Probability Foundations for Electrical Engineers
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Lecture - 45
Cumulative Distribution Function (CDF)

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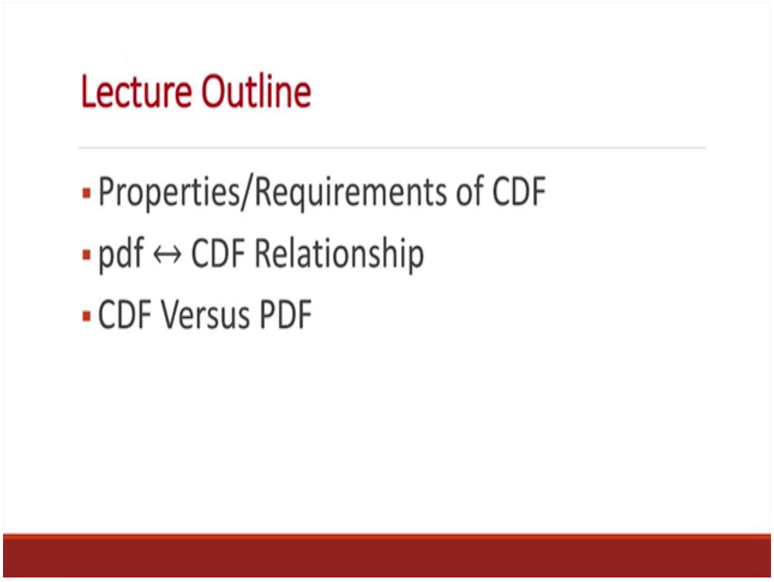

So, now this is I say is one way by which you can assign probabilities to these Borel sets. At different way of or a more general way in fact, it turns out more general way of assigning another way another more general way of assigning probabilities, is through what, is through. What is this? Those who are taken studied probability before to give the answer to this question. What we have, what we call a Cumulative Distribution Function CDF.

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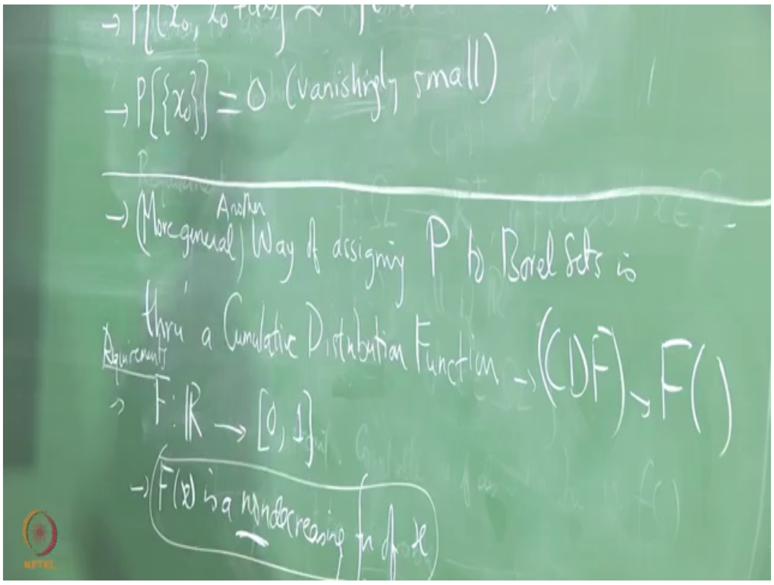

And here we will always use uppercase capital letters to specify the CDF to specify the acronym will always be in caps right. So, what is this CDF? These are unlike the pdf the CDF is actually a probability. It is not you do not have to do anything to it, it is like a instance or some sense a the counterpart of a pmf if you will fine.

So, here we have, let us say a function in general the function F. So, let me call this CDF as we will use capital F. So, what is the domain of this F? Typically we will write define it for the entire real line and let us say it is a probability so its range is only the interval 0 or 1. So, and what are the properties of this F? It is going to be a probability its domain is the entire real line, basically we want it to be a non decreasing function of x F of x, it is a non decreasing function as x increases this k should not decrease, it is a non decreasing function of x. that is very very important con constraint. Why because we will see. Before I look at the exact meaning of F let me write out two more conditions.

