Probability Foundation for Electrical Engineers
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Lecture 12
Part 2
Trinomial Joint PMF

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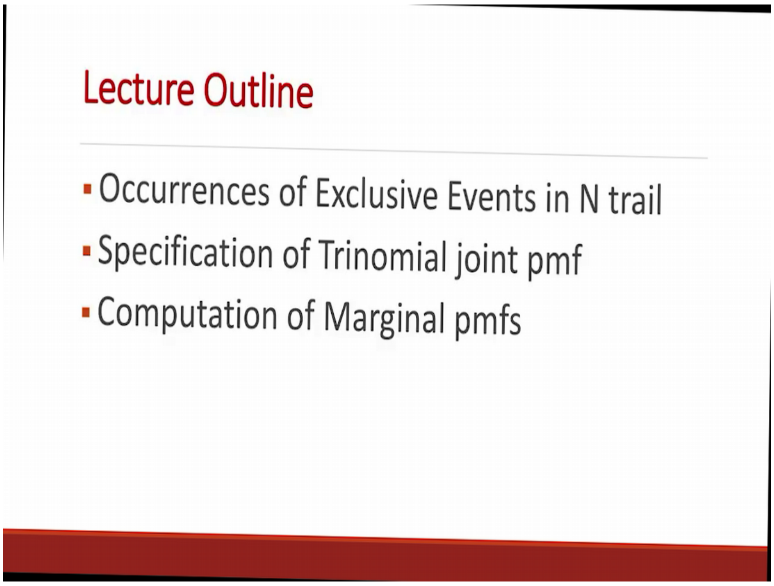

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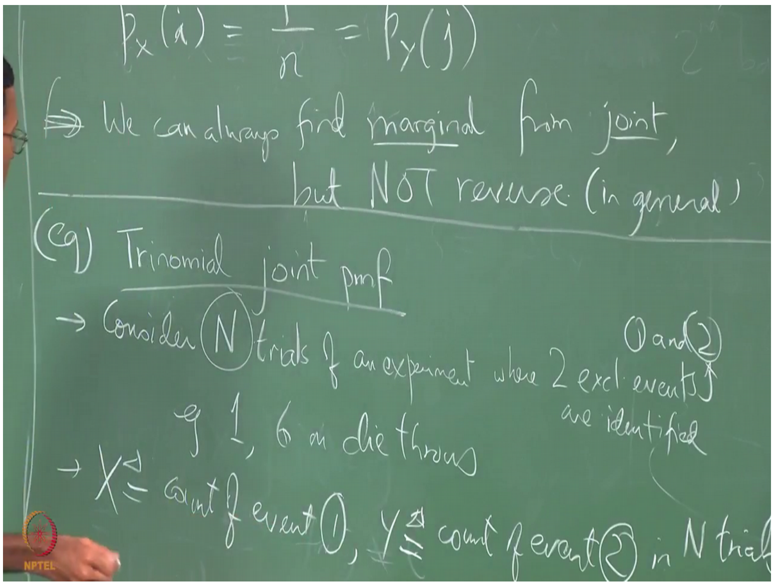

Which is eg what we call the trinomial joint pmf, so what is this this is go going to look like an extension of binomial, right. So, it is trinomial. So, you consider an experiment right, let me say I am going to use capital N which is I have been using it in my notes which I do not want to change although it is more commonly written with small n, but does not matter . So, any independent trials of an experiment on which you can get let us say 2 events 2 exclusive events you have identified. So, you all I am trying to say is that you have identified 2 poss; 2 exclusive possibilities for example, getting a 1 and getting a 6 on the throw of die right and you are considering N different trials of that that is just let me write it here eg 1 getting 1 6 on die throws.

So, I am doing N trials now how do I how do I define X and Y X is going to be the count of one of those events and Y is going to be the count of the other events here out of in N trials. So, X by definition is the count of let us say event 1 1 and 2 say events right one and 2 count of event one and Y is count of event 2 in N trials. So, you have these 2 random variables defined in that manner clearly the space is discrete right 1 X and Y can only take a discrete collection of values accountable. In fact, is a finite collection now alright. So, is the I wanted to keep it as general as possible right for example, in this case you could say X is the number of times you see one and Y is the number of times you see 6.

So, now we are looking at the joint probability that the number of ones is some number between of course, 0 and N and Y is again another number between 0 and n, but the 2 numbers have some restrictions right the count of event one is the count of event 2 cannot exceed N at maximum they can be N they cannot become more than N because if I throw a die at ten times if I right, I cannot ask for the probability of getting what is it seven ones and nine sixes in individually I may be able to get seven ones and nine sixes, but I cannot get them jointly right. So, this is it turns out that this joint pmf arises in a number of situations which is why I have left it general.

So, this is one of those general right a general joint pmf right which I said we do not have we do not have too many examples of, but this is one of them I am talking of a nontrivial joint pmf which cannot be obtained from as we will see later it is a very important subdivision where this this operation can be done and may cannot be done in general right we cannot do it, but there is of course, a very important subclass where we can we will come to that later, but this is an example why we cannot as we will see and this fits into this.

