Probability Foundations for Electrical Engineers
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Lecture – 35
Example: Compute Marginal and Conditional PMFs, Probability of Events

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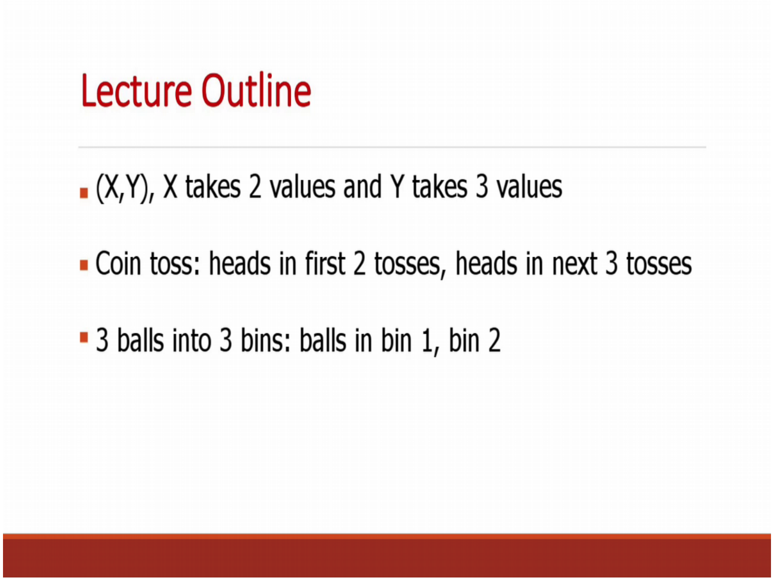

Welcome to this lecture on computing with joint PMFs of 2 random variables. So, we must have seen Professor Aravinds lectures on how to do computation of marginal PMF from the joint PMF, and how to do computations for probabilities of different events involving 2 random variables. So, these 2 are extremely crucial ideas, once again in this lecture I am going to take some simple examples and illustrate how to do these calculations.

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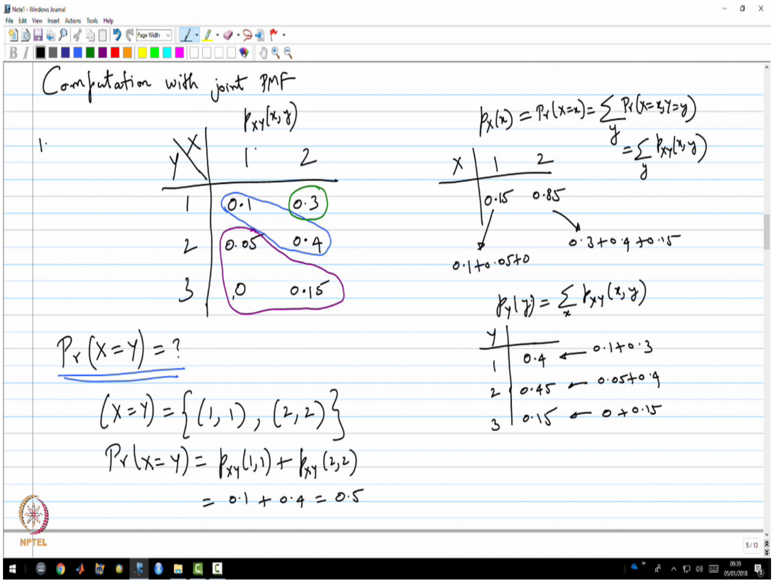

So, let us begin this. So, this is a computation with the joint PMF.

So the first example I will take is some sort of an artificial example where I give you the PMF, and then I ask you to compute some basic things. So, that is also very important to see let us take a joint PMF once again I am going to have this notation for the joint PMF where I put a table and then I put probabilities corresponding to the table. So, I will take a very simple case where x takes 2 possibilities, and y takes 3 possibilities and I will then put probabilities here.

Now, remember I have to put probabilities, so that they all add up to 1 and I will do it in a certain way I mean 1 can do multiple things here nothing is really critical here. So, let us say I put 0.1 here, 0.3 here, 0.05 here, 0.4 here I am just making it up as I go. So, maybe 0 here and then I have to put here something to make it a valid PDF PMF.