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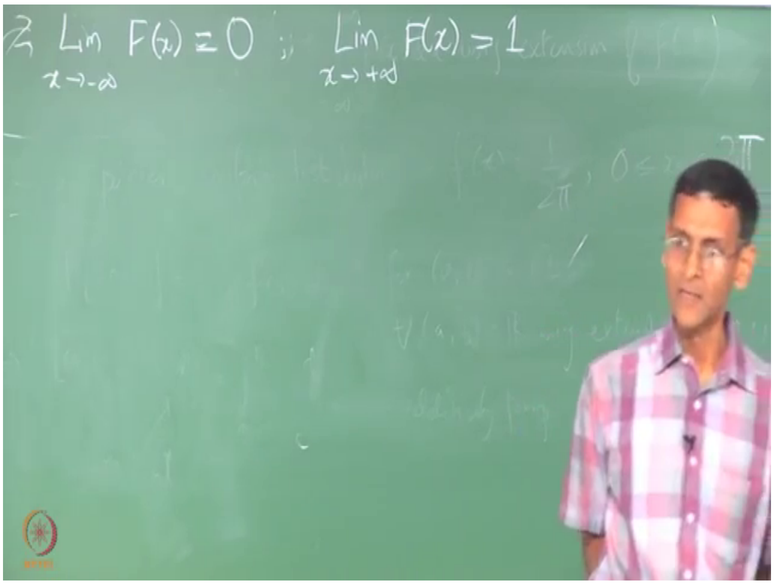

I want its limit, as x goes to minus infinity I want these are requirements by the way sort I want the F to satisfy these. So, these are requirements, this, this if you are you can add the word requirements here. What is what do we want this F to look like. This should be what, what value should be assigned to this it should be 0 as x goes to minus infinity I want F of x to go to 0.

Remember by insisting on if you look at the pdf, if you want the pdf to be integrable all the way from minus infinity, infinity small f also has to go to 0, but that is not the sufficient condition. For integral fine you know that there are functions which go to 0 which still cannot be integrated that is a typical example is 1 by x. So, just keep that in mind, but even the pdf has to go to 0 if you write, if it does not go to 0 you cannot it will not have a finite integral, but anyway we are not going to integrate this F away anyhow we just want to put this as a as a condition and as x goes to plus infinity we want this to be 1.

In other words the F starts out at 0 at minus infinity and ends up at 1 and plus infinity and in between. What does it do? It does not decrease. So, monotonically increases if you want, but it can stay flat also right. So, an example of such an F, let me just make sure I do not. If somebody were to plot this what will it look like?

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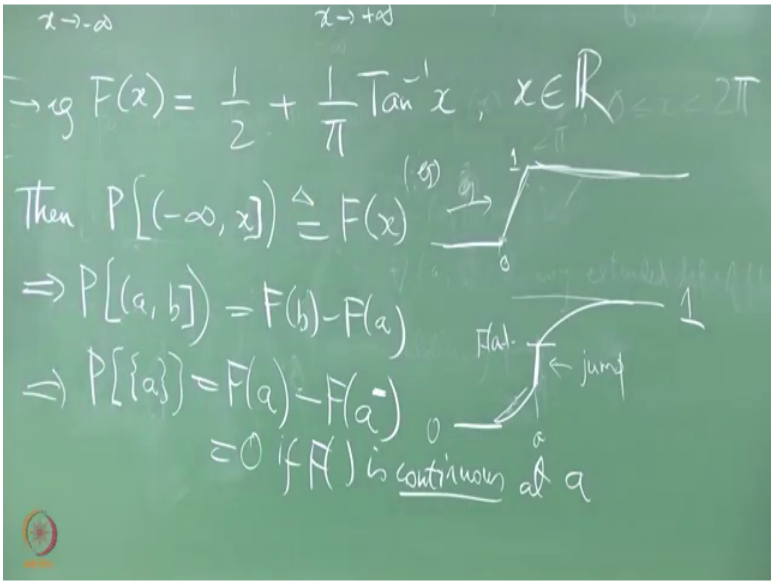

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What does this look like? Does it satisfy my conditions? Yes, it does. So, if I have such an F then I can said it turns out this probability I sorry this event that is the portion of the real line from minus infinity to x, I will define it to be or I will define F of x and should maybe I put it the other way around, F of x at any point is going to be the probability of this event.

Now you may ask me what if minus infinity is not part of omega whatever so obviously, this could be a combination of an ally this interval could be a combination of a null event and I useful portion, but it is. So, I let me not get into too many nits not split hairs over the exact orders I mean the role of capital omega in this and so on. You can always it turns out that even when omega is only 0 to 2 pi you can always look at this event and meaningful assign probability is no matter what number you put up here. If you put for example, any number less than 0 this will become a null event and F of x would be will be 0. So, there is nothing to prevent F of x to be 0 all the way from 0 to minus infinity to 0 for example, because we said it has to be non-decreasing. So, you could have something like this, this is one example, this is another example. It can be proper to 0 it can be flat and then you can go to one and then remain flat here, this is also possible.

