Probability Foundations for Electrical Engineers
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Lecture – 50
More Continuous Distributions

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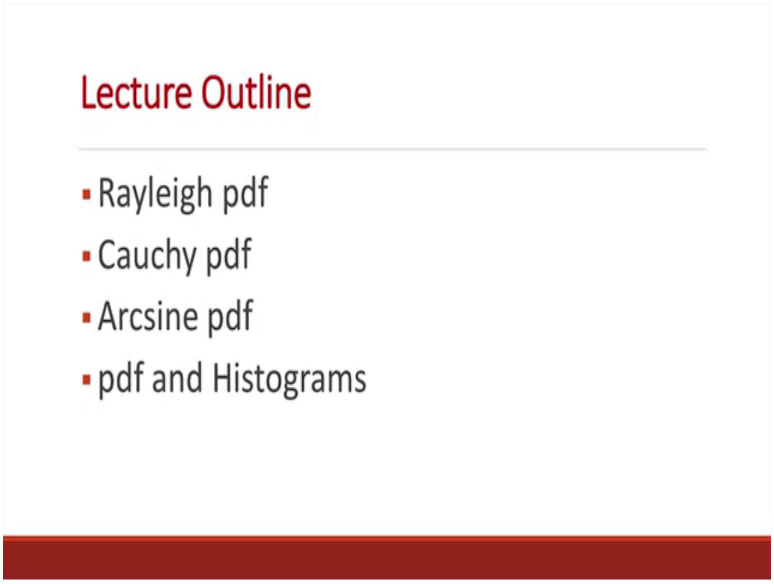

So, yesterday we went up to the Gaussian or normal pdf. So, today we will look at a few more examples. I use number 4 for Gaussian that is what I have here. So, number 5. So, is there any is a Gaussian pdf is you understand this nothing much too much else to say about it this remember the role of m and sigma. What is the role of m? It shifted from left to right, if m is positive. Where does it go? It goes to the right, if m is negative it goes to the left and what goes to the left the peak. And the maximum of the pdf occurs at m and also the pdf is symmetric about m. So, these things you have to mentally keep in mind.

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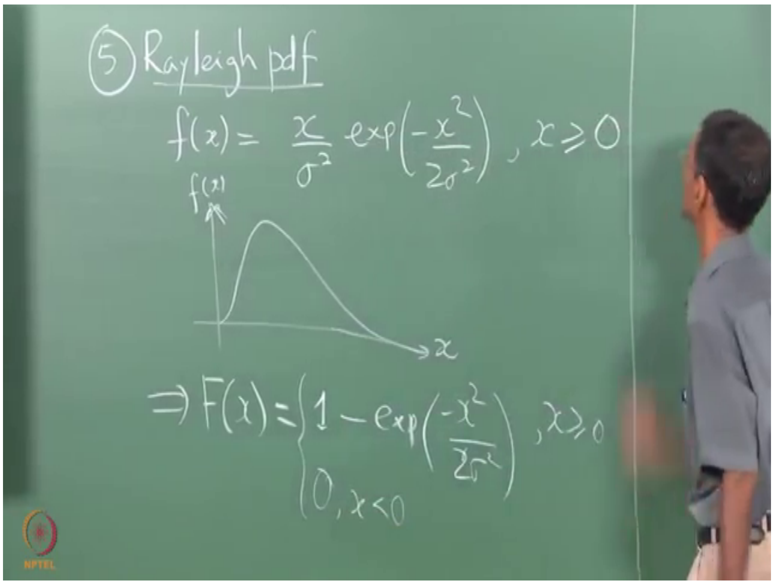

Then now this turns out this is very, this period is highly related or very very closely related to the Gaussian pdf. There is something I have to say. So, this is defined only for positive values with the argument or non negative values with the argument. So, whenever some such thing like this comes it automatically means what? The pdf is 0 for x less than 0. So, if you plot this what does it look like? Looks like this. Note that there is an x up here with the crucial x. Why is that x so crucial? It turns out that this the presence of this x makes integration of this pdf and this finding the CDF is it makes enables the finding of the CDF and so the CDF is immediately written down.

What is the CDF? Is integral from 0 to x of some you put some dummy variable here. So, if you do that what do you get? You recall how to integrate by parts I hope it is not a big problem or not even integrate by parts, I mean I am assuming that your write you can easily write this is 1 minus exp per what. What is it? What will come here?

Student: (Refer Time: 03:03).

Minus x squared the same as this minus x squared by 2 sigma square. So, this of course, again for x greater than 0 and obviously, F of x is 0 for you have to say this if you want. Now, there is one item I skipped over which is what, can we write an F capital F of x for the Gaussian pdf in closed form. Answer is no.