So, I have put some values here for example, for the joint PMF of x and y I put, 1 as 0.1, 2 comma 1 is 0.3, 1 comma 2 as 0.52 comma 2 as 0.4, 1 comma 3 as 3 as 0 what should be 2 comma 3. In fact, I can find out what should be this guy; why is that because all of these guys should add up to 1, what are they adding up to 0.7, 0.8 0.85. So, this should add up to this should be 0.15. So, this makes it a valid PMF. So, this is one check like I mentioned you have always done particularly when you calculate complicated joint PMFs should ensure that they add up to 1 that is a valid condition alright. So, now, let us do computations now. The first computation that you should it is important to understand this marginal PMF. So, this like I mentioned is the joint PMF of x and y, this is a notation that I have for it, now the marginal PMF of x has only 1 entry here x is 1 2. So, basically this tells you what is the probability that x is x right. So, this is simply summation over all y probability that x equals x comma y equals y and that is simply the probability joint PMF of x comma y. So, for every x what should I do? For every x suppose I want to find the probability that x equals 1, I have to add the joint PMF over all y. So, I should simply add up all the entries in the column here, I will get the probability. So, this will just be 0.15, this will be 0.85. So, how did I get it? 0.15 is 0.1 plus 0.05 plus 0.85 this 0.3, plus 0.4, plus 0.15 what about the joint PMF the marginal PMF of y.

So, here once again if you do this calculation carefully, I have to add up the values in every row right. So, I can repeat the same thing you will get the final formula as p x y of x comma y; you add up over all values of x keeping y as same. So, y is 1. So, I have to add a 0.1 and 0.3, I have get 0.4 here, 0.45, 0.15. So, this like I said it is 0.1 plus 0.3, this is 0.05 it is 0.4, this is 0 plus 0.15. So, this is a way to calculate marginal PMF from the joint PMF we add over columns or the rows. So, in this simple example it works out quite nicely more complicated examples like say the trinomial example that Professor Aravind did for you in the lectures, you have to do more sophisticated computation, but essentially the idea is the same. So, you add up over one of the variables to get the marginal PMF.

So, that is important to understand. So, the next interesting calculations that Professor Aravind does is, he asks questions of events based on this joint PMF then asked for probabilities of it. So, let us do a few events. So, first event I want to do is X equals Y. So, what is the probability of X equals Y? So, suppose somebody asks you the question what is the probability of the event X equals Y. So, here, the first thing to do, when you given a problem like this in multiple random variables, one random variable is to first describe the event very nicely. So, in this simple situation one can de describe the event as the set of possibilities that x and y can have. X can take 2 values 1 2, y can take 2 values 1 2.

So, if x has to be equal to y, x comma y should either be 1 comma 1 or 2 comma 2. So, it is easy to list out this possibility here. So, let me do maybe a blue calculation here, to show where the event would be x equals y is this here right all right. So, this is the event 1 comma 1, 2 comma 2, x and y are equal x squared x and y could be both 1 or x and y could be both 2. So, the probability of this is simply p x y 1 comma 1, plus p x y 2 comma 2 you have to simply add those 2 guys this 0.1 plus 0.4 plus 0.5. So, that is how you do events. Now you can do any other event of this form there is nothing sacred about x equals y, for instance you might want to do probability of x greater than y ok.

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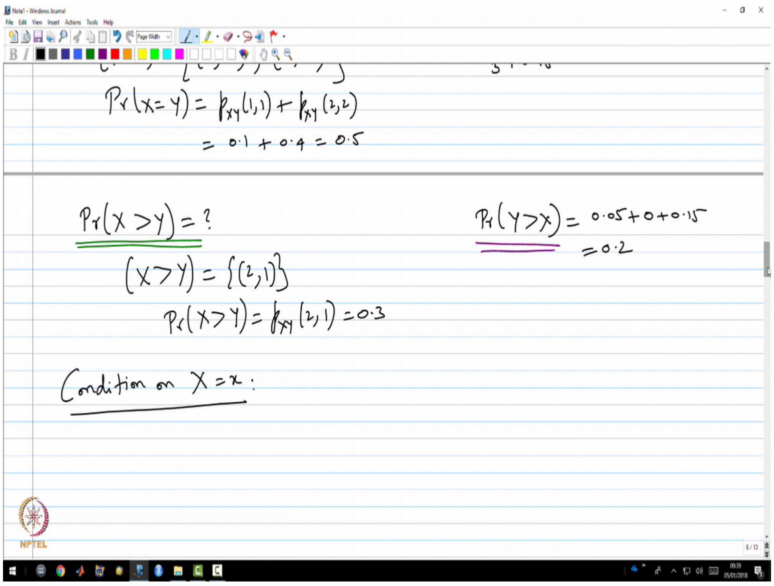

