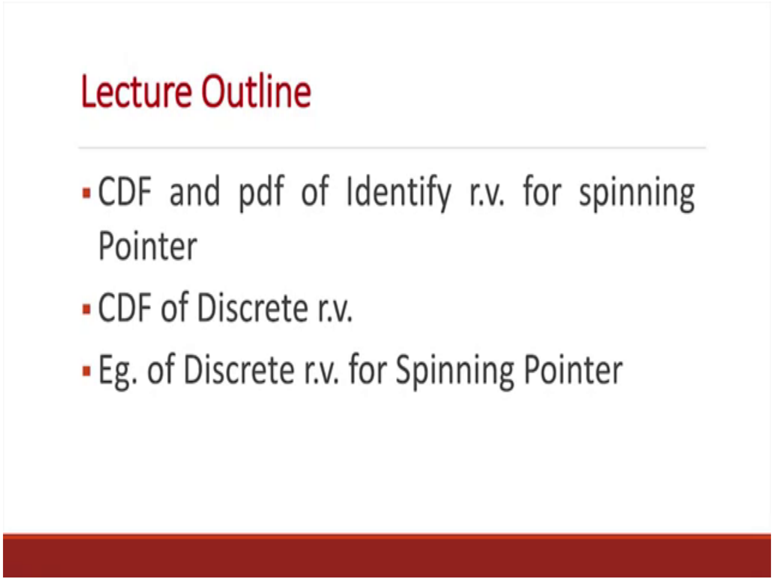
Probability Foundations for Electrical Engineers
Prof. Aravind R
Department of Electrical Engineering
Indian Institute of Technology, Madras

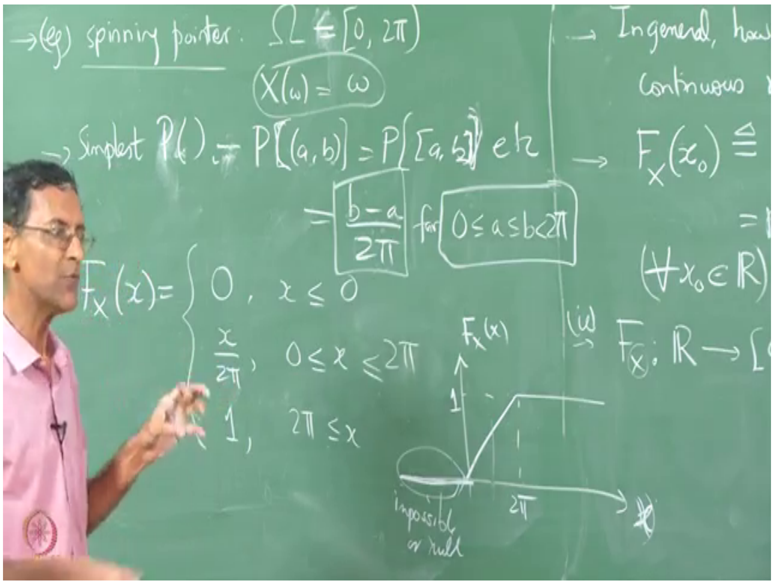
Lecture – 48
Spinning Pointer Example

(Refer Slide Time: 00:13)



So, now we want to go back to the spinning pointer where we have the pointer take you know coming to rest at angle between 0 and 2 pi and we are as I said we are using the open interval 2 pi to indicate the 2 pi essential is identical as 0.

(Refer Slide Time: 00:22)



But the pointer obviously, there are no negative numbers as the though as the outcome of the experiment there are no numbers bigger than or a bigger than or equal to 2 pi. I say output of the expense. So, the number there 7 numbers we can get there is a angles we can get are limited between 0 and 2 pi.

So, this only tells us on x sorry this is only give tells us the sample space no. Of course, then I am going to also let X, capital X of omega also b omega itself again this is the simplest definition of the random variable x in this in this particular case right. So, now on the, now on the third mean before to proceed further to calculate the CDF of this X we need to put a probability measure on omega. So, how are you going to do it again it tells out that the simplest probability measure is a so called uniform probability measure simplest P is defined is basically you take any interval a b whether open or closed where these two numbers a and b are between 0 and 2 pi. So, we say that we will make this probability assignment to this interval.

