Probability Foundations for Electrical Engineers
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Lecture - 14
Part 2
Addition of Random Variables

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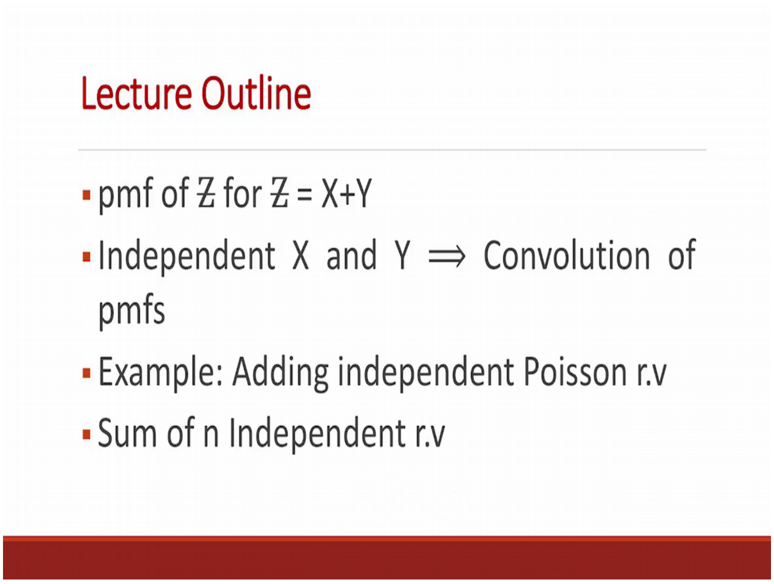

Let me look at its manipulations of a sum.

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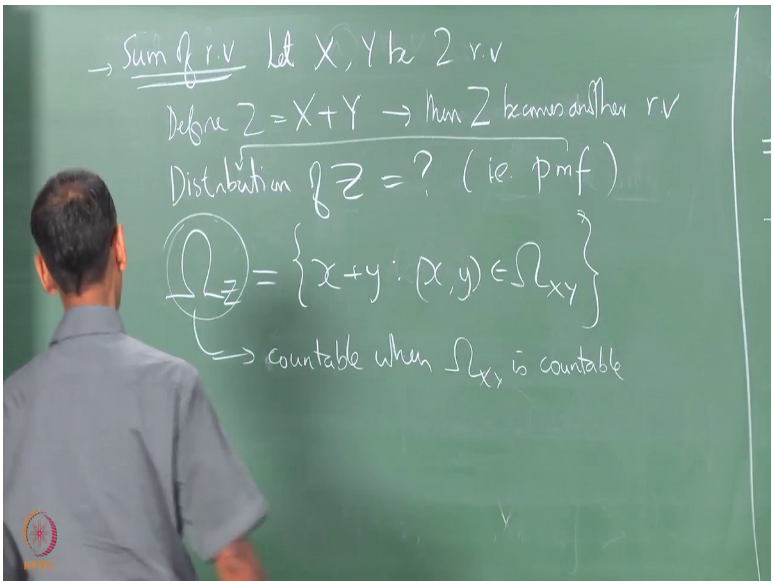

Let X and Y be any 2 random variables I do not care, that is we define say Z equal to X plus Y. What do a I mean by adding?

Student: Consecutive.

By adding I just mean since X is after all going to be number on as it is all of an experiment Y is going to be in other number I can always add them. So, if I do that then Z becomes another random variable defined for each point of the sample space right, remember right at you do composite experiment whatever experiment it is which out an X is space or Y we are saying that then the Z the value Z is going to be the sum of those 2 values X and Y always.

So, Z equal to X plus Y means that for every realization which is just correct word or X and Y realization of Z is obtained by addition. So, now, it is interesting that in the case that X and Y are both discrete valued random variables, which we all will be studying all this I mean classes. So, what can I say about let p m f of Z. So, I said I will ask a question distribution of Z by distribution I mean p m f now for now.

Student: (Refer Time: 02:05)

First of all if X and Y are discrete it is clear that Y Z also can take values only in a discrete set. So, how do why defined omega Z. So, omega Z is basically obtained by saying all possible values of x plus y such that x y right, I mean wherever.

Student: (Refer Time: 02:36).

X and wherever this x y point occurs in omega x y in other words where there is nonzero probability of the joint distribution occurs then you add. So, all possible such values will comprise the set omega Z. And so if this is a discrete set countable set this also has to be countable, you cannot get something this in sum cannot in bigger than this. And so this is countable when that is omega Z, countable when omega X Y is countable.

So, how do I write out the p m f now? So, if I want to take let us say just to introduce some verity, if I want to look at p Z of let us say w instead I instead of putting Z I am going the problem with the letter Z is that it is very difficult to distinguish between the lowercase and uppercase.

