability

Now, in the continuous case is a very similar analysis construction holds that is if you want to combine two independent experiments, like for example, two different random number generators u 0 1 random number generators which is what we are going to you know look at is a very important example of this. But anyways it is true in general that if you are given the x and y separately you can always and you know, that they are coming from independent sources you can always write a joint pdf which is a product of the marginal pdfs, and all points on the plane actually using the external definitions of the individual pdfs. So, this is true even when this is 0 or this is 0 it does not matter, because 0 is always multipliable and the product is 0.

One other consequence of doing this is that this omega xy as I was saying at the end of last class is an aligned rectangle always an aligned rectangle it cannot be multiply sorry rotated with respect to the axis. So, even here, what was the nature of this omega xy in the discrete case? It was an aligned grid of points you could, not just have some arbitrary crazy points which did not fall into a nice grid. In the independent case it always omega xy, always has to look aligned with respect to the x and y axis it is there is no other way of forming it. So, here you had a grid of points nice grid here you have a rectangle itself or union of align rectangles, yes, let me just say aligned with respect to x and y axis, otherwise you can get a simple rectangle like this from x 1 to x 2, y 1 to y 2 this could be your omega xy. This is just an example of what I mean by align.

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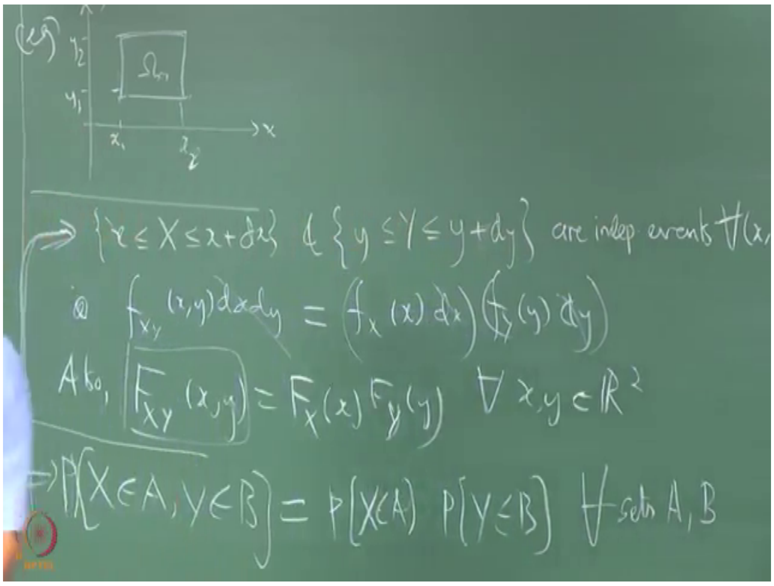


Why is this such a necessary condition for a independence? It means that essentially for any value of y the x takes the same set of values x 1 to x 2 the this is profoundly in contrast to what we had for the darts example is not it when we restricted the x and y to be in a circle as we change the values of the what x could take the y value were shrinking or expanding. So, if that happens then you know for sure that xy cannot be independent, they have to be dependent. So, this is a necessary condition, this omega xy looking like this is a necessary condition for the independence of x and y.

Of course it is not sufficient it is just necessary just giving you a mental just a picture to start with. And if you have the independence to begin with then; obviously, if you form omega xy as a Cartesian product omega x into omega cross omega y you will always end up looking like this you cannot end up looking you know having any other weird shape is not it, think about it right. And of course, a aligned includes this for example, the entire first quadrant for example, or any portion of the xy plane whose boundaries are parallel to the x and y axis of course, the whole xy plane is obviously, also an extreme example of an aligned omega xy itself, is not it.

So, what happens here there is a consequence of this is that, so the consequence of this guy implies that the events x that is these two events X axis between x and dx and y is between small y is capital Y it takes values between y plus dy. If you look at these two events these are independent events now for every value of small x and small y. So, there this joint if you look at the joint probability of this. So, in otherwise this, by the way the Joint probability is going to be f xy, xy, dx, dy for this event it always has this joint probability by the definition of the joint density function itself. So, this if you say it is independent then this, then it will be or the product of these two which is what, f x x dx into f y y dy. So, that is f y. So, this factorization happens which is equivalent to saying of course, you can always cancel these two and say this must be equal to this times this. So, that is the justification for making this statement out here in terms of the density functions.