This it turns out now here we are actually assigning a probability directly based on the interval length which is b minus a, we are not using the CDF as such I mean being more general than CDF, in the CDF we would go minus infinity to 2 x naught. But I am doing it more generally now between for 2, any interval 2 points a and b of course, I am going to assume that. For any a and b which is between 0 and 2 pi. So, if I pick 2 angles between a and b between 0 and 2 pi I am going to say that the probability of observing that the pointer is stopping in that interval is basically this b minus a divided by 2 pi. Why divided by 2 pi? Because it is a normalization, we want this probability to be between 0 and 1 or to have an action while have unity. When does it have a maximum value of unity? When you consider the entire dial, when we take b to b 2 pi when you take a to be 0 then you are asking for the probability that it stops somewhere on the dial which is obviously, always unity it. Has to stop somewhere on the dial, is not it.

So, it turns out that this P of a b is equal to b minus a by 2 pi only if this a and b are both are real angles that you can observe, but when you talk about CDF we have to now define the CDF for any real argument and without you know getting into too much of explanation about this, let me just de define the pdf CDF for this particular random variable identity random variable. So, here they turns out that this F x of x basically can be defined, it is basically 0 for x smaller than 0. If x is negative or even 0 the CDF has a value 0 because this basically is saying that you cannot get any negative numbers to the output of the experiment and any interval of the type minus infinity to x when x is negative a 0 probability is basically a null event or in possibly event.

Now, if you take values of x between 0 and 2 pi it becomes x divided by 2 pi for numbers between 0 and 2 pi anyway, is again put a even put an equality here because if you say if you put x equal to 2 pi it basically becomes 1. I can put less than equal to everywhere here because I am only defining the function and I am it is actually continuous, if this continuous everywhere.

Therefore, it is perfectly to write it with the less than equal to although some people might leave what some of these equal tools in some ways, but does not matter I think it is better to you know for understanding sake to appreciate that it is a continuous function. So, even though you are defining in as 3 different things you are defining them to be as 3 different pieces are all continuous. So, essentially it looks like a simple ramp, it goes like this. So, basically here you get impossible events, impossible or null. Why impossible or null? Because you can never observe negative angles, the angles you can observe are always between 0 and 2 pi.

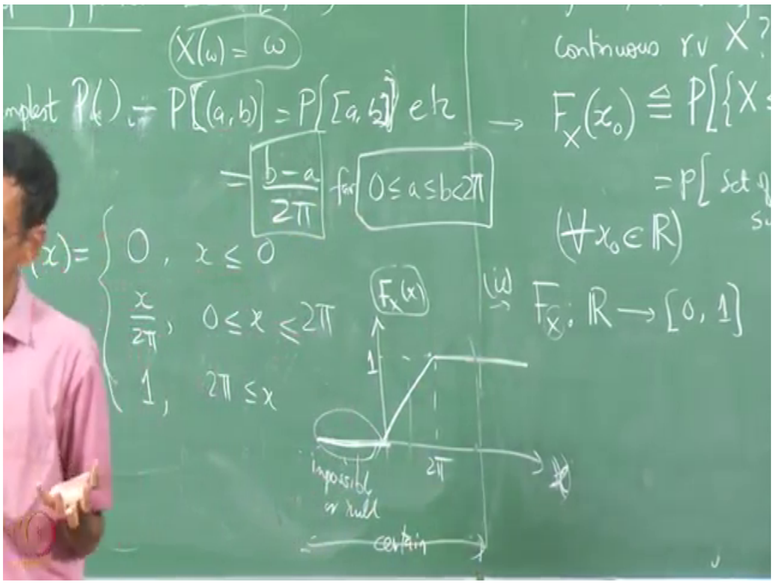
Now, once you observe an angle I mean once you say well I pick a number x which is this x is entirely under your country you can ask for the problem CDF of 0.5 or CDF of 1 or the CDF of whatever between you know for any real number any real argument x. So, remember that the CDF of any random variable, these are, is specified to be defined for all real numbers, not irrespective of their omega. So, this may it is impossible or null here and here in the when your argument is between 0 and 2 pi if F x increases linearly the CDF increases linearly because you are assigning probabilities in this fashion if you do not assign probabilities like this then it is not going to be this linear increase.