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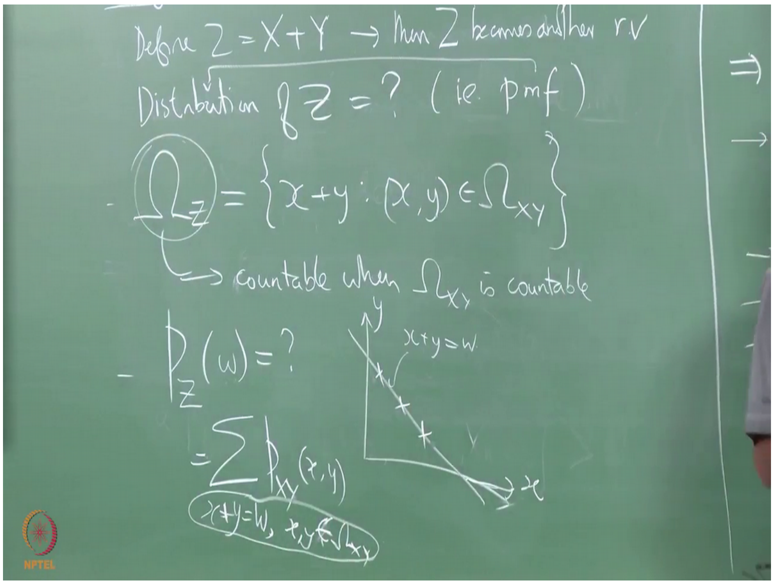

So, let us force a distinction and say ask for put for the argument is w and not a Z, which is the whole reason for right introducing this notation, this is the name of the random variable this is the value that you can take there is no reason why they should be always some you know look like this or just conventionally people have been writing low upper lower upper case here and lower case here that is all.

So, what is this, how do I go about finding?

Student: (Refer Time: 04:25) with the joint of.

It is not joint it is. So, it means this right you look at this diagram here x and y, you look at this straight line x plus y equal to w which is the deterministic straight line that you can draw and you look at all the points are fall on that straight line that also fall in omega x y. So, omega x y it is we have assumed that it is countable. So, therefore, there will be some collection of points here which will. So, if this omega if this w is in supposing this w is in omega z that is there is at least 1 point.

Student: (Refer Time: 05:10).

On that in a omega x y which is on this straight line typically there is more than 1. So, the probability that z will take the value w is probably that x y can take either this or this or this just common sense, you will have been in tuition that way remember these are all exclusive events that if x y takes this value we cannot the pair cannot take this value, but the sum will take the same value. So, we are looking at all possibilities which give this give this same value for the sum. So, therefore, this will be equal to if you want to write it mathematically you have to write it as p in terms the joint p m f p x y x comma y here you will write x plus y equal to w and with the understanding of course, that x y.

Student: (Refer Time: 05:58).

Are in ok put it here comma x y in omega x y also. This whole thing is little somewhat longish subscript here, but I do not know what how else to say right, but you have to identify all these points the situation becomes a little easier to visualise when you have only integer valued random variables. So, what do I am what I know it is a good time to.

Student: (Refer Time: 06:36).

define them what is them integer valued random variable any random variable whose omega I mean the space is it consists of set of integer is called integer valued. So, since the sum of an 2 integers in always another integer no matter what.

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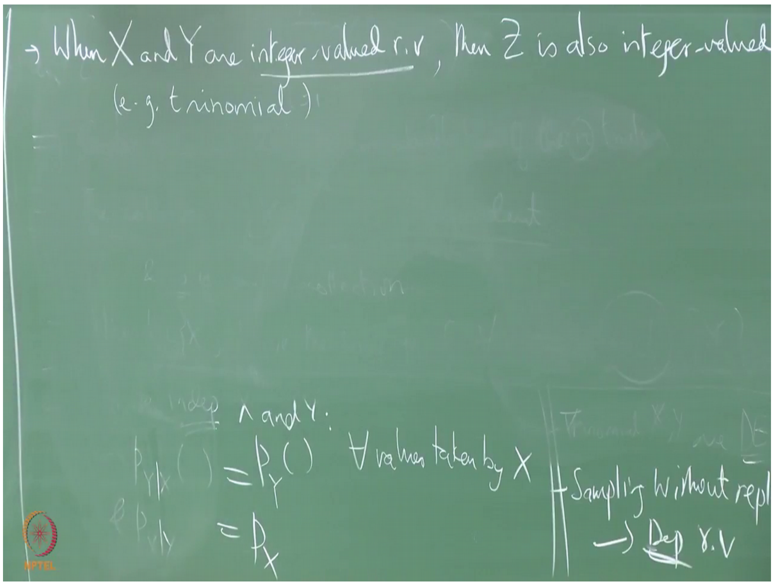

So, this is all of those terms which is self-explanatory I do not think we need to write out an explanation for this right so then. So, this is of course, are standard trinomial case right not just that in main right, but that is the case that the most interesting case that we looked at. So, for let me just make sure that this is the interesting property I did I did this sum of; we can also do that little later, but let me see just make sure that I put it in the right place.