Maybe I will use a different color here; I will use green color for this event. So, f x needs to be greater than y, if you if you look at this possibility the only way in which X can be greater than Y is this guy, there is only one possibility by which X can be greater than Y. So, once again if you want to write that down, X greater than Y only possibility is x equals 2 and y equals 1 and. So, probability of X greater than Y simply p x y of 2 comma 1 just 0.3, it is very simple example and you can also do maybe one more example maybe I will do that on this side, probability of let us say Y greater than X just to make it a little bit more interesting.

So, maybe I will use a violet color for this, and I do that if I am going to skip a few steps go to a bit higher a bit quickly. So, if I want Y to be greater than X the possibilities at these 3 right and the probability of that is simply going to be 0.05 plus 0 plus 0.15 that is 0.2. So, this is how you do simple events in this easy case if of course, this case the situation becomes more complicated, you have to do more sophisticated calculations. Like instance in the trinomial case the calculations are not so simple, but nevertheless the idea is the concept is exactly the same. So, let us also do a few more examples, the next example is conditioning right. So, conditioning is another thing that Professor Aravind talks about, suppose I want a condition.

So, you condition on X well x right. So, this is something that Professor Aravind talks about. So, if you look at this joint PMF, I could condition on x equals 1 or x equals 2 condition on x equals 1. So, I will have a conditional PMF of y, conditioned on x equals 1 ok.

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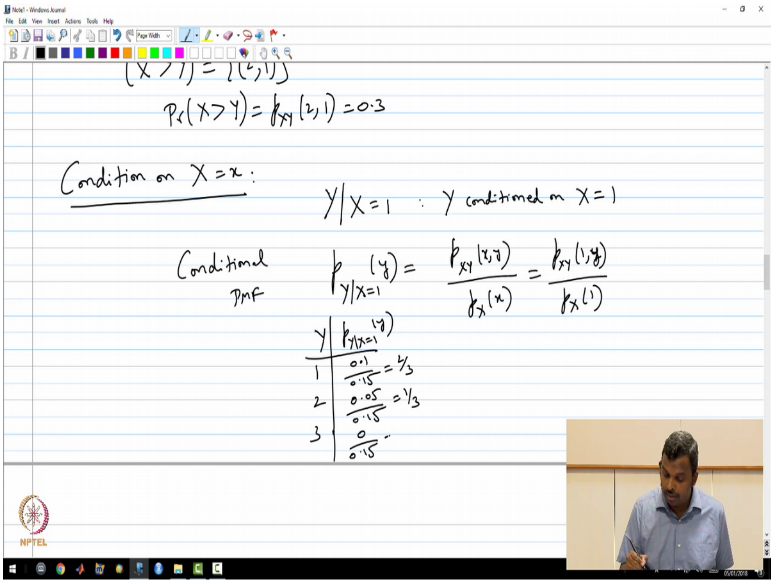

So, Y conditioned on X equals X. So, this is Y conditioned on let me just do this 1 X equals. So, if you want to do this sort of PMF. So, if you condition on x equals 1, then you do a PMF. So, this is the conditional PMF this would be p y x of x y I am sorry of x comma y divided by p x of x right. So, this is the formula here. So, since x is 1. So, this is just p x y of 1 comma y divided by p x of 1. So, we have calculated most of these things here p x of 1 is simply 0.15, and p x y will work out as 0.1 and 0.05. So, if you do that you will simply get p x p y given x equals 1 of y be the ratio of these 2 guys for different values of y this will work out differently. So, it is good to write down like before you put a Y here 1 2 3 and then I want p y given x equals 1 of y. So, this is going to be 0.1 by 0.15, 0.05 2 by 0.15 0 by 0.15. So, this will be 2 by 3, 1 by 3 and 0. So, that is the conditional PMF of y given x equals 1 is that. So, basically the idea is you look at a particular column that will be the same proportion as the conditional PMF except that you have to normalize it to 1.

