Probability Foundations for Electrical Engineers
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Lecture - 42
Example: X+Y, X-Y, min(X, Y), max(X, Y)

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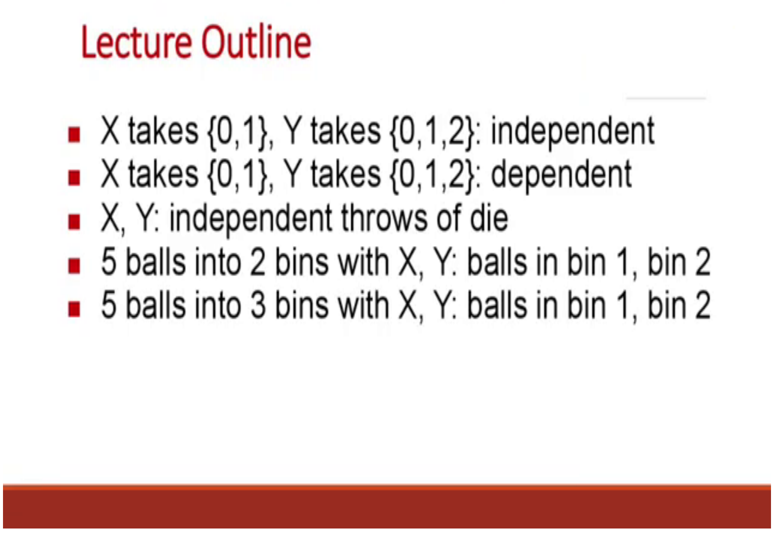

Hello and welcome to this next lecture involving examples in probability. So, we are going to start looking at examples of functions of random variables, supposing you have 2 random variables which have a joint distribution and I have a function of those 2 random variables, the function could be the addition of those 2 random variable, subtraction of those 2 random, it is multiplication of those 2 random variables, the minimum of those 2 random variables, maximum of those 2 random variables, any number of functions that you can think of and given the joint PMF of 2 random variables, how do you find the distribution of the function of those 2 random variables?

Now, this is a very, very important example it comes up again and again and again in any probabilistic model and it is very important to know how to go about doing it. So, I am going to take a few simple examples you saw probably Prof. Aravind do slightly more complicated examples and I am going to take simple examples and start reinforcing deities.

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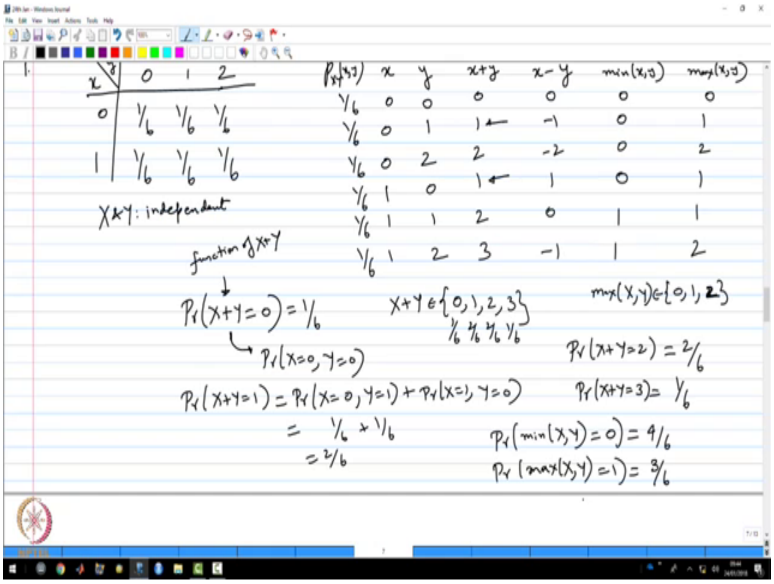

So, let us start with the first example I am going to take. So, I am going to consider an example where x and y together take values in this form. So, x takes 2 values 0 and 1, y takes 2, 3 values 0 and 1, 2 and I am going to say all the probabilities are 1 by 6. So, this is my joint PMF, so x and x are this is my p xy, x comma y, there are 2 random variables x and y and the joint PMF is given like this.