Now, why are you assigning probabilities like this? Because we are saying that the probability of an interval is proportional to the length of that interval right. So, if you take a longer portion of the dial it has that proportionally higher probability the entire dial has probability unity. A small portion of the dial will have a smaller probability which is scaled down by the length of the, or the interval the of angles that you are that that you specify out here. So, if I specify pi by 2 to 0 then it will be one-fourth or if I specify pi by 6 to 0 it will be y by 12, 1 by 12 sorry or if I specify pi by 3 to pi by 2 it will be whatever.

So, this assignment of probabilities is just our way of recognizing the fact that the pointer stops uniformly on the dial and does not favor any particular interval over any other interval. So, all intervals of the same length are treated or identical as far as the pointer is concerned which is what we expect in practice, unless we have a funny point spinning mechanism which is not, which is deliberately not or whatever for whatever reason not uniform, but anyway that is a very common more much more complicated situations. We take the simplest case again which is nicely well oiled machinery which means that the pointer stops at any point regardless of where the point is on the dial.

So, we have an impossible or null event for small x between 0 minus infinity 0 then we have this linear increase and then we have unit once it once the CDF hits unity then it stays you at unity you cannot increase any further. So, and what does this unity mean? This now becomes a certain event.

(Refer Slide Time: 08:53)

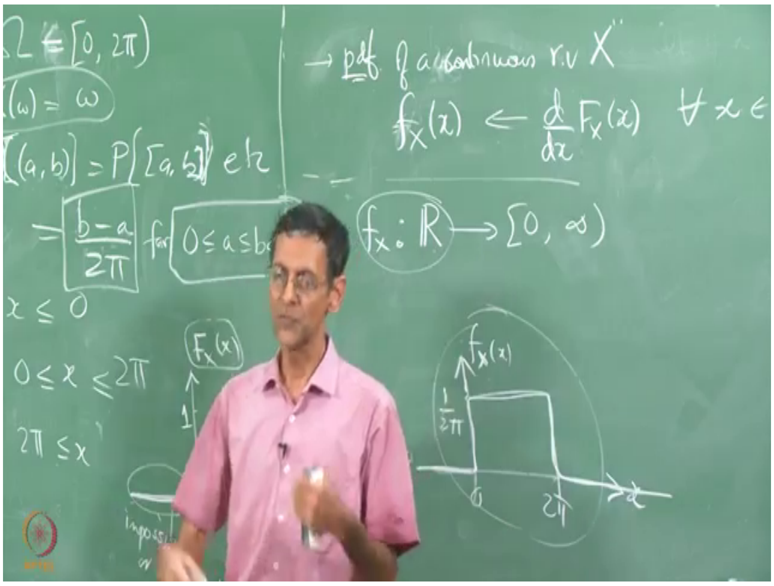


In other words it is certain that you will assign, you will observe a real the point of value will be between minus infinity and 10 for example, if I pick this number is 10 or 9 because 2 pi as you all know is 6.28 or whatever it is and therefore, any number bigger than; that means, the probability of the interval from minus infinity to up to that is always unity. So, big this becomes, it becomes a certain event once you go beyond 2 pi and it impossible event if the upper limit this x here is below 0.

So, this is a very simple example of a CDF assigned to this identity mapping. But in general this calculation is not, gain I have to warn you it is not an easy calculation to do and cannot and you cannot very easily extrapolate from this, you cannot claim that in general its always going to be an easy thing to do in general it is not. And I think it is better that we do not consider more complicated examples now because they do not really serve in much of a purpose that is what I.

So, we will see how far we can push this. And most of the things that we will do in the immediate from the immediate for your next lectures is basically to consider this situation. We take some portion of the real line not necessarily 0 to 2 pi could be any I know 0 to infinity for example, the omega could be any non negative real number and we define x of omega, 3 omega. So, then we can immediately not by we do not have to worry about you know bigger spaces or anything of the sort and calculations become a lot simpler. So, the last thing that we will you need to clarifies now the density, the pdf of a continuous random variable.

(Refer Slide Time: 10:48)



So, pdf the density function probability density function of a continuous random variable its basically f x of x once you find the CDF you can end at the CDF is continuous. You just define it for sorry differentiated for all I mean through everywhere the real line and then you get they by differentiation can go from the CDF to the pdf which is exactly what we did earlier. So, in other words this is going to become the uniform pdf between 0 and 2 pi. So, this is going to be capital F, this is going to be the density function and to obtain probabilities of intervals as we did earlier we have to integrate this pdf over a to b or whatever right.