So, you divide by the sum of all the values in that column so, 0.1 by 0.15, 0.05 by 0.15, 0 by 0.15. So, you get that the same thing on this side this is going to be a little bit more messy in calculation, but it is the same thing. So, you have to divide 0.3 by 0.85 to get the conditional PMF of y equals 2. So, let me do that also here. So, if you do y given x equals 2 of y 1 2 3, you were going to have 0.3 by 0.85, 0.4 by 0.85, 0.15 by 0.85.

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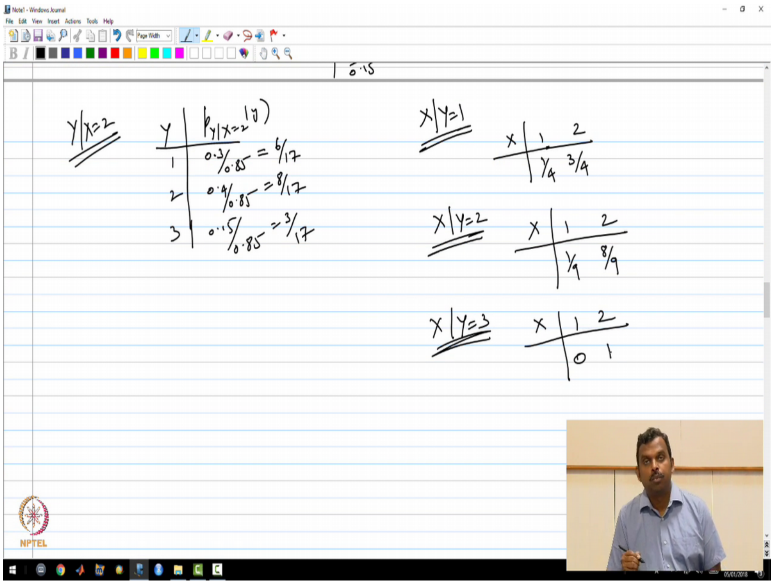

If you want you can simplify you can divide by 5 etcetera. So, you would get 6 by 17, 8 by 17, 3 by 17. So, that is the conditional PMF of y given x equals 2. So, this conditional PMFs differ based on what you condition on and the idea is to simply take the particular column and then divide by the sum of all these columns. Now I can also condition x on y. So, let me do that also; if you look at. So, this is again Y conditioned on X equals 2. So, I might want to look at X conditioned on Y equals 1.

So if I do that x is 1 2. So, I have conditioned on y equals 1. So, this is just this row I should take 0.1 divided by 0.4, 0.3 divided by 0.4. So, it is 1 by 4, 3 by 4, you also do a conditioned on y 2. So, this would be 1 2 conditioned on y equals 2, you were going to get 0.05 by 0.45, 0.4 by 0.45; that is 1 by 9 and 8 by 9 and the last case I might condition x on y equals 3, in this case actually you get a very very simple answer if I conditioned on y equals 3, it is 0 and 0.15. So, 0 divided by 0.15, 0.1 5 divided by 0.15.

So in fact, actually I get a trivial random variable which is 0 1, probability that X equals 2 given Y equals 3 is simply 1. There is no other possibility given Y equals 3. So, hopefully this simple example showed you the various types of computations that one does with joint PMF and these are all very important to understand in a simple conceptual way.

The first type of computation with joint PMF is computing the marginal PMF, second type of computation is computing probabilities of events involving 2 random variables, which is greater, which is equal or it may be even more complicated events may be we will see examples later on. And the third one is computing conditional PMFs given one of the random variables takes a particular value.