In any case if you start out with such a function which behaves like this then its values can be assigned to be the probability of this event where we by convention we look at this x as a, we include this they are the upper limit the x. This for if for all x which is not infinity usually as I said yesterday we do not usually think include the infinity as a closed interval, but anyway for all, but that is it says if we do not worry about that in this definition anyhow. We can just say for all finite x, but x here this is a well defined interval and the probability of this interval is given by the value of this function. So, why is this more general than this, than the density? It is more general because I can have discontinuities in this F which I cannot get by having a density functions. They will we will see that let us go, you know be a little careful with this.

So, therefore, this is the definition, which implies then what? That the probability of the interval a to included b, not a but including b will be what? Will be F of b minus F of a this follows directly from this, from the definition of the CDF. So, this is just by definition given this function right. In fact, I think I have it right, I am saying that I am starting out with some such function either like this or like this and then I am saying given this I can always do this then if I do this then this follows.

The probability of this a way of this interval open at this end a closed at this end b because this is closed here. So, how does this follow from this? Because this is the union of minus infinity to included a and not including a to included b those are disjoined. So, that will be F of b this would be F of a. So, again by a you know sorry basic if you take the F of a on this side clear that it has to follow. It is a, this is the union of two disjoint events as obvious from the construction of this all right.

What about the probability of the isolated point a itself? This is going to be F of. So, if I make this b arbitrarily close to a or so, supposing I want to write I want to include that I the point a itself I just want to look at that point then all I have to do is look at the value at. So, supposing let me say if I do something with a jump out here if I have some something like this and it jumped and something like this, this could also be an F this could be 1, this could be 0 I have a jump here let us say, let us say I have a jump and this is a value. So, what I am saying is this is the value F of the, F of a includes the value of the jump at a if there is a jump.

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Yeah, basically in I was just going to say that F is defined to be continuous because we are including them this end point in the interval. So, this difference between that is F of a minus which is the limit coming from here and the value. So, this difference is base is a value is a probability which comes from this if you make or maybe I should have put be here let us say a approaches be in this could maybe I mean it is better to be put b here, but does not matter. Maybe I am getting confused because I wrote a and it is probably better to have started b you know if it becomes clear if you put b and write this is b minus a b minus is not it, instead of writing a its anyway I do not want to erase this right. So, I will just leave it as it is. So, nothing wrong with this right, but this does not so directly follow from this you know with a, but it follows with b.

So, what is this saying? This is saying that this model can actually include a combination of points with some finite probability and have smoothly increasing F at other places also. It can actually do the CDF model is capable of being more general than the pdf model. What? So, this when is the 0, this is 0, if what, if F is continuous at a right. So, this is 0. So, at every point where F is continuous you do not have a finite probability of that particular point, but the model allows you to it says nothing about having jumps in F it is. In fact, a countable number of F jumps in F is in fact, allowed right. So, let us go by a stay that out here.

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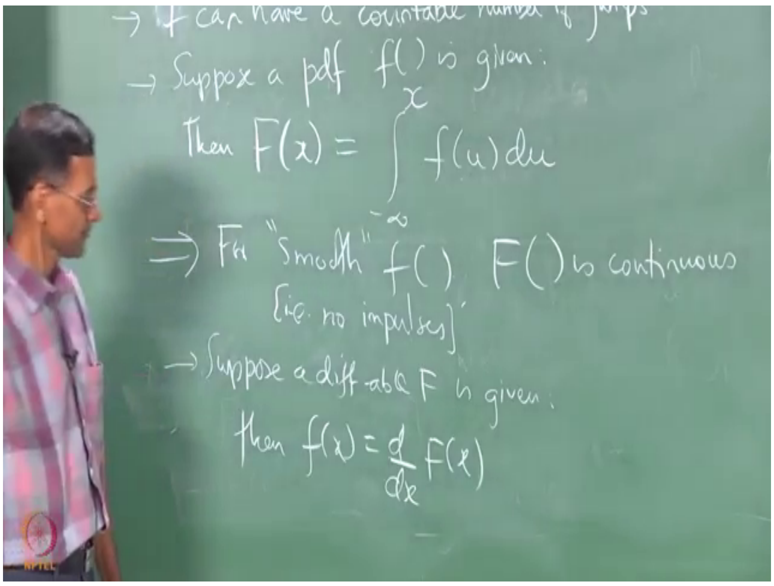