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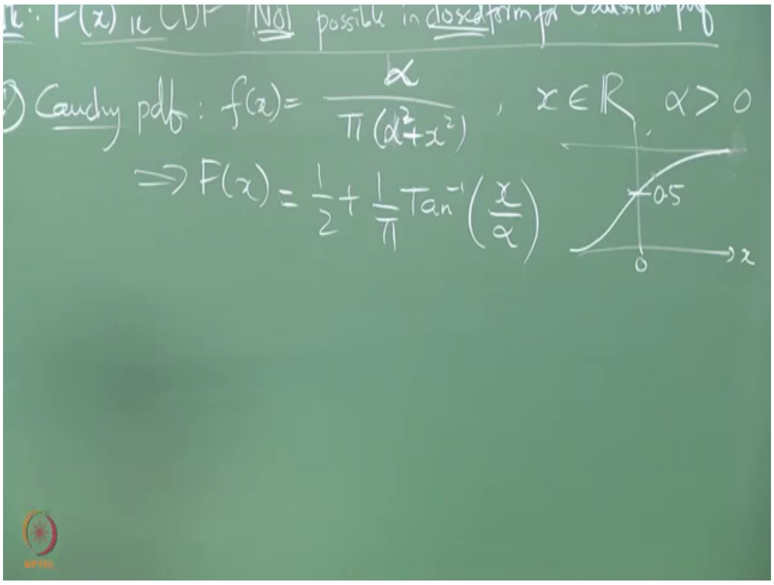

Note, this is not possible in closed form. I should have written say said this before I start that, but well I think it is good to remind ourselves when we can, we cannot, all I am trying to say is we cannot take the closed form expressions for CDF what always for granted which is another reason why pdfs are mathematically more convenient to manipulate than CDFs.

Something as important as the Gaussian pdf you have a closed form expression only for the pdf, but not for the CDF. But does that mean that capital F of x does not exist. Absolutely not, it has to be a number between 0 and 1, it is just that there is no nice expression for it that is all. So, later on we will see some upper bounds and approximations and so on for the Gaussian pdf which have been used to either approximate or bound anyway. So, we will come back to this issue after once we go through some of the examples right. But I have to emphasize this.

Then let us look at an example we have already seen the Cauchy right. So, I think the Cauchy example or let me just for save completeness also give it a name maybe I did not call it in now Cauchy pdf is basically the 1 by we saw this already. So, let me actually puts an alpha here, alpha by I will write it as alpha by pi into alpha square plus x square rather than 1 by and this is for all x. So, this we have already seen as an example of a two sided pdf long back or not last week when we started talking about pdfs and we know that the CDF is what? This is in some sense my very first example, this is the classical.

Earlier I may not have reduce the alpha, but you can clearly have the alpha also and obviously, this alpha has to be greater than are has to be a positive parameter cannot be a negative parameter. Earlier I believe I took may have taken the alpha to be unity, the previous encounter with this.

So, this is again true for I do not have to say it in even if I do not qualify this, this is understood x is any real number. So, here we have, there is some similarity between the Gaussian CDF and this CDF. In that both of them do something like this and it turns out that no matter what value of alpha you pick the value of this capital F of x is exactly 0.5 were at alpha equal as x equal to 0, is not it. Similarly if you take the Gaussian pdf and set m equal to 0 you always get capital F of 0 to be half because pdf is perfectly symmetric about the origin. Similarly this pdf is also symmetric about the origin and the Cauchy sorry the Laplacian pdf the two sided exponential that we wrote was also symmetric about the origin. But what is the fundamental difference between there are some similarities, but there is a big difference between Cauchy and Gaussian or normal.

The difference has to do with how they tail of the pdf decays to 0. The Gaussian pdf decays is e power minus x squared which is very fast where is, what is the rate of decay here? It is just barely sufficient to be integrable. If I had something like 1 by x I would not let us say not counting or 1 by 1 plus x or something let us say you know you can if you want to construct a pdf like that what is the immediate problem you run into; yes, they say you want to go all the way to infinity.

Student: (Refer Time: 08:51).

It will not be integrable, it will not have finite integral you only have a finite area. So, this x squared is almost like the smallest power of x in the denominator that is required for having a finite area. So, as a consequence of this x squared we will see later on that this Cauchy pdf has no moments we are going to see that, as not even the first moment if you look at the careful definition of the first moment whereas, the Gaussian pdf as every single moment you can. So, that is for later. So, this pdf is called a heavy tailed pdf, is an example of a heavy tailed pdf. It is a rate of decay decays as 1 by. So, all these pdfs have some mathematical property that we should be looking at.