So, these kind of calculations of bread and butter is very very important to understand at least in some cases like this. So, I am going to go ahead and give some examples from some sample spaces, that we have considered the past of something very similar, and ask you to as let us see the calculations of similar calculations, like what is the possibility if 1 thing being greater or lesser etcetera second example I am going to see.

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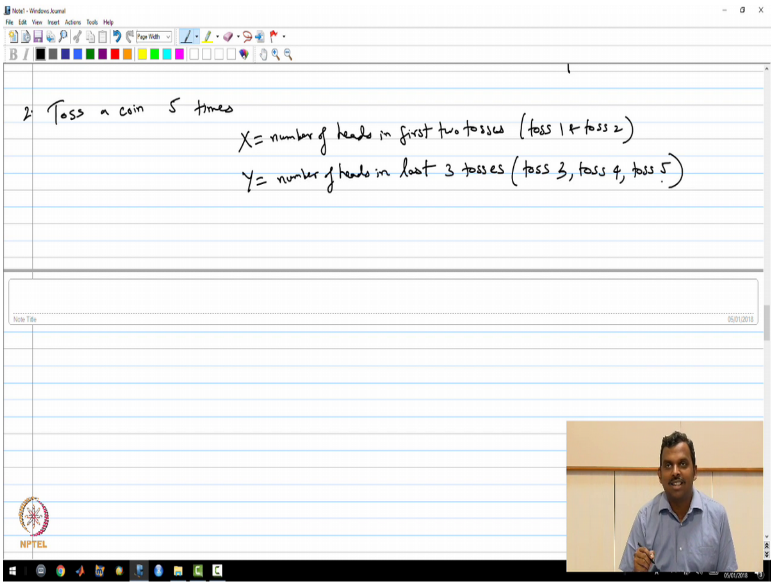

So, let us say we toss a coin 5 times and. So, this is a interesting sort of situation, but 1 needs to be a little careful here. So, now, I have to start defining multiple random variables each toss can give me a head or a tail, and I am going to define the random variables in the following fashion I am going to say, X is number of heads in first two tosses and y is number of heads in last three tosses.

We have 5 tosses toss 1, toss 2, toss 3, toss 4, toss 5 X is the number of heads in the first 2 tosses toss 1 and toss 2, and Y is the number of heads in the last 3 tosses. So, this would be tosses toss 3, toss 4, toss 5. So, that is the 2 random variables X and Y once think about how what these things are and I have to.

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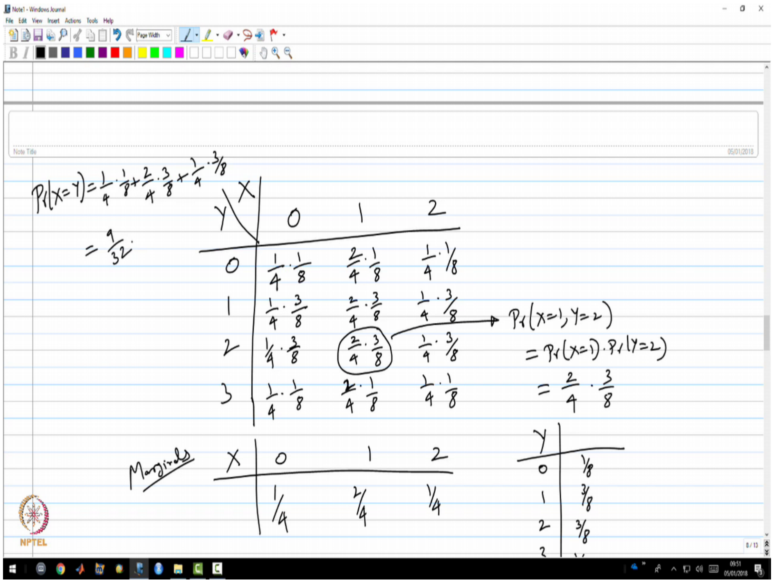

So, let us start by doing the PMF like we did before the joint PMF. So, X is the number of heads in the first two tosses it could be 0 or 1 or 2, Y is the number of heads in the last 3 tosses it could be 0 or 1 or 2 or 3 alright. So, what is the probability that I get 0 comma 0. So, I should have 0 heads in the first two tosses, remember the first two tosses what happens in the first two tosses is independent of toss, what happens in the last three tosses then what the same there is no overlap.