So, that I can talk about it here along with this, but first let us focus on the independent case because it is why is independent case easy to manipulate in such cases. This quantity has to be expanded first one.

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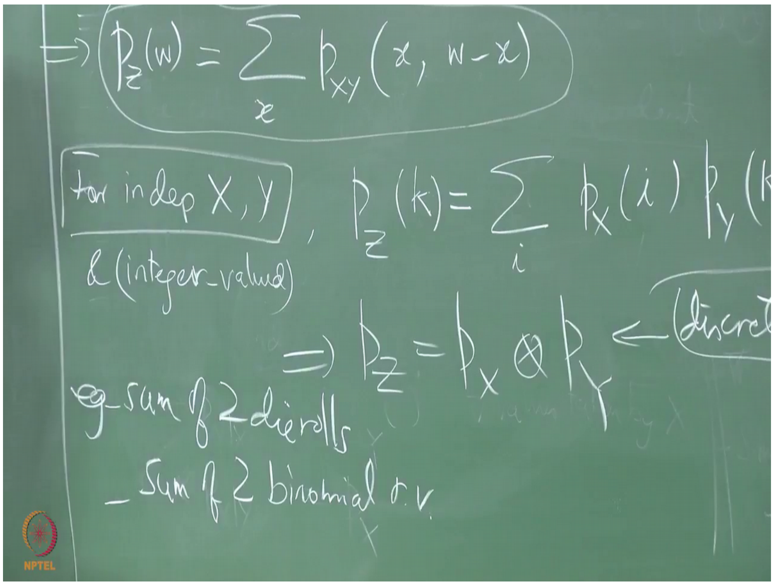

So, which implies that now I can write p z of w w has to be an integer, but before that we can we will write it like this p x y in place of y I can put w minus x going from here, because y always has if I say x is now the variable right over the summation just becomes x I do not need to worry about y because y automatically is going to be w minus x or I could have written it as w minus y comma y. Of course, understanding is that you only look at right those points in this summation such that this joint probability is nonzero.

So, this is the standard formula no matter what of course, it is conceives a lot of things like what are the limits on x and so on right.

Student: (Refer Time: 09:25).

So, when you have integer value; that means, x is going to be small x is going to be integer w minus x is going to be an integer w is going to be an integer, but I am writing it just to continue the notation and I am writing it like this, but in the independent case supposing will come to the trinomial example a little later if they are independent right what happens. First let us look at this which is; obviously, right not trinomial right, but I want to develop a very important diagnity here, I can write now I do I can even dispense with the w and I can I am going to put k and I and so on.

So, independent X Y and integer valued. So, independent and integer valued not necessarily I d at all, but just integer valued.

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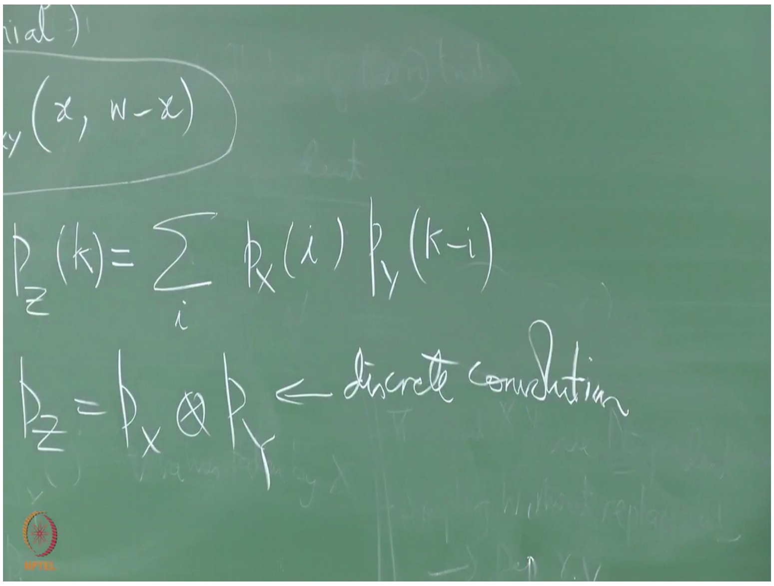

So, p z of k will be sigma if I write this what happens to do this this is the marginal sorry joint p m f, which can be written as a product of marginal. So, how will it be written it is p x of something into p y of something now I put k here. So, if I put I here and if have to put K minus i out here. Instead of that in otherwise I am writing I in place of x and k in place of w.

Student: (Refer Time: 10:52).

Maybe, but that I wanted to do only after saying the very integer valued, before that I it is better to keep it as some keep it as whatever x y whatever, but once I say that integer valued I can write this and the summation will be over values of I have you seen this can sum before.

Student: (Refer Time: 11:12).

It becomes convolution exactly.

Student: (Refer Time: 11:15).