So, when you are dealing with functions of random variables and the random variables are very small in some since they take only a very few values, it is useful to write tabulation. So, you can write a simple table and then deal with all functions in a very nice way. So, I like that method we could call it a table method if you like. So, you can just list out all the possibilities and write the functions and find the probabilities it is quite easy to do that. So, let us do that, so you have x here, y here and then it is convenient to write say p x y of x comma y in this fashion.

So, x could be 0, y could be 1, x could be 0, I am sorry, x becomes 0, 1, 2 or x could be 1 and then y could be 0. So, all the joint PMF is just 1 by 6 for all these possibilities.

So, the now if I want to find the function, let us say I am interested in the function x plus y. So, if I am interested in the function x plus y, see you write like this and then you write down what x plus y is 0, 1, 2, 1, 2, 3 or maybe I am, so now, supposing I am in I want to have or ask the question what is the probability that X plus Y equals 0. So, this is my X plus Y; this is a function I am interested in.

And, what are the values that X plus Y takes? From this table you can see that X plus Y is another random variable and it takes 3 values 0 comma 4 values 0, 1, 2 3, so I know that for sure. Now, I can ask the question what is the probability that X plus Y is Z. So, in this table you see 0 occurs only in this first position and that occurs with probability 1 by 6. So, it is quite easy to read out, this is the probability of this, so in fact, from this table what do you read out you can see that this probability is actually the same as probability that X equals 0 comma, Y equals 0 and that is 1 by 6, quite easy straightforward.

Supposing, you asked the question was the probability that X plus Y is 1, from the table you can see there are 2 possibilities here, it is a possibility here for which X plus Y is 1 is a possibility here for which X plus Y is 1. So, X plus Y is 1 actually corresponds to 2 cases here X is 0, Y is 1 or X is 1 Y is 0, now this is 1 by 6, this is 1 by 6, together you get 2 by 6, if you want you can write it as 1 by 3. So, that is the probability for X plus Y is 1. So, now, we can keep continuing supposing you say I am interested in X plus Y is 2, again from the table you can quickly see that there are 2 possibilities here. So, we get 2 by 6 and you want to look at probability that X plus Y is 3 this is 1 possibility, so 1 by 6.

So, if you write it all together you have here 1 by 6, 2 by 6, 2 by 6, 1 by 6, this is a very easy method for small distributions. So, if you notice here in this case actually X and Y are independent. So, in the independent case for some you also have a simpler expression for computing the sum. So, it is the convolution of the 2 marginal's, so that is also possible you can do that also in this case, but you know it is a small example things like this are very, very easy to do.

Now, suppose somebody asks you to find x minus y. You can just keep continuing on this table, so what is x minus y here minus 1, minus 2, 1, 0, minus 1, so that is. So, and then you ask the same question; what is the probability that x minus y is 0, you simply add these 2 guys, you will get 2 by 6 what is the probability that x minus y is minus 1, you will get 1 by 6 plus 1 by 6 again, 2 by 6 what is the probability that x minus y is 1, 1 by 6 what is the probability that x minus y is minus 2 in 1 by 6.

So, this is just easy to write this down and one can see them this is the stable method, this is quite easy to do for any function that you want. So, supposing I want min of x comma, y here, you have minimum is 0, 0, 0 here, you have 0, 1, 1; supposing, you want max of x comma y, you have 0, 1, 2, 1, 1, 2. So, suppose somebody were to ask a question just from this table you can answer any question you like, supposing you want to ask what is the probability that min of X comma Y equals 0. So, you have 4 possibilities here 1 by 6 plus; 1 by 6 plus; 1 by 6 plus; 1 by 6, that 4 by 6.

Supposing, you were to ask what is, the probability that max of X comma, Y equals 1. There are 3 possibilities here 1 by 6 plus; 1 by 6 plus; 1 by 6 that is 3 by 6. So, for small examples like this, it is very, very, very convenient to simply write down the table, write the joint PMF and then evaluate whatever function you want; whatever the function may be you can quickly write down the answer and we can write down all the things. So, if you notice the table also tells you everything. So, for instance the table tells you that max of X comma Y is a random variable which takes 3 values 0, 1, 2 and then it is PMF can be calculated quite easily using the table.