So, we can have or let me say F can have a countable number of jumps and I say that countable can actually be countably infinite. But supposing your F is small f as the pdf is defined what is the relationship using calculus between the pdf and the CDF? Supposing f is specified, then capital F of x is what is this in terms of small x is; obviously, the integral from minus infinity to x of f u du. Again here we are using the external domain for small f and by definition the domain of capital F is the entire real line regardless of the actual portion of the real line that we are interested. In just conventions that we are using that this is the way the CDF is defined and a pdf also, if suppose. So, can this model give you ever it, can it ever give you discontinuities in capital F? This cannot write the integral can never give you discontinuities unless this F is what, some crazy impulsive type of thing which we are not going to consider in this course right.

Let me also be very clear in this flow of course, we are not going to ever invoke the dirac dirac impulse right. So, do you agree with me that this capital F you know is in a way more general because if you start with this small f you can always get this capital F using this equation, but the constraint is that this capital F is going to be its continuous which implies that for smooth F turns out to be continuous. So, if you start with the smooth when I say smooth I mean I am including the case of countable number of jumps no that is no impulses. So, we are not going to consider this case at all right.

So, as I said some time back the constraint or small f is that it at best what we will look at in our models will be a continual finite number of jumps and that we will include in our definition of smooth for example, 0 from minus infinity is 0 and then 1 by 2 pi from 0 to 2 pi and then 0 afterwards. That is perfectly ok, that is to what we call a smooth F right. So, the point is that with such as such an f you cannot get any discontinuities happen that that is consistent with our earlier observation that if you use probability, if you define small probabilities based on small f all individual points will have vanishingly small probabilities no point can have finite probabilities. However, if you want to for some reason as we will see in some homework problems if you want to have points with some finite probability more than non 0 you cannot use pdfs you have to in fact, use the CDF to specify that probability distribution right.

So, if you want discontinuities like this a jumps like this you have no choice, but to, I mean you can say some textbooks do in fact, allow some impulses, but impulses actually it turns out more they cause more trouble than they are worth right. So, there are good reasons not to bring them in right. So, we will use pdfs only for continuous CDFs.

Student: (Refer Time: 17:43).

No, pdf is allowed to have jumps, but no impulses. CDFs can have jumps, if this F is continuous then if F is or the let us say for differentiable F then we can write what. So, supposing a differentiable air and if you have a differentiable F is given then how do you get the pdf from this then you differentiate. This is the exact opposite of this and calculus will tell you that these two are. But I want to make this little more general I do not want to insist that small capital F should be differentiable in other words I want to allow kinks. For example, here mathematics will strictly say oh this point you cannot differentiate capital F with that I am saying it is does not matter.

You take some you we will assume that you know there is some ambiguity in the exact derivative at that point you know, but it really does not matter as far as probability is concerned right. So, you take either for exactly at 0 you can take either 0 or the value of a value from this side its right. So, this differentiable we will relax a little bit and say that actually in practice we can do, we will do this operation even when you have kinks right. So, that we will try to have the maximum use of I mean this equation. Apply also to things just say continuous F.

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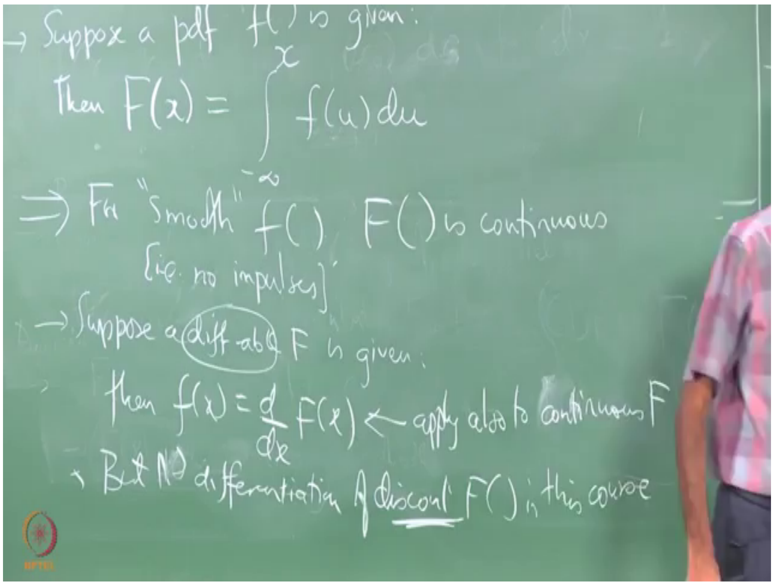