Say discrete convolution of p x and p y, note that the convolution is best written without any arguments you do not write it as p z of n equals p x of n convolve with p y of n which is nonsensical way I believe of writing it right is you are actually convolving 2 sequences to give a third sequence, what are the 2 sequences you are convolving p x and p y the p m f is the sequence and the output is another sequence whereas, we are using this p x p z of k to specify that particular value only.

Student: (Refer Time: 11:55).

Not the whole sequence. If you coded this in MATLAB for example, you would use only a statement of this kind, you would not use a statement p x of z equals right you try coding that in MATLAB it is going to give you an error you cannot call convolution with p right p x of k in has to be just input as just p x I do not know how python does it, I am sure it is very similar.

So, this is the discrete convolution. So, it is a very interesting result now right that is why

Student: (Refer Time: 12:30).

It is preferable to do this in a couple other place.

Student: (Refer Time: 12:36).

We should typically go through the d s p course before coming here, because that in addition to giving making a mathematically more prepared it gives you concept such as these which are important here. So, what we are saying is that this the p m f of the sum of 2 independent random variables provided their integer valued is the convolution.

So, they have not only do they have to be independent they also have to be integer valued. If they are not integer valued then you do not get a nice straight formula for convolution. For example, if x takes the values 0.25 0.75 or around 0.3. So, something and y take some other 0.4 0.6 you cannot right the value is z will take will not follow any nice grid for this convolution to work out and anyway even convolutions over such a domain is also not. So, easily defined I believe right I it is only defined when the domain of p x and p y other same typically integer.

So, for this convolution to work then therefore right you required both the independent first and foremost and then we also require the integer valued nature which is actually very commonly encountered in practise. So, how do you apply this results straightaway right what is that p m f is sum of 2 dipoles you get a nice triangular distribution do you not.

Student: Yes.

From 2 to 12, which is the convolution of 2 rectangular uniform discrete uniform p m fs sum of 2 binomial this is a very important example which we have to look at, now for this I am going to let me just write it here sum of 2 binomial r v, but not any 2 binomial r v which we will do here. So, I continue over here.

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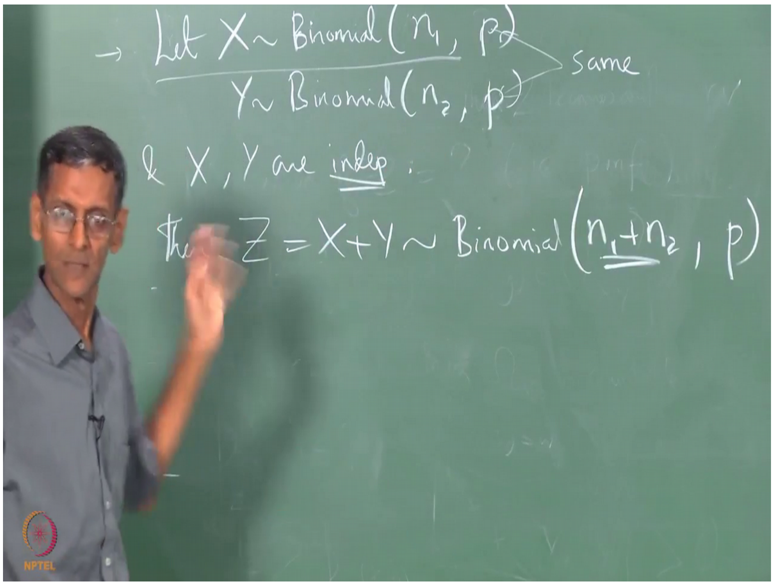

So, let X be have their p m f binomial let say n 1 comma p and Y have the binomial p m f with n 2 comma the same p p must be the same, in other words we are looking at the number of successes in n 1 trials being x, then n 2 other trials being Y and X and Y are independent so; obviously, these trials have to be disjoint in some sense there cannot be there cannot be any overlap in those n 1 and n 2 trails.

Student: (Refer Time: 15:45).

They if there were, then x x and y could not be independent then what can you say about Z just by reasoning without going into any mathematics. If you look that n 1 plus n 2 trails together if right that is what we are doing

Student: Sir, what is the argument of that n 1 and n 2.

N 1 is the number of trials for that defines X and n 2 is the different set trails define Y.

Student: (Refer Time: 16:23) about attention for that being independent.

These 2 trails have to be different sets of trials.

Student: That is it, sir.

There cannot be any overlap. Most important right I mean when you say that independent I mean I am just telling how the how this happens in practise that is all as far as the maths is concerned you just simply write this, but to get the answer you has to visualise the situation what you are talking is what n 1 plus n 2 trails now. So, what is what is answer going to be without any calculations p m f is z going to be 1.

Student: Binomial of z 1 and z 2.

Because, p is the same if p were not the same there it is a painful horrendously you know which you cannot even do by hand have to feed it your computer for different values p and q whatever right.