So, what this method is? It is kind of bread and butter, it is a very simple method, it works in small situations when the random variable becomes quite complex and you kind enumerate it quite so easily, but this is not a bad way to visualize, how you are doing the functions of random variable. So, I will keep coming back to this method, we are going to be dealing with simple examples in this nicely.

So, you can consider other PMF. So, instead of 1 by 6 throughout you could take some other PMF and the same method would continue. So, there is no real problem here, so every time you have a case, you just go look up the PMF probability and then keep adding the corresponding values, then you will get your answer. So, hopefully this method is clear in that fashion.

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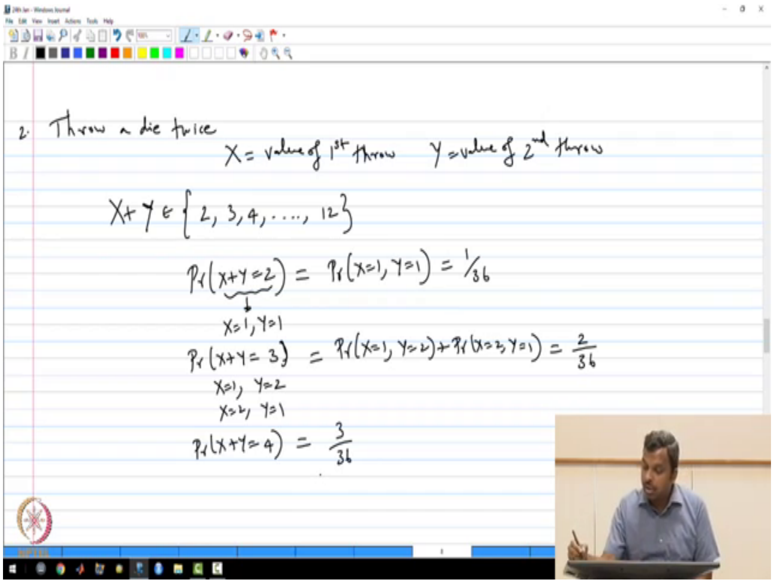

So, let us look at 1 more example this may be slightly bigger, so may not be that easy to do, the second example we take let us say we throw a die twice . So, I will say X is the value of first throw, Y is the value of second throw and I am interested. So, remember X and Y are both independent, they take values from 1 to 6 with uniform probability and joint PMF is also simply the product. So, no joint PMF is just 1 by 36 for all the pair of possible values. So, if you want you can actually write down a table, so it is not very hard, but it is going to become big, it is going to have 36 values. So, it may be it is probably a little bit better to try and see if we can do cut shot the calculations a little bit in some case.

So, let us say we are interested in X plus Y. So, this is the function for which we want to compute the PMF. The first step always is to understand the range of values that your function takes. So, X took values from 1 to 6, Y took values from 1 to 6 and the joint PMF all possibilities are there. So, X plus Y will take values from 2 to 12, is that ok? Think about it, it can take value to 1 over 1 plus 1 2, it cannot go below that, cannot take a value 1 it so you have to visualize this thing and understand that X plus Y would take all these values and then for a particular value of X plus Y you need to find out the probability. So, for instance I might be interested in the probability that X plus Y equals 2.

So, important thing is able to break it down into the possibilities for X and Y individually. So, X plus Y is 2 means X has to be 1, Y has to be 1 that is the only possibility. So, this event corresponds to X equals 1, Y equals 1. So, this is the same as probability that X equals 1 comma, Y equals 1 and from the PMF I know that is just 1 by 36.

So, if you want to push ahead and see what is X plus Y equals 3. So, here you will start having more possibilities, you could have X equals 1, Y equals 2 or X equals 2, Y equals 1 there are 2 possibilities here. So, the probability simply probability of X equals 1, Y equals 2 plus probability X equals 2 Y equals 1. So, that is 1 by 36 plus 1 by 36 it is 2 by 36.