So, when there is no overlap like this it is easy to compute probabilities, because I can multiply the events independently; probability of x and probability of y can be multiplied whose this what is what happens is independent of this. So, that is that something that I can use. So, what is the probability that I have 0 here, it is 1 by 4, 0 heads in 2 tosses, and 0 heads in 3 tosses it is again 1 by 8 is that and then what is probability of getting 1 toss 1 heads in the first 2, that is actually 2 by 4 and then 0 tosses. This would this would again be 1 by 4 times 1 by 8 is that ok.

Think about it and then I would have the same 1 by 4 multiplying here, 2 by 4 multiplying here, 1 by 4 multiplying here because the probability of x is the same, probability of y becomes 1 out of 3 tosses, I get 1 that is actually 3 by 8 right. So, 3 by 8 3 by 8; so, likewise here I would get 1 by 4 times 3 by 8, 2 by 4 times 3 times 3 by 8, 1 by 4 times 3 by 8. Here I would get 1 by 4 times 1 by 8, 2 by 4 times 1 by 8, 1 by 4 times 1 by 8. So, this is the joint PMF think about how I got this once again. So, for instance this probability I will just show you how I got this probability that x equals 1 comma y equals 2, this is probability of X equals 1 times probability that Y equals 2, because what happens in X is independent of what happens in Y this is intersection. So, I can multiply these 2 things and this is 1 toss 1 heads in 2 tosses it is 2 out of 4 possibilities, 2 heads in 3 tosses that is 3 out of k. So, this is just binomial calculations or another ways. So, this is the idea.

So, this is your joint PMF, it is a little bit more slightly more complicated, but you will see some simplifications will happen in. So, supposing from here I want to do marginal, I have to just add up everything right. So, suppose I want to do marginal here marginal PMF X is 0, 1, 2. I have to add along the columns; if I add a along the columns I see that this 1 by 4 is common, and if I pull this 1 by 4 out I get 1 by 8, plus 3 by 8, plus 3 by 8 plus 1 by 8, that is just 1.

So, I will have here 1 by 4, 2 by 4, 1 by 4, it is easy to do this calculation and if you do Marginals for y likewise you will have 0, 1, 2, 3 and you can add up along the rows to get the marginals, you will get 1 by 8, 3 by 8, 3 by 8, 1 by 8 see it is enough to do that that is interesting. So, now, let us do other calculations, you can do more interesting calculations. So, for instance you can do let me just keep this here and do calculations around it, I might want to calculate probability that X equals Y ok.

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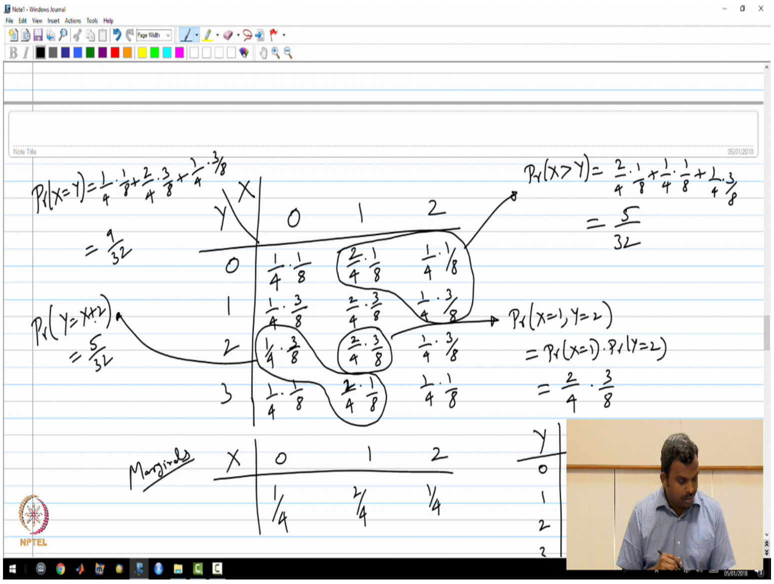

So, you can see that is the event that you have along this diagonal right this diagonal here X equals 0, Y equals 0, X equals 1, Y equals 1, X equals 2, Y equals 2 this diagonal. So, this will just be 1 by 4 times 1 by 8, plus 2 by 4 times 3 by 8, plus 1 by 4 times 3 by 8. One can simplify this calculation this 6 plus 3 is 9, 9 by 8 4 s are 32, 9 by 32 is that. So, that is the probability that X equals Y.