So, for the same p which means the same experiment is being repeated the basic experiment this is going to be binomial parameters n 1 plus n 2 comma p. Now this same thing this result can be obtained using convolution just to make sure, but let me not do it do it for the convolution for the binomial case, I will instead do it for related p m f which is the poison. I do not have 2 parameters I have only 1 parameter right in the first distribution. So, this is binomial keep this a side or to prove this formally using convolution I write I am going to leave it you to do that because instead I said we will do the conversation of Poisson.

Student: (Refer Time: 18:02).

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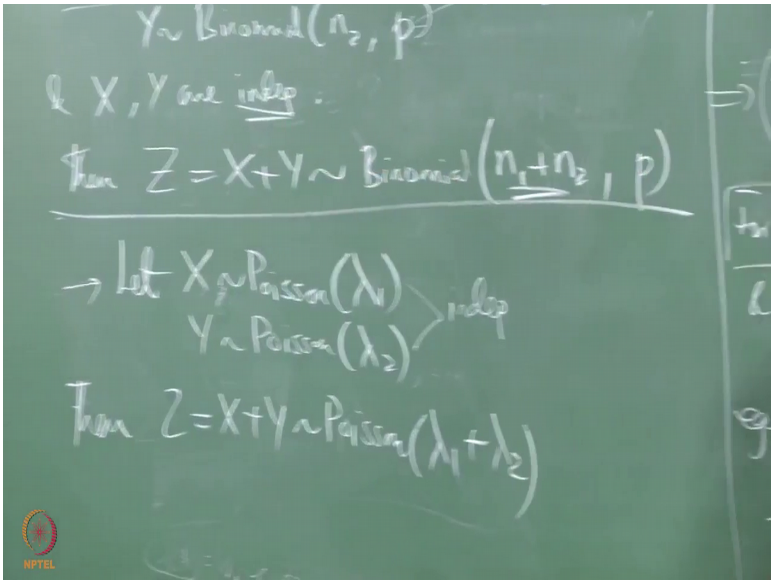

We have let us say we take 2 Poisson random variables again integer valued, but lambda 1 and lambda 2 at can be any 2 positive real numbers independent.

Now, again we look at Z equal to X plus Y here what do you think the answer should be in line with that what is the sum of 2 independent Poisson it is going to be another Poisson, how do we get this mathematically this is what I will just to make sure that you people have not forgotten your convolution you will do this in the next 2 3 minutes. First of all right what when I say Poisson lambda 1; that means, omega X is entire set of nonnegative integer from 0 to infinity and omega Y is also entire set of nonnegative integer 0 to infinity. So, the some will be again set of nonnegative integers from 0 to infinity right. So, at least that part is easy to see.

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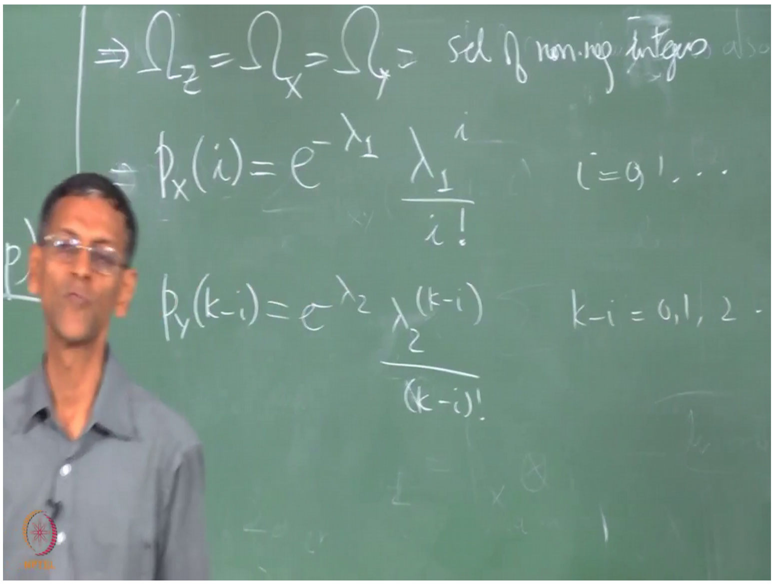

There is a notation for this, but I am not writing it this set of non-negative integers is has some Z plus I can some script Z plus or something, but since we are already using z here I do not do not want to repeat use.

Student: (Refer Time: 20:01).

So that notation. So, which implies that what now we have p x of i is going to be what for the Poisson please tell me what is p x of i it is e to the power minus lambda 1, lambda 1 power i divided by i factorial, then p y now we want k minus i. So, this is i equal to 0 1 2. So, on I am going to directly put k minus i here, because I do not want I do not want it in any other form right I want just k minus i is not it.

So, this is going to be e power minus lambda 2, lambda 2 to the power of k minus i divided by k minus i factorial now k minus i is between, but note that. So, between these 2 what is the allowed values that, you are going to look for I can cannot go negative cannot go less than 0 cannot not exceed.

Student: K.

K either. So, if you freeze k and say what is p Z, now what is the probability that Z takes a value 5 or 10.