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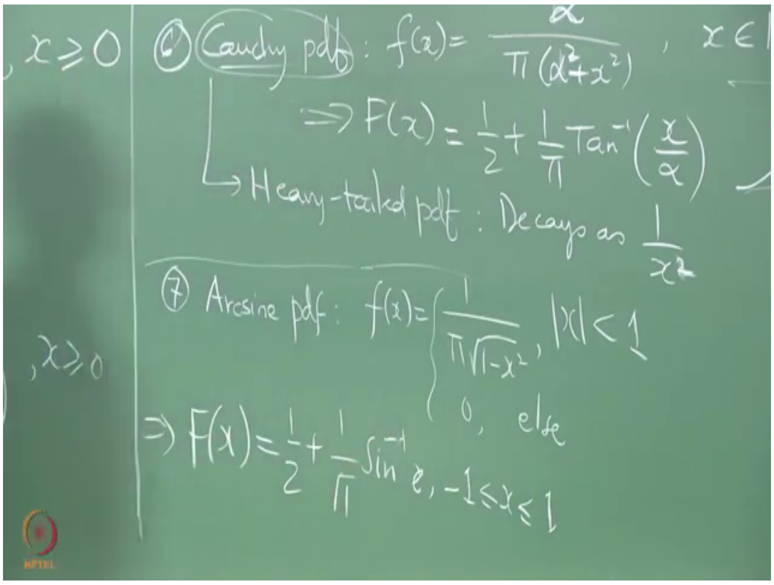

One more interesting example I promise not to go any further than this. It is so called arc sign pdf you might wonder where this thing might occur we will actually derive this you know find an example real life example of where this can in fact occur, say mod x less than 1. What is interesting about this? Why is it first of all, first of all does it have finite area. Does it?

Student: Yes.

Does it have a closed form CDF? What is F of x? Again you have, exactly somebody said sin inverse, it is exactly like this except that this tan inverse becomes a sin inverse. So, we will write these inverse trigonometric functions with just the minus 1 superscript instead of arc sine and arc tan and so on. So, arc sine or sin inverse of just x and this expression is valid only for minus 1 less than x you can say even equal to. What is the value at minus 1? It turns out it will be 0. So, what is this? Just sketch these things. So, this as a sketch will look like what, is a pdf bounded I said as no its not bounded pdf can go up to infinity at x equal to minus 1 and plus 1, but that does not prevent it from being integrable having finite area.

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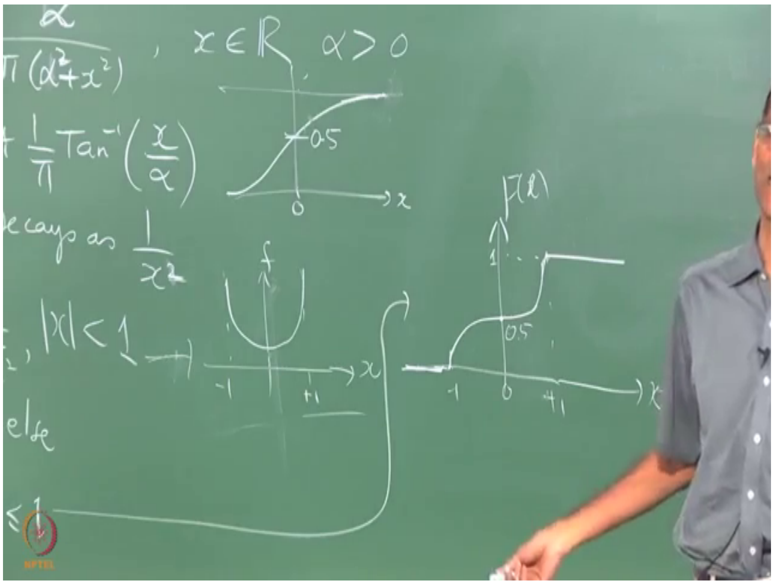

So, this thing looks like this, and as a consequence of this going to infinity plus infinity here in here what is this capital F look like? It will take off, it will go up is go up to minus 1, this is the portion where we saw its 0. So, if I want to fully specify this I have to say f of x is this between minus 1 and 1, F of x is 0 for x less than minus 1 or F of x is 1 or blah blah blah, I am not going to write all that stuff here. I write, I do not think that is needed we will just plot it.

What is the slope at minus 1? It takes off vertically and then does this. Ends at plus 1 also it ends vertically, so in this portion from minus from, so this value is unity again as usual as with all these the symmetry about the origin says what, that the value at 0 is 0.5 if I put x equal to 0 note this is sin inverse x here. So, if I put x equal to 0 obviously, I get 0.5. So, this is this portion corresponds to this expression.