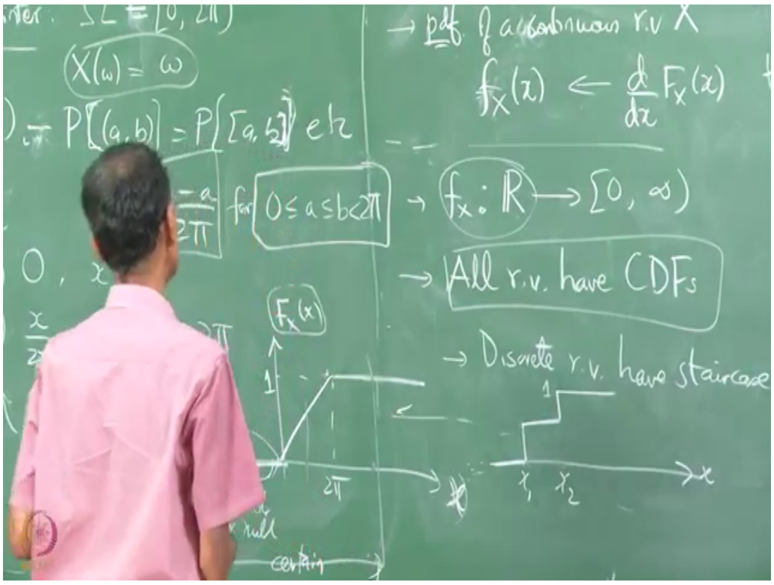
So, it becomes you know in this particular case it is very easy to go back and forth between this the pdf and the CDF. So, all that we require to obtain this pdf is just that basically where we are not even bothered so much with these corner points where as long as the CDF is continuous and as long as these points are few in number you can always differentiate this to get this. Let us not worry about the exact value derivative here and here. We can define them in some consistent way, that is the exact value of derivative at 0 and 2 pi is not of any importance at all.

So, therefore, for simple case we have looked at how going from a given probability assignment we can go we can calculate the CDF and then you know get to the pdf by differentiation. In many examples that we are going to look at we are going to start from the pdf we are going to say x hat this pdf density function plus small pdf oh yeah of course, there, thankfully p and c are different right, pdf. So, we are going to start with some pdf and then say what can we do that of course, we can always obtain this a CDF my integration going in this direction that is again as we did earlier. The only difference between the earlier lecture, now is that we have attached this the name of the random variable to a CDF of pdf. And how are we doing it? We are going we are saying that the CDF of any random variable is basically the probability that the capital X takes values between or no bigger than the argument.

So, I mean let me not, I mean I have said that enough some now going to write, I am just going to define the domain of f x. So, f x is basically a function which takes you from R to 0 to infinity as always right. So, again by domain extension over you know it depends on how you view this is defining the entire, I mean you are defining the CDF or pdf sorry the pdf over the entire real line from minus infinity to infinity. It could be 0 over most portions of the real line. Wherever it is nonzero and this particular is been 0 to pi that interval of the real line is called the support of the pdf. So, basically it is that the p the pdf tends to in many case in many examples tends to be nonzero only or a support interval of the real line. In the rest of the real line it typically it takes the value 0.

So, that means that the random variable takes values only wherever the pdf is nonzero. But again as I said earlier the value of the pdf is not a probability the value of the pdf is never a probability all only by integrating the pdf do you get actual probabilities. Then I said earlier also that the concept of CDF is more general than pdf all random variables basically they have CDFs whether continuous or discrete or any random variable basically has a CDF.

(Refer Slide Time: 15:04)