In particular you also have the classic textbook definition of independence that is CDFs are also factorizable. So, this is the factorization of the joint pdf. The joint CDF are also equally well factorizable into the product of the marginal CDFs, the individual CDFs. So, you have the joint and also and also we have this will be this. Why do I get this? Because this is a joint probability that capital X takes values up to a not exceeding this number x here and a jointly y takes value is not exceeding capital Y. So, that event is a joint event and its joint probability is a product of these two p ab equals pa into pb just like here.

So, these are all equivalent definitions this is a least useful of them as I said we do not work with CDFs that much you just know that they are there in the background if we need them we will call upon them otherwise we do not usually use them right. So, this is also true for all xy in the plane actually right. Because we are using extended definitions we are using a certain event capital F x of x can be a certain event or it could be null all those things are always included in the definition of these guys. Similarly this, this guy could be null or certain or whatever. No matter what if if x and y are independent continuous or anyway this is true even for these two discrete random variables you can go and check that this definition will be perfectly valid for this discrete case also. So, generality of the joint CDF is what? Proms many authors and many treatments to start with that, but then they do not point out the problems of dealing with it, that is all.

So, this is you can think of this as a kind of a master statement which is true for all kinds of situations and in fact, this joint CDF itself is generally enough to accommodate some funny jumps in two dimensions or you know if you look at for example, the what is joint CDF of two discrete random variables it is a very complicated function, maybe you can try to plot it with MATLAB in a 3D plot or something, you will have some crazy up jumps in our every point where every point in the plane where there is a non finite and nonzero probability it will jump. So, you can it is just kind of you know crazy three dimensional staircase going from 0 to 1. But try plotting it for a simple case to see you know how many of you have you; have you tried MATLAB law 3D plots, have you ever had occasion to do them? Anybody? No, this is a good ah.

Student: (Refer Time: 13:45).

Good exercise for you, to take a very simple discrete example like this. You could take a simple let us say x could be a taking even the independent case or think of just two Bernoulli iid random variable 0 1, 0 1 and then plot the in fact, the (Refer Time: 14:08) you do not need MATLAB of course, you can plot the joint CDFs yourself because it is just 4 points, anyway.

So, the point is that this statement for independent random variables is true no matter what x and y you have, could be discreet, could be continuous, could be neither fully not fully, not what did we call it purely continuous and so on whereas, this statement in terms of pdfs is invoked only if the pdf itself is well behaved well right, if it is well defined.

Now, the interesting twist to this is that sometimes it turns out that when you derive a joint density function it turns out to have this property. Just like how did we first start to look at independence. We took a look at a die experiment and said well. If you look at the probability, joint probability of getting a square and in even number odd number or something it turned out you got 1 by 6, and 1 by 3, it 1 by 3 without having two different experiments begin with. Like that it is possible we will see some examples of that to get independence by accident. We saw in the discrete case also. I think one of the homework problems you would have seen x for example, what was that z equal to xy for uniform 0 one in that third problem you said you find that as x and z were actually independent. Even though a z was equal to x into y. Is not it, you did that thing. Like that you can get independence by accident also, it is not so commonly encountered, but it does happen once in a while. One important example is coming way maybe in a week from now or something right. But you do get this equation satisfied this, these both these conditions satisfied without explicitly, without it happen, without construction in other words.

So, in that case also obviously, the math is same, if the math is same they are independent, you do not care whether they started out in life is independent quantity and independent experiments or whether they came out of the same experiment does not matter, if this satisfied then they are independent.

So, basically what happens is that, but the bottom line is very simple for independent X and Y, you have this the probability that X takes values in some set A while the joint probability that X takes values in some set A which and A of course, is this X element of A bottom; it is just like an interval on the x axis without regard to y and the probability Y take some values as a set B which is some interval in the y axis without any reference to x. So, this will be what? This will always be equal to the probability that X is element of A multiplied by Y element of B for all sets A and B. What are these sets? This is a Borel sets basically, intervals essentially.