Probability that the number of heads in the first 2 tosses equals the number of heads in the last 3 tosses is 9 out of 32, and you might want to do more calculation, you might want to say probability that X is greater than Y, this would be 2 by 4 into 1 by 8, plus 1 by 4 into 1 by 8, plus 1 by 4 into 3 by 8 right. X is greater than 2, here X is greater than Y is this part right top side maybe I should show that here this is this calculation, and that works out as 6 by no 5 by. So, will do that 5 by 32, likewise you can do other events so in fact, you can do more sophisticated sort of evens, I am I might want to define some something slightly more tricky.

So, for instance I might say I need two more heads. So, for instance you might say probability that Y is equal to X plus 2, I might want to say that. So, this will correspond to these two guys right. So, you just mark out that event and add. So, you will get 5 by 32. So, likewise you can define any other relationship you want between X and Y, Y is less than X plus 2, Y is greater than X plus 2 or any other way of de marketing this area that you want to have you can write, and you can compute probability the way in which this probability is being done.

You can also do conditional probability exactly like I described before and in this case you will see this interesting thing that the conditional probability conditional PMF is the same as the unconditional marginals. So, that comes from the independence as well. So, this was a slightly more interesting example that comes from our examples. So, you can also do similar examples. So, I think I will just finish with one more examples, this is the third example I am doing.

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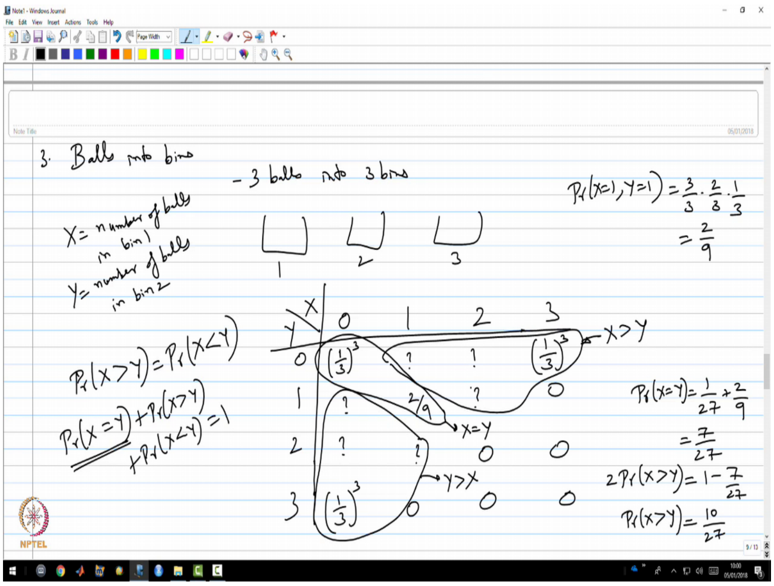

This I will do balls into bins, always ends up being a little bit tricky and we will not do it in great detail, but anyway. So, I think it is good to finish up with slightly more complicated. Let us take 5 balls into 3 bins. So, you have 3 bins 1, 2, 3 and you are throwing 5 balls. I am going to define 2 random variables here. So, let us say X is similar to before number of balls in bin 1, Y is number of balls in bin 2 similar to before nothing a very fancy. And if you want to do X Y here, maybe I want to make this even simpler. So, I will just say 3 balls into 3 bins I will make it very very simple so that you do not have to write down long calculations.

So, there are 3 balls. So, in the first bin could be 0 or 1 or 2 or 3 likewise, it could be 0 or 1 or and we know that this you know that is this; there is only 3 balls that we have thrown. So, you cannot have too many balls beyond this also. So, some things will go to

0. So, for instance 3 comma 1 will go to 0 3 comma 2 will go to 0 3 comma 3, will go to 0 2 comma 2 will be 0 this will be 0. So, this will be upper triangular part lower triangle part will go to 0 and some of these other things are easy to write down for instance 0 comma 0, all balls should go into the third bin right.