So, we can proceed like this, if you look at probability that X plus Y is 4, we will have 3 possibilities for X and Y together 1, 3; 2, 2; 3, 1. So, that will become 3 by 36.

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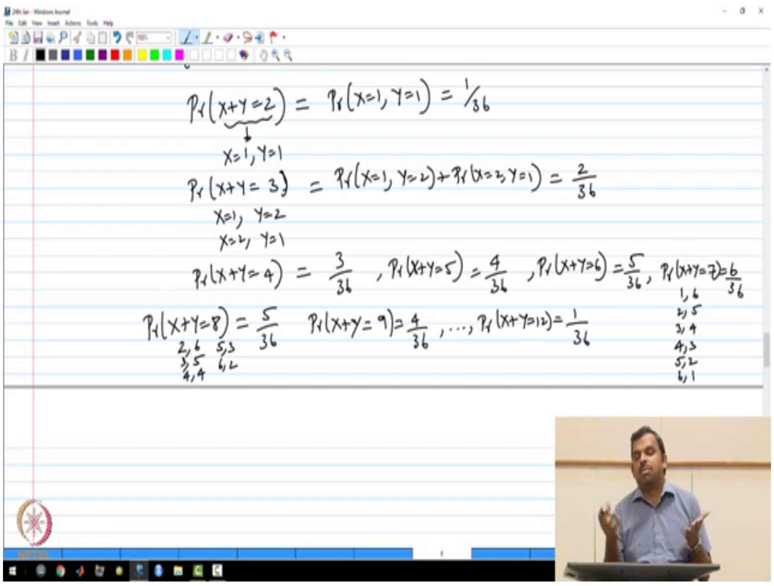

So, so on you have to go, but you have to be slightly careful. So, this will go for probability that X plus Y is 5 also. So, many possibilities would have 1, 4; 2, 3; 3, 2; 4 1, so that is 4 possibilities. So, likewise 6 is also 5 by 36, 7 is also 6 by 36. So, up to 7 you will have 7 would be like the maximum possibilities will come. So, 1 comma 6, 2 comma 5, 3 comma 3, 4 comma 2, 5 comma 1, no yeah 5 comma 1.

So, did I get that right yeah. So, I am sorry X plus Y is 7, 5 comma 2 and 6 comma 1. So, let me repeat this. So, X plus Y equal to 7 you will have the maximum possibilities 1 comma 6, 2 comma 5, 3 comma 3, 4 comma 2, 5 comma. So, let me repeat this once again X plus Y equal to 7, you will have the maximum possibilities 1 comma 6, 2 comma 5, 3 comma 4, 4 comma 3, 5 comma 2, 6 comma 1. So, these are the 6 possibilities for X plus Y equal to 7. So, now, when you go start going a little bit higher than that if you want say X plus Y equals 8, you will see X cannot be 1, if X is 1, X plus Y is not going to be 8, it is Y maximum is 7. So, the number of possibilities will decrease.

So, here you will have for a instance 2 comma 6, 3 comma 5, 4 comma 4 and then 5 comma 3 and then 6 comma 2 there are only 5 possibilities 5 by 36, it will start falling off. So, you can notice all these kind of patterns it is also possible to use convolution in some sense to compute this sum though I will not talk about it here. So, 9 will be 4 by 36 and so on.

So, all the way till 12 will again become just 1 possibility 1 by 36. So, this is I mean I just wrote it down exhaustively, but you can see how to think of functions and do it in this form. So, you will have to understand how many possibilities are there in X and Y and simply just write down that probability there is nothing much more to happen here.