So, you can write endlessly debate does this function of a derivative at x equal to plus 1 and minus 1. As I said we are expanding or concept of derivative to include such cases also. So, there is nothing peculiar about this pdf at all, it is a very healthy pdf and it just turns out that it is unbounded at plus 1 and minus 1, that is all. So, this kind of thing can never happen in a discrete distribution. You can never again a pmf, a pmf always has to be bounded between 0 and 1, impossible for a pmf to behave like this. So, this is one more strong or should I say illustration with a point that a pdf value is not a probability.

Remember that for forever and ever especially if you go for I was and stop. So, on you may be asked this question, immediately say, no, pdf is not a probability. However, of course, CDF values each and every one of them is a probability. So, I am not going to there are; if you go and look in tables and books there are lots of more examples like a long pdf and gamma pdf and this and that, again more and more mathematically complicated. So, I am not going to do any more any more of them. I just wanted to give you of the simpler ones. Most of these we will use the examples later on in the course also, so please be familiar with them. Especially the Rayleigh turns out which I just erased has some very interesting applications in wireless communications right.

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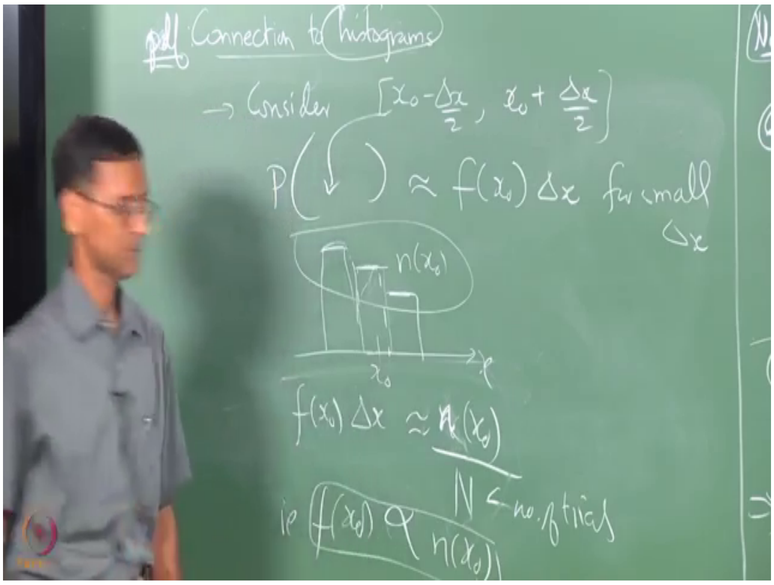

So, what is the connection? Let me just say one small thing before going ahead with them. So, connection to histograms or probabilities well the pdf connection, let me say that the density function it is connected to histograms in the sense connection in the sense there is a relation between a pdf value and a histogram. So, supposing you consider an interval it does not matter closed or open x naught my say minus delta x by 2 to its not. So, you pick some value and pick some delta x. So, the width of the interval is delta x it can I am just taking a symmetric interval for convenience I could take it non symmetric also. So, what is the; so the probability of finding of encounter or getting this interval is basically probability of this interval as we said first it will be f of x naught into delta x for small delta, small delta x and assume of course, delta x is positive.

So, this, if I plot this x naught and I look at if I run the experiment many times what is going to be assumed that at, this number occur that this interval occurs some n of x naught times right. This interval occurs n of x naught times will occur. So, this would be a histogram that you could have drawn of the experiment. So, this is a different. So, I am assuming that I have carved up the x axis into small small bins and plotting a histogram of continuous quantity this is very doable which is what most people do when they encounter some data. One of the things that people do and they get a lot of data of some experiment is to do something like this.

So, this particular n of x naught and let us say you have the total number of trials are some n. So, what is the relationship now? So, this f of x naught delta x is supposed is supposed to be roughly n of x naught divided by number total number of trials which we will call say. So, I will use capital N just to make I do not want you small n here just use capital N. This is this n of x naught.

So, this kind of a plot again as I said is a histogram. So, what is it saying? That the pdf value at any x naught is what, is proportional to the histogram, to the histogram counter. So, this delta x will write if you want to remove the delta x you have to say this is proportional to that. So, there is obviously this connection of, the pdf is not totally unconnected from probability this is exactly it models this probability of getting an observation in this interval.

So, given some unknown, say unknown distribution how would you go about if I deciding which pdf is a good model. You have to fit; you have to look at the histogram carefully and fit various pdf models to the histogram and see which one best fits the data, and that is not a trivial exercise I have to tell you. I just say f of x naught is proportional, alpha is proportional to n of x naught assuming you have N, this N is fixed right. So, just keep this also in mind this could also be I am obviously, not going to ask this kind of thing in the exam, but somebody you know it could easily ask this question in the viva or interview.