So, I am going to clay; I am going to say for the purposes of this course any continuous function can be differentiated. Although mathematicians will scream I say I will say fine does not matter you will live with some discontinuities. After all what are they going to, what is it, how, what is it derivative going to be? It is going to be discontinuous and you have said anyway small f is allowed to have discontinuities. And are those points of discontinuity you clearly cannot define the precise value of f, but does not matter and anyway small f is not a probability. It is only its integral is a probability and how you define it at a point of discontinuity, does not change any probability calculation whatsoever.

So, therefore, I am saying that this differentiation idea we will apply you know as long as we can find some meaningful derivative at some point either from the left or it does not matter. So, jumps in other words we are perfectly with jumps in f we are perfectly right. So, if there are jumps in f and you integrate you will get kinks in is the word kink I suppose when I say kink I mean I am talking about point where the slope changes abruptly like this. So, you get kinks in capital F which is fine. We do not have any problem.

But jumps and capital F cannot be differentiated. We will not attempt to differentiate discontinuous serious, but no we will not differentiate, no differentiation of in this course I am being very specific. In some other course you may want you write some other teacher might be quite happy doing it, I am not happy doing it I am not going to do it. So, in other words what I am trying to say is that I will not use a pdf model which requires me to put finite probability at a specific x in capital omega, is this clear so far.

The p small pdf model is perfectly fine as long as you want a smoothly increasing CDF, for all such cases you can think of a pdf. If you want to have a pdf which is smooth in some portion and smooth the increasing some portion suddenly jumps in some other portion and then again resume smooth increase then a pdf model runs into trouble and by itself you cannot expect a smooth expect any standard pdf to solve that or to be useful in that situation. So, it is best to stick to the CDF and not invoke things like impulses and all that which are more trouble which eventually come back and bite you in different ways.

Why do impulses bit? Impulses bite because they are not real functions. Impulses bite because you cannot square them. Are you aware that you cannot square an impulse? It is meaning a totally undefined you cannot look at f squared of x f, as f contains an impulse. So, because you cannot manipulate them as generally as you can man as you can manipulate other functions I prefer not to do, not to bring them in I think it is just to keep the discussion more sane I think yeah you know let me say my piece my prerogative and I do not want to bring them in here that is all, as simple as that. So, is this everybody so far?

So, we mean all and I am again trying to say we need both, we cannot I mean as a concept. I may be in a particular situation we may not need to bring in the capital F anywhere, we might do everything all the project manipulations we want to the small f which is fine, but do not forget that there is all if you ever need it the capital F is sitting there under underlying which you can always bring it. And this capital F is a in some sense is bounded this is a probability, but analytically it turns out for example, what is a small f corresponding to this capital F, this is a perfectly smooth differentiable function. So, what for this capital F what is the small f? What is this small f here? What is it? What is it please tell me.

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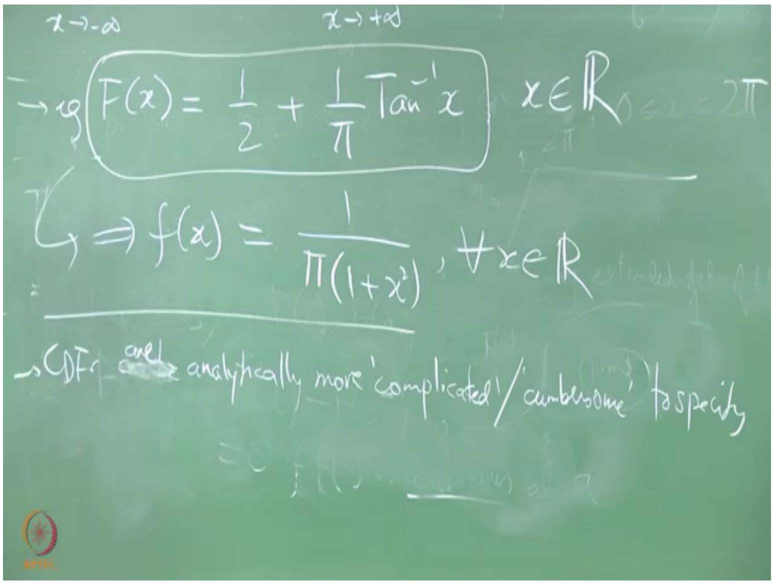

Student: (Refer Time: 24:37).