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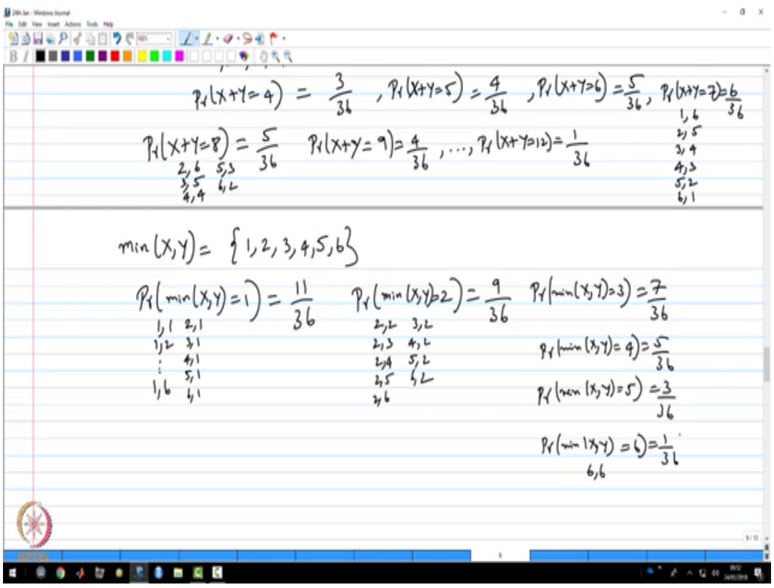

So, let us look at a slightly more interesting maybe a function let us say min of XY. So, now, min of x y if you think about it is X X takes values from 1 to 6, Y takes values from

1 to 6, the min of x y will take value 1, 2, 3, 4, 5, 6. So, this is the possibilities for min of x y and then you would ask the question what is the probability that min of x y equals 1, here again you know smartly count how many possibilities are there.

So, if you look at it 1 comma 1 comma 1, 1 comma 2, so on till 1 comma 6 all of them will give you min 1, not only that you can also have 2 comma 1, 3 comma 1, so on till 4 comma 1, 5 comma 1, 6 comma 1 all of these possibilities are also there. So, there are 5 here, 6 here, 6 here, 6 plus 5 11; 11 by 36 it is the possibility of min of X comma Y being equal to 1.

So, even for 2, you need to look at it min of x y equal to 2, remember it is equal to 2. So, I could have 1 come no I cannot have 1 comma 2. So, it should be 2 comma 2, 2 comma 3, 2 comma 4, 2 comma 5, 2 comma 6 and also 3 comma 2, 4 comma 2, 5 comma 2, 6 comma 2. So, it is 4 plus 5, 9 possibilities it will go on like. So, you can have probability of min of x y equals 3 would be 7 by 36 probability of min of X comma Y equals 4 will be 5 by 36, probability of min of X comma Y equals 5 will be 3 by 36, probability of min of X comma Y equals 6 will just be 1 by 36, 6 is the easiest because you only have 1 possibility 6 comma 6.

So, one very useful thing to do when you compute probabilities like this is to make sure that they all add up to 1. So, that is a good check to do can see 11 plus 9 is 20, 27, 32, 35, 36. So, that adds up to 1, so you know that you have not made very glaring miss statements still be wrong, but this is less likely that you are wrong. So, this is a good way to compute probability. So, you can see I am actually using only the table method this is just enumerate you know and whenever required in some smart way and quickly observing some patterns and going about doing it.

So, when you have functions of random variables, this is what you do. Now, there are some generic methods Prof. Aravind refers to those methods, you should look at those methods also very closely, when you have like min or max there are general ways in which you can think about how to compute these things. So, that will also help you, so those are very, very important to understand as well, but you know. So, some simple examples are also nice to see.

So, the final example; final couple of examples I am going to do a slightly more complicated. So, this is the 3rd example I am going to do.