Student: (Refer Time: 21:13).

I can only go between 0 and 10 or 0 and 5 you can get the pairs 0 10 1 9 2 8, but you cannot get 11 towards you cannot get add 2 non negative numbers 1 of them being 11 you cannot get 10.

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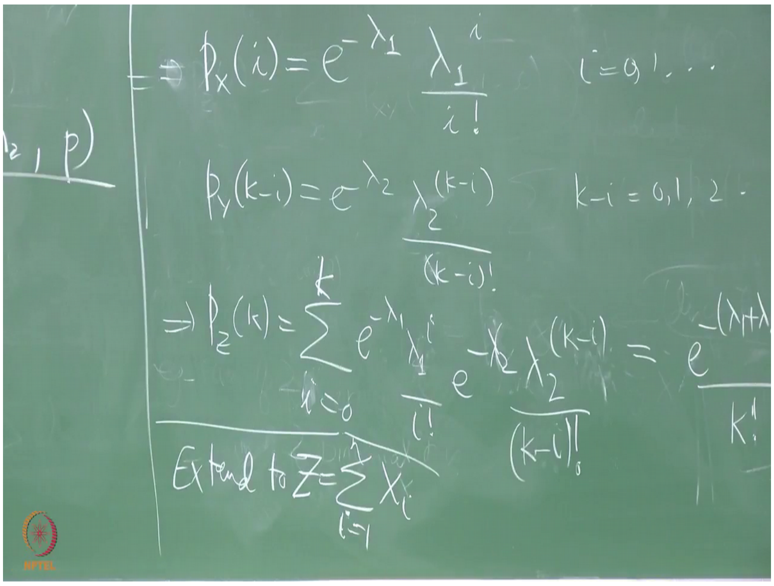

So, therefore p Z of k is the sum again this just goes back to convolution right you have seen this kinds of things this is the convolution of 2 causing sequences, were you write sum from 0 to n remember the limits do not stay as minus infinity they become 0 to n in this case 0 to i. So, e power minus lambda 1 lambda 1 i by i factorial e power minus sorry lambda 2 see now you get the e power minus lambda 1 plus lambda 2 straightaway you get that lambda 2 to the power k minus i by

Student: (Refer Time: 22:17).

K minus i factorial.

So, now let me make sure that I can manipulate I mean do not want to spend too much time manipulating this, because actually extremely straight forward all I have to do is pull out the I.

Student: Sir (Refer Time: 22:40).

It is actually extremely straight forward.

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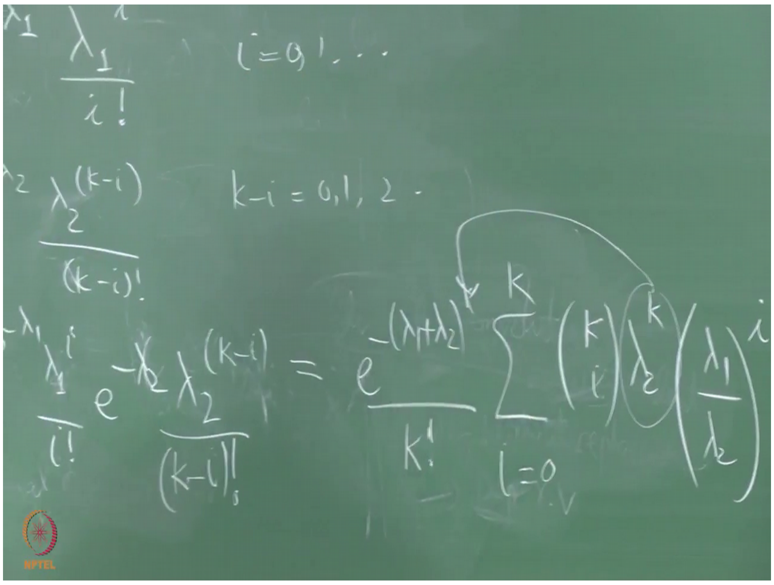

E power minus lambda 1 plus lambda 2 will come out and I multiply and divide by k factorial. So, that what do I get if I multiply if I divide by k factorial and multiply by k factorial and keep it inside I will get I equal to 0 to k, I have k k factorial in the numerator I have I factorial I have k minus i factorial, I have lambda power I and I lambda 2 power lambda 2 to the power 1 power I lambda 2 to the power k minus i. So, what is all that all of that going to become it will be just become?

Student: (Refer Time: 23:18).

Just want to make sure it is it is lambda.

Student: K.

So, k choose I so there will be a k choose I term here this lambda 2 to the power k also I if I pull out this 1 more thing I can pull out which is this lambda 2 to the power of k, I can pull out the lambda 2 to the power of k is not it; I will pull out this lambda 2 to the power of k, but does not matter I will write it out here separately.