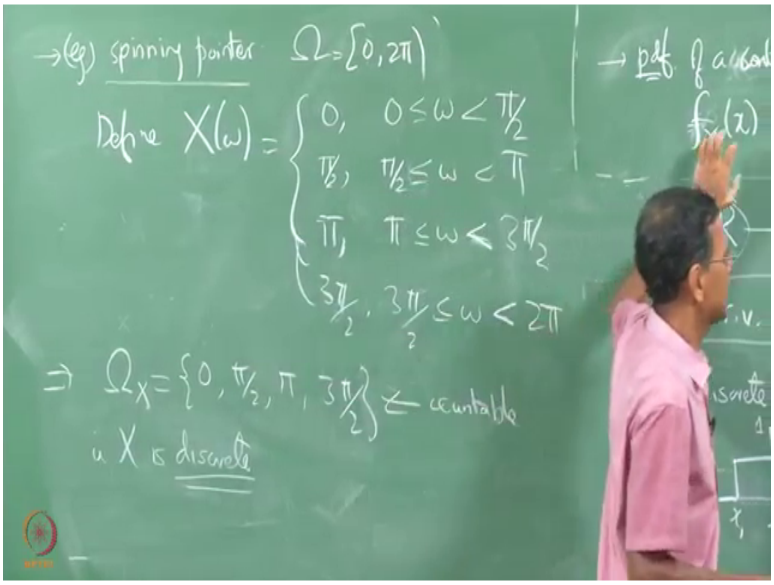


So that means that CDF now turns out to be the mathematically unifying concept of photo random variable whether the continuous or discrete right. So, what kind of a CDF does it discrete random variable have? Turns out that discrete random variables have staircase CDF. Why staircase because, let us say these are the values that the random variable takes x 1, x 2 for example, just I said let it they only take two values. In other words it goes from 0 to a, the random variable takes only value you know values to only two values like head or tail, 0 1 whatever right. So, up to x 1 you have a null event and so the probability of the random variable x take of taking any values in interval is 0.

If you include x 1 then it jumps the CDF of jumps to some value between 0 and 1, and then at x 2 because now it becomes a certain event that there are problem the random variable will always take well is between minus infinity and x 2, x 2 included either x 2 the CDF jumps to 1. So, the CDF ends up being continuous and a staircase which is actually quite messy to a specify analytically that is why for discrete random variables we typically do not specify their probability sigma here in terms of their CDF because this type of function is very messy to write down.

I think we have said the enough things today. So, one last point again I mean before closing is that just because you have a continuous space you need not or necessarily always define continuous random variable. You can also define discrete random variable by the process of quantization. Let me just end this lecture with the example of how define a discrete random variable on a continuous space. So, if I go back to the spinning pointer and if I do not define x of omega equal to omega, in other words if I define, omega capital omega is still 0 2 pi but I define x of omega in this fashion, I define it to be 0 when let us say omega takes values between 0 and is strictly smaller than pi by 2.

(Refer Slide Time: 17:35)



Now, here note I cannot put less than equal to 2. Then I say that the measurement takes the value pi by 2 for this range of output angles. So, if this pointer is going to stop at any number between 0 and pi by 2 not, pi by 2 not included I am going to say that the value of the random variable is 0. Then I am going to take, so I am going to extend this. So, clearly I have to specify the entire range here, the entire sorry entire omega here. So, I have to let me see I continue this in this fashion, pi less than equal to omega less than I say three pi by 2 and then 3 pi by 2, for 3 pi by 2 to 2 pi. So, this is now a full specification of a random variable because I have defined an x of omega for all omega that can come out of the experiment.

So, this is exactly something which fits our definition of a water variable it should be should do. It should take some real value based on the output of the experiment. So, that is what is doing here. But the omega x in this case is just the set of 4 values 0 pi by 2 pi three pi by 2 which is certainly a countable set and therefore, exist in this case x is discrete it is not continuous and you can talk here in terms of the pmf of this x. What is a pa probably the x takes a value 0? That is the probability that you observe this interval.

What is the probably that x you take x takes the value pi by 2? That corresponds to probability of this interval and so on.

So, you need not necessarily define it only defined continuous random variables on uncountable omega or and omega is a portion of the real line you could define something like this, and then based on the probability measure you want to give to these intervals you can you know that will induce the probability mass function and induces specific in individual probabilities of these points.

So, I think, I hope that with this the idea of continuous random variable as little is more as clear and we have we have covered the we have understood how to marry the idea of a CDF with and pdf of a random variable. We are not talking about them in some generic sense, but we are talking about them in conjunction, with a random variable. And remember that the key point for that I said earlier which is which is erased now that the CDF of a random variable is the probability and its consistent, it is basically equals the probability that random variable takes values between which is no bigger than the argument that you give. So, the argument of the CDF of pdf is entirely deterministic there is nothing random about the argument that is entirely specified by the user.

Thank you.