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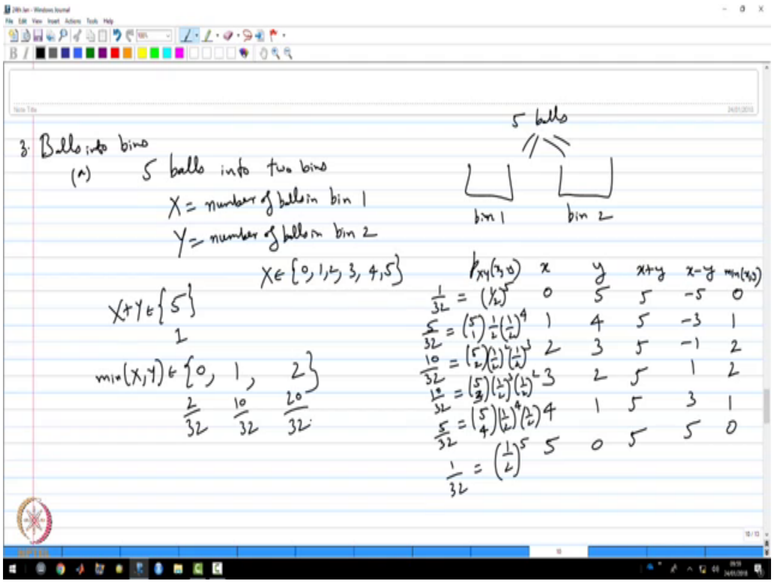

We will look at balls into bins. So, the first case we considered is let us say we are throwing 5 balls into 2 bins, there are only 2 bins remember this is a bit critical sometimes you can get misled by this problem, there are 2 bins bin 1 bin 2 and you are throwing 5 balls into these bins, so 5 balls. So, let us say X is the number of balls in bin 1 and Y is the number of balls in bin 2.

So, now we might start asking for questions about what is this distribution. So, one needs to be a little bit careful here with this distribution, it is not very straight forward. Now, X takes how many values you can see X takes it could be 0, 1, 2, 3, 4, 5, there are 6 possible values that X can take and if you look at the joint distribution, one needs to be pays pay a little bit of attention here. So, supposing you put x here y here and say x is 0, 1, 2, 3, 4 or 5, what are the possible values for y?

Remember, there are only 2 bins, the ball either went into bin 1 or went into bin 2. So, if I say the number of balls in bin 1 is 0, what should be the number of balls in bin 2? It has to be 5. So, there cannot be any other possibility, there are only 2 bins and this restricts the number of possibilities that there can be. So, if the number of balls in bin 1 were 1 this has to be 4, this has to be 3, this has to be 2, this has to be 1, that is to be 0 . So, x and y are kind of heavily dependent in fact, they determine each other right. So, X is 5 minus Y, X plus Y has to be equal to 5. So, that is the important thing about this experiment that we are doing, it is a very simple connection between x and y.

So, if you were to look at p xy of x comma y here. So, what is the probability that all the balls fell into bin 2, so this would be just like half power 5. So, 1 in this case, so you would have like you know this is just binomial you know you think about it how many what is the probability number of balls that fell into bin 1, this is binomial 5 comma half. So, that is the kind of thing you would do, here you will have 5 choose 1, again half times half power 4, well it is just writing it in a laborious fashion this is hopefully you see where it comes where this is coming from right.

So, this 1 exactly 1 ball in bin 1 that is with probability half, there are 5 different ways in which it can happen it could be the first ball or the second ball or third ball or fourth ball, fifth ball if I choose 1 and then half power 4 means all the other ball is going to bin 2, so this is by binomial.

So, actually this distribution is binomial, so you will have your 5 choose 2, half squared half power 3, I am just writing it in a slightly laborious fashion. So, you can see where this is coming from. So, then this is 5 choose 1, 5 choose 4, if you like 3 here if you like, half power 4 into half and here again you will have just half power 5. So, you can simplify this, now this is just 1 by 32 n, this is 5 by 32, this is 10 by 32, this is again 5 by 32, no this is 10 by 32, this is 5 by 32, and this is again 1 by 30. Again a good thing is to count 1 plus 5 is 6, 16 26 31 and 32, ok.

So, that is that is a good thing to do. So, this is the joint distribution and interestingly if you look at x plus y I will simply get 5 5 5 5 5 5. So, X plus Y in this case this takes one value and if it takes 1 value that should happen with probability 1 it is a constant. So, X plus Y is easy to write down what about x minus y that may not be the same, I mean x minus y can be a bit tricky. So, if you look at x minus x 0 minus 1, minus 5, minus 3, minus 1, 1 3 5 ok.