We can always pull it out later right I will put it write it here then this will be lambda 1 divided by sorry lambda, lambda 1 divided by lambda 2 to the power of I do I or do I not get this I get it right lambda 1 power I divided by lambda into lambda 2 power minus i is same as putting it that way and then this k factorial if I put it here there is an I factorial k minus I factorial it just is becomes k choose I, what is that summation you pull out this lambda 2 power k outside, that summation is nothing, but 1 plus lambda 1 by lambda 2 whole power k,

Student: (Refer Time: 24:25).

What?

Student: but sir i 0 to k: if i equal to 0 to k.

This should be sorry i equal 0 to k i wrote it correctly over there.

So, 1 plus lambda 1 by lambda 2 whole power k so if manipulate all that what do you get.

Student: Lambda 1 and lambda 2. (Refer Time: 24:50).

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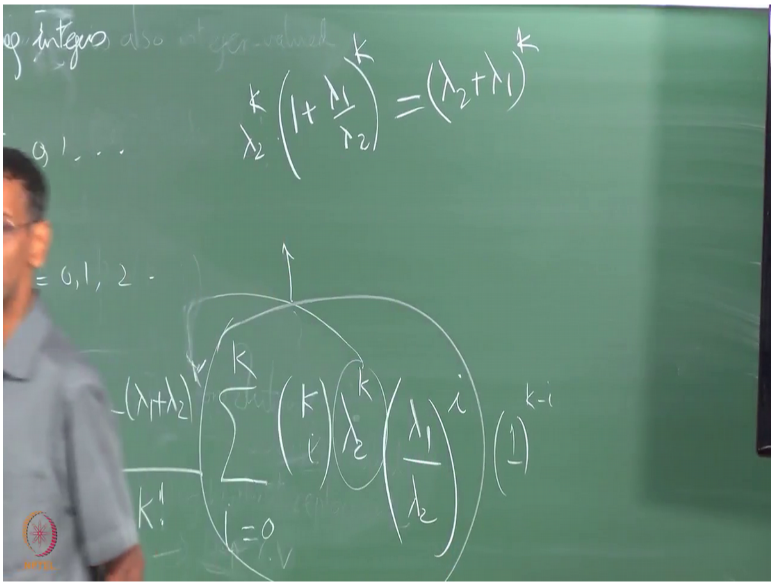

So, this whole thing becomes lambda 2 to the power of k multiplied by.

Student: (Refer Time: 24:59).

1 plus lambda 1 divided by lambda 2 to the power of k, what is this equal exactly.

Student: (Refer Time: 25:06).

So, this becomes nothing, but lambda 2 plus lambda 1 whole power k which is exactly what we want. So, actually I do not need to pull this outside I can leave this as it is right.

Student: (Refer Time: 25:25).

In other words I am saying I am let me add this 1 power k minus i for those of you that want tosee the binomial expansion, but actually it is not needed right 1 plus x power n does not need, the extra term 1 plus x power n is what 1 plus all those terms. So, the 1 1 power k minus i is not needed, but if you want to write it you can write it.

So, convolution works now convolution of 2 Poisson is another Poisson in this does not require that the advantage of showing it for the Poisson is right I do not have to worry about an extra p I just have 1 only 1 parameter which is lambda are or lambda 1.

Student: (Refer Time: 26:12).

Lambda 2 whereas, here this critically depends on writing the same p in both places of course, mathematically you can do it. So, I am going to urge you people to complete this derivation by yourselves, but it is actually obvious I mean if you look at the physics of the situation it is there is no other way for it to happen right, if it is a same p then it is basically the same experiment repeated n 1 to n 2 times and the sum of X X plus Y can only be binomial with the increased n there is no other nothing else it can be. So, these are actually. So, this Poisson is a very interesting case right you can think of a big Poisson now.

I can now break I mean I can extend this to multiple summations right I can look at Z equal to sigma i x i, I can do instead of just looking at X Y X plus Y, I can consider I can extend to Z equal to sigma i equal to 1 to n.

Student: (Refer Time: 27:16).

Or if you do not like does not matter I gives as I here again i equal to 1 to n x i. In the sense that I can do pairwise supposing X i s are i d Poisson with some s some lambda, what is that sum going to be you can add it now remember lambda 1 lambda 2 can be 2 2 totally different takes. So, if I add 2 Poisson’s with lambda and lambda I get 2 lambda. So, if I add it over n I will get n lambda.

So, let me write it here.