So, that is just 1 by 3 power 3. So, the same thing will happen for this guy right 3 comma 0 means all the ball should go into the second bin. So, that will also be 1 by 3 comma the power 3 right. So, some of these calculations are easy to do right all ball should go into the first bin, all ball should go into the second; all ball should go into the third bin this is that post for probability here, but this one is all ball should go into the second bin, and this guy here one on the side is all the balls should go into the first bit right. So, that is this that is an easy calculation to do all other calculations are a bit tricky. So, let us try; I mean you can try it I am not going to I am not going to do this in great detail here. So, all these calculations I have to be done there is symmetry here. So, you do not have to really calculate everything exhaustively, if you calculate the ones on the top right you will get the ones on the bottom also exactly to be the same symmetry is going to help you, but nevertheless you have to do a calculation here of at least 3 or 4 numbers, they are a bit messy, but if you do that you will get a joint PMF, but with this joint PMF you can do a similar calculation there. So, for instance you might ask question, what is the probability that X is greater than Y or X is equal toY? So, so let us ask that question. So, what is the probability that X will be greater than Y? So, on the face of it, it looks quite difficult to compute this, but one can say some interesting things about this situation. So, for instance, so if you think about this.

So, even though we do not know much about what is probability of X greater than Y we can say for sure that this will these 2 will be the same right probability that x greater than y, and probability that x less than y will have to be the same why is that? This symmetry in this situation right so, whatever is on the right top right side will be the same as what is in the bottom left. So, if you look at X greater than Y, Y greater than X those are those two events right. So, x greater than y is this event x greater than y and y greater than x is this event right. So, we can say that these 2 will be equal because of the symmetry of the situation. So, this kind of reasoning is very interesting to do with many problems. So, even though you do not know much about these probabilities actually, you can compute them I am not saying it is very hard, but still without even computing you know that these 2 would be true. In fact, you can even say something more.

If you look at probability of X equals Y right this plus probability of X greater than Y plus probability of X less than Y have to be equal to 1 right. So, that is everything. So, what is X equal to Y it is this guy right, this is X equals Y. Together they should all add up to 1. So, these 2 are all the same and this X greater than Y and Y greater than X, X less than Y are also equal. So in fact, if you find the probability that X equals Y, you can find probability that X is greater Y and X is less than Y without doing too much other work based on the symmetry. So, this kind of reasoning will help your calculations a lot. So, the only thing I need to really find to find probability that X greater than Y is this guy. So, what is the probability that X equals 1 comma Y equals 1. So, I have one ball in the first bin, one ball in the second bin and what should happen to the third ball it should go into the third bin. So, what is the probability of that?

So, the first ball can go anywhere right it does not matter where it goes, the second ball should go only into the other two. So, we have 2 out of 3 and the last ball should go only into other 3. So, you will just 6 by answer you simply get 2 by 9. So, this probability is 2 by 9. So, once I know that I know probability of x equals y. So, what is the probability that X is equal to Y? It is 1 by 27, 1 by 3 power 3, plus 2 by 9. So, that is 7 by 27. So, what is the probability that X is greater than Y. So, if you look at this equation these 2 are equal and this is 7 by 27. So, I should do 1 minus 7 by 27 divided by 2, two times this is 1 minus 7 by 27 probability that X is greater than Y if you do this you get 20 by 27, you can divided by 2 you will get 10 by 27.

So, that is the probability that X is greater than Y. So, I mean of course, I can do all these calculations and come up with the answer it is not very complicated in this case, but nevertheless it is nice to be reason like this. So, supposing if I were to increase the number of balls to 5 balls, 10 balls this kind of reasoning will still help you I mean it is not impossible to do, I mean it is still hard, but not a little bit easier. So, I will stop with this as far as computations with joint PMFs is concerned, this is quite important like I said given a joint PMF in this simple form like this, you should be able to find the marginal, should be able to conditionals, and you should be able to find probabilities of units.

Thank you very much.