So in fact, x minus y is the same as X in some sense right. So, it takes this 6 different values minus 5 minus 3 minus 1, 1 3 5, but you know they are all different. So, probability that x minus y is minus 5 is simply going to be 1 by 32, probability that x minus y is minus 3 is simply going to be 5 by 32, there is no real operation I mean 2 values do not occur do not repeat all 2 values all values are distinct.

So, x minus y and X are more or less the same I mean they take the same the probabilities which they take values are the same except that the values are different. So, that is all so, but if you look at min of x y you might have a slightly more interesting situation you have 0 1 2, 2 1 0. SO, min of x y has some repetition. So, min of x y can take just 3 values 0, 1, 2 and one can quickly see 0 happens at these 2 possibilities.

So, the probability is just 2 by 32, one happens at these two possibilities probability is 10 by 32 and 2 happens the middle 2 possibilities probability it is 20 by 30. So, you can quickly do this, even though the experiment looks a little bit complicated any function you can easily do this is actually quite a simple example and X plus Y it became very very simple, but the other functions are slightly more complicated that is all ok.

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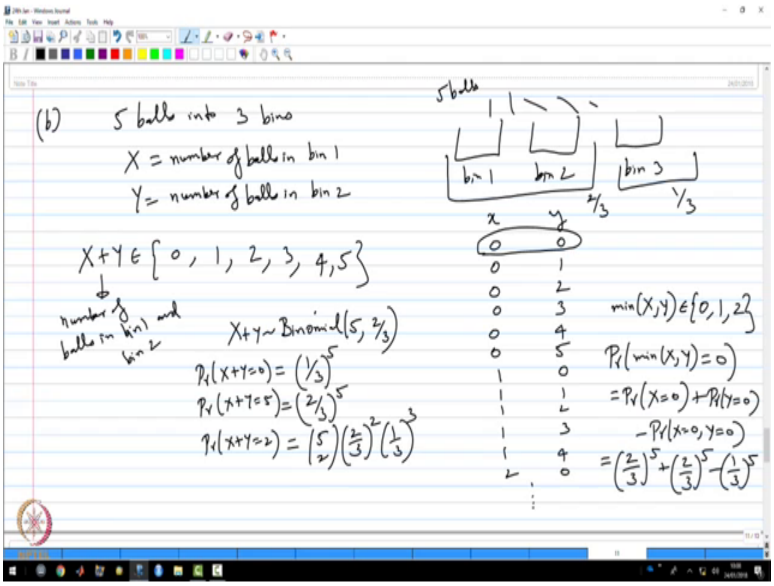

So, the next example I am going to see is it is going to be a small slightly different from the previous one. So, you look at 5 balls into 3 bins, the third bin is the additional thing that happened here, but let X and Y be the same thing X is the number of balls in bin 1 and Y is the number of balls in bin 2, that we would not change remember those 3 bins now it is not just 2 bins, and then you are still throwing 5 balls. So, this could go anywhere. Once you have 3 bins things start becoming a little bit more interesting because you cannot say X and Y determine each of them. If I give you X you do not know why right. So, mean Y could be 0 or anything right. So, some things you can know, but not the not a whole lot ok.

So, this is a slightly more complicated situation. So, if you were to start building a table it might become a little bit more difficult than what you bargained for so, for instance if x is 0, y can be 0, if x is 0, y can be 1, if x is 0, y can be 2, right. So, all of these are possible all the way up to here is possible or x is 1, y could be 0 1 1 is possible, one is possible 1 2 is possible, 1 3 is possible, 1 4 is possible, 1 5 of course, is not possible. So, x is 2, 0, so on. So, number of possibilities is a bit too much you can mean you could enumerate, I am not saying it is impossible, but it is just the number of cases just starts blowing up a little bit too much. So, you will have to use some more clever ideas, we will see you will see how to go about.