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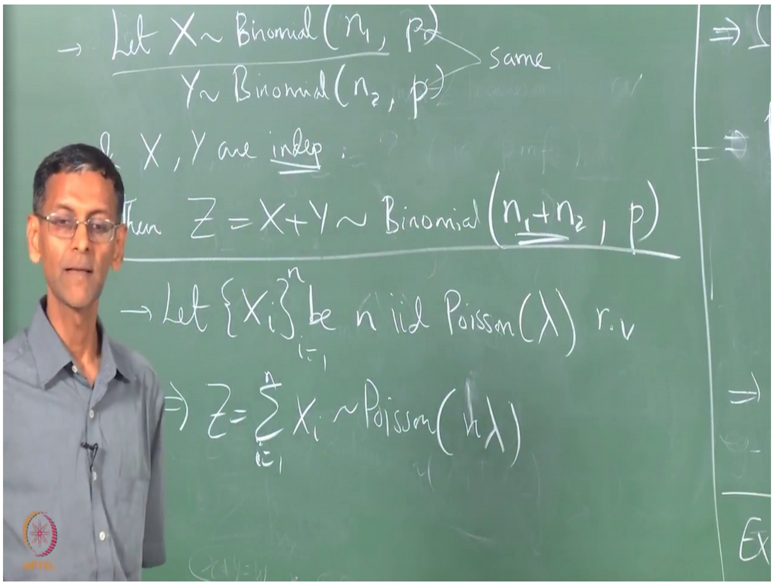

Be n I equal to 1 to n be n i i d Poisson lambda r v, then Z equal to the summation of all of them this limits are important because if I take a smaller collection I will not get n lambda. So, this is going to be Poisson this is 1 of the most interesting results in the theory right which says, now at the how many Poisson's you keep adding.

Student: (Refer Time: 28:38).

As long as they are independent the independent is built in here i i d itself you get the sum of the parameters n lambda. For Poisson, for binomial we have a similar result that it is the last I thing I want to say for today a binomial can be regardless is sum of n Bernoulli's a binomial n with parameter n why not what is the if I want to break this kind of summation down to the atomic form with each trial I define X instead of X i i will instead of starting with binomial I start with Bernoulli.

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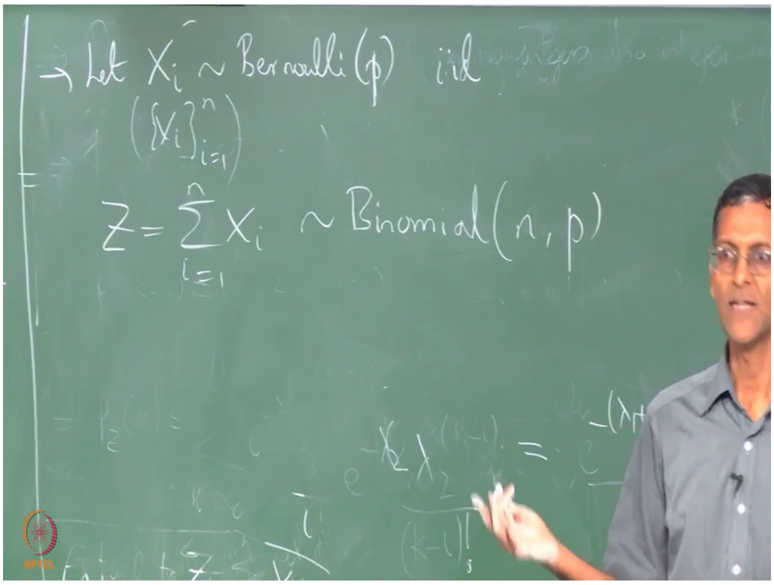

Right this is a basic head tail success failure experiment you are repeating n times right you have i equal to 1 to n. So, what is now Z this will be binomial n p this is no longer; obviously, if you add 2 Bernoulli's you are not going to stay within 0 1 you are going to become 0 1 2, but why do I get this results

Student: (Refer Time: 30:19).

As long as the p is the same in all, what is this summation exactly. It will be the number of total number times you get 1 I mean Bernoulli automatically 0 1, I do not have to describe this there is no longer success failure it is 0 1, when I say X I is Bernoulli I am constraining X i to be either 0 or 1 nothing else because random variables have to be numbers. So, if I add a bunch of 0es and ones here I will add the only ones.

Student: (Refer Time: 30:48).

And that ones is the number of ones I get in the sequence of entries the sum is automatically the number of ones I get.

So, addition of i i d Bernoulli's gives you automatically it gives you binomial. So, this is the connection between binomial and Bernoulli. For Poisson it stays as Poisson of course, this is more interior see this is somewhat less interior to see where in what is the sequence of experiments you have with n i i d Poisson that is a different issue I do not want get it into that, anyway if you really want think about it you can think of right even this makes introduced sense, you know if you keep augmenting a long sequence of observations right and the number of events in a particular time in time period being Poisson.

If you keep looking at more and more observations periods in your sort of looking at that situation right X 1 plus X 2 plus X 3 over some 3 different time intervals. So, that they are independent X 1 is the number of observations you make it 1 interval X 2 take an if the in underlined process gives this right is same sound by the same physics then they will all be x i s will have the same parameter lambda and therefore, X 1 plus X 2 plus X 3.

Student: (Refer Time: 32:08).

We will be equivalent to observing for such a long period of time.

Student: (Refer Time: 32:12).

So, that also make sense is this clear I simply will be looking more in that direction let me also look this side right give some visual feedback are you happy with all this.