So, starting with this capital F we get small f is what? You are going to tell me.

Student: 1 by pi (Refer Time: 24:50).

1 by pi into.

Student: (Refer Time: 24:55).

Right, what is it domain?

So, this is a pdf which is defined over the entire real line right. So, and it has nonzero values everywhere in the real line not just you know small in a portion of it. So, for this model to be valid we are basically saying that that I that the experiment can spit at you any real number from minus infinity to infinity right. But CDFs are why do we not use a CDF everywhere? Because it turns out even though the CDF idea is more general than a pdf, the CDF is analytically more painful to deal with like for example, it is much nicer to deal with this rather than deal with this. Can be analytically or I should say they are in general analytically more complicated to specify.

When I say more complicated I mean more cumbersome, not complicated in a conceptual sense they are cumbersome. So, maybe all these words pretty much have to reproduce in entire text book on the blackboard at this rate. You go look at textbooks you see all these words coming up they are cumbersome to specify than pdfs. Why because you look at this look at the CDF corresponding to the spinning pointer. How do you specify it? You would say F of x is 0 for minus infinity to 0 and then F of x is some theta x by 2 pi for x to 2 pi and then one you have all these you know you have a typically what happens is you get a different bunch of cases case 1, case 2, case 3, case 4 like that, where this CDF has pure cases and you can say 0 elsewhere thinks of the charge. So, CDFs are turns out sorry a pdfs at it turns out, are more compactly written and easier to write down and work with compared to CDF.

And you can also have well we will come to that maybe a related aid a staircase CDF, but before that I have I want to. So, I think maybe this is enough for today right. So, we have looked at today what have we done. We have looked at two different ways of assigning probabilities, one with pdf alone, one with CDF. And we have seen that CDF is in some sense more general than pdf because whatever we can do in the pdf we can also do with the CDF, but the jumps in CDF cannot be matched with anything in a pdf. Just keep that in mind and we will convene next week with the lectures.

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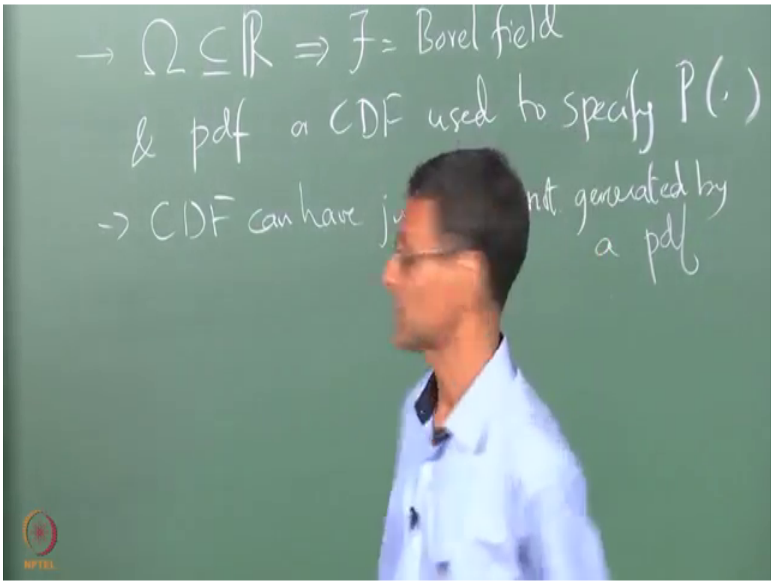

Last week we looked at, or, this is essentially what we did last this discussion. I think we have looked enough at pdfs and CDF. So, I am not going to look at them again I keep, but they are very important. So, please do not miss lay that portion of the discussion all we will be needing it all the time from now on.

But just acronyms a small pdf and small letters is probability density function, CDF in capital letters this is this cumulative distribution function. So, the D is not the same in one case it is density in the other case it is distribution all right. So, in general this we talk of probability distributions, so the CDF in some sense is more general because it allows jumps and anyway. So, CDF jumps these jumps cannot, not reflected or not capable of not what did I want to say here not generated by these jumps cannot be generated by pdf, by integrating a pdf you cannot get a jump in the CDF that is the basic. So, that is a problem we have with pdfs, there is some sense limited, but as we will see it is not a very strong limitation right.