So, supposing you say X plus Y. So, clearly if you think about it X plus Y it takes values 0 1 2 3 4 5 all of the possibilities are there. Now if you want to assign probabilities here, you can do the PMF and try looking at the cases of X and Y and adding, but that will not be very efficient let us look at the little bit more closely, what is X plus Y plus X plus Y represent? It is the number of balls in bin 1 and bin 2 together. So, what you cannot do is you can kind of fuse these 2 bins together, and think of it as one super bin, and probability that a ball lands here is 2 by 3 the probability is that the ball lands here is 1 by 3 right.

So, every ball if it lands in either bin 1 or bin 2, I have some sort of a success when I want to count X plus Y right. So, I want to count X plus Y, I have a contribution of 1 to X plus Y every time a ball lands in either been 1 up in 2, and the probability of that happening is 2 by 3 in every drive, and I have five such possibilities. So, when can quickly see that this X plus Y is actually binomial with 5 comma 2 by 3. So, this is a nice observation you can make based on how the problem is set up. So, you can compute our probabilities quite easily. So, what is the probability that X plus Y is 0, no ball lands and bin 1 and bin 2, that is going to be 1 by 3 power 5. So, that is the binomial calculation here. So, what is the probability that X plus Y is 5 all the balls land here. So, that is going to be 2 by 3 power 5 ok.

So, any other value you pick. So, let us say X plus Y is 2. So, there are 5 choose 2 different ways in which this can happen and 2 balls land in bin 1 or bin 2 and 1 ball lands in I mean 3 balls land in the other way. So, this is possible. So, one can do this is quite, but any other function is going to be a little bit more tricky. So, for instance if I do min of X Y. So, this is not necessarily easy. So, if you do min of X Y. So, what is the least number of balls that fell either in bin 1 or bin 2. So, it is. So, one needs to do some thinking right. So, what are the various possibilities here X and Y could be 0 0 0 1 etcetera. So, 0 as possible as a minimum value you can see these things, 1 is also possible as a minimum value there are several cases in which 1 will be the minimum. 2 is possible as the minimum value right, between just X and Y so, you could have 2 here and then 3 in the next one right.

Now, once you go beyond 2 actually it is not really possible, you cannot have 3 as the minimum between X and Y right because if one of them had 3 the other one is not 2 or less. So, then that will become the minimum. So, minimum will take just 3 possibilities. So, computing probabilities for even min of x y this is quite a bit complicated, it is not very easy. Supposing you want to focus on just one value, min of x y being equal to 0, this can happen if I mean either X should be 0 or Y should be 0 and or both can be 0 right.

So, I have to worry about probability that X is 0 right that you can compute, you will be able to compute that and then you have to worry about probability that Y is 0 and then I have to subtract the case in which both of them is 0. SO, that I get the count of min of x y being 0. SO, this would be a probability of X equals 0, plus the probability that Y equals 0 ok.

So, this will give you, but there is a case in which both of these could have happened. So, I have to subtract the case where X is 0, Y is 0. So, think about why this is true. So, if min of x y is 0 if the X has to be 0 or Y has to be 0. SO, if I simply add the probability that is the case in which both X and Y could be 0 together. So, I have to subtract that out. So, I will get the all the events. So, this is basically the event 0 0 0 1 0 2 0 3 0 4 0 5 and then 1 0 2 0 3 0 so on. So, if I just compute X equals 0, I will get one answer, Y equals 0 I will get one answer, but this event 0 0 would have been countered twice. So, I have to subtract that it ok.

So, the reason why I write like this is probability that X equal to 0, I can easily calculate what is the probability, that X is 0 no ball landed in bin 1 all the band ball should land in bin to up in 3. So, that is just 2 by 3 perfect and probability that Y equals 0 I can also easily compute that is again 2 by 3 power 5 what is the probability that both X and Y are 0 that is 1 by 3 power 5. So, this will be the answer for min of x y equal to 0 ok.

So, in various cases as I just in the last example I just wanted to suggest that if you look at the function very carefully and understand what you need to compute, you can go about doing it is not really difficult, but there are there can be no generic method you know in some the experiment can be really complicated and your function can be a very very complicated function of all the events that happen, in which case you have to really sit down and think about what you are doing it.

Thank you very much that is the end of this lecture.