It is very important note and once again I repeat this fits exactly into this kind of situation where you know the jo marginal, but you do not know join how do I know the marginal in this case suppose if I close my eyes to what is happening in Y, then I look if I look only at X I will get a marginal pmf of X right I can always do that although the experiment will give me both values of X and Y if I close my eyes to Y I can evaluate only at X what do you think the marginal pmf of X is going to be on this it is the success rate for;

Student: Binomial.

Binomial.

Now I want to be a little more careful, here I will say that event one has probability small p and event 2 has probability small q of occurring on any one trial right.

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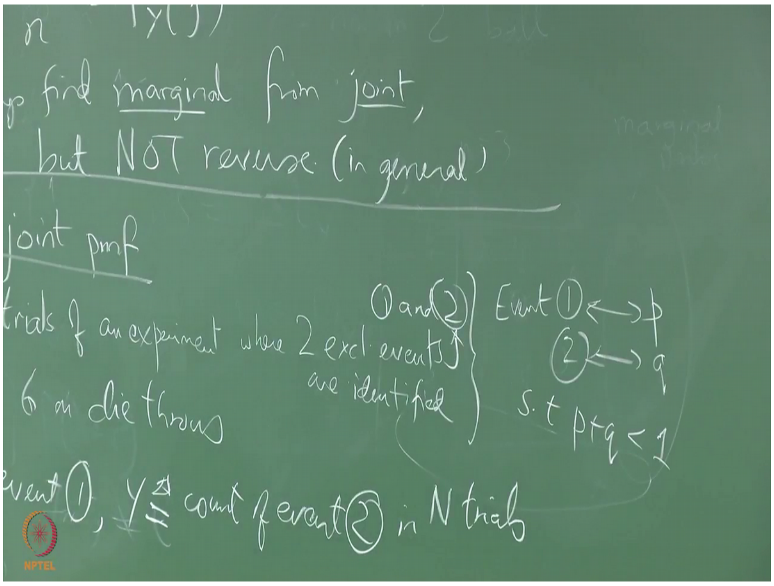

So, these 2 exclusive events, right event one has probability p and 2 has probability small q such that of course, p plus q must be let us say strictly smaller than one right not equal to 1. So, there is space for something else neither 1 nor 2 and of course, p and q are both bigger than 0 both of them can happen, but they do not tile the space right.

If they tile the space then you really do not have much o of freedom right why if you say X; X occurred k times and Y must occur in minus k times if u if v plus q equals 1, it is like counting the number of heads and number of tails on just on a coin on a coin throw there is no point in defining a Y in that case because you know the value of the number of tails why would you want to unnecessarily define a random variable for that which is when you know the right when it is not really I mean it is a Y.

So, what I am trying to say is knowing X, you cannot where we have come all that this is a much late in great detail, but at this point you should just keep in mind that knowing the value of X you really do not know exactly the value of Y right that is that is also one very important thing about these joint distributions, right in general right. So, these 2 have some independent you know some degree of freedom.

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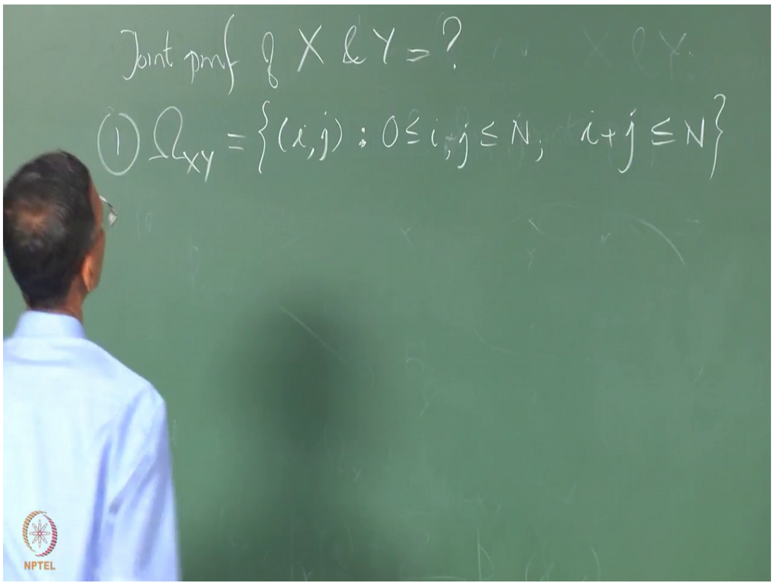

Now let us look at the joint pmf of X and Y how would I write it; first step is omega X Y identify this X and Y can individually take values from 0 to capital N and I have put capital N I am going to stick with this cap you know and earlier we may have used small n, but from for now I am going to for this thing i please right stick to I mean I am going to use capital N only. So, what is omega XY the set of all points ij such that i and j are both between 0 and n, but you also want to do what say why do you want to say i plus j must be;

Student: Less than or equal to N.

Small yeah less than or equal to N.

Of course, I can equal N that is right this is just i plus j equals N is possible, but not more. So, what kind if you do if you plotted all these points on the XY plane it would be what it would be a triangle actually right.

A triangle bounded by these straight lines i greater than 0 greater than equal to 0 j greater than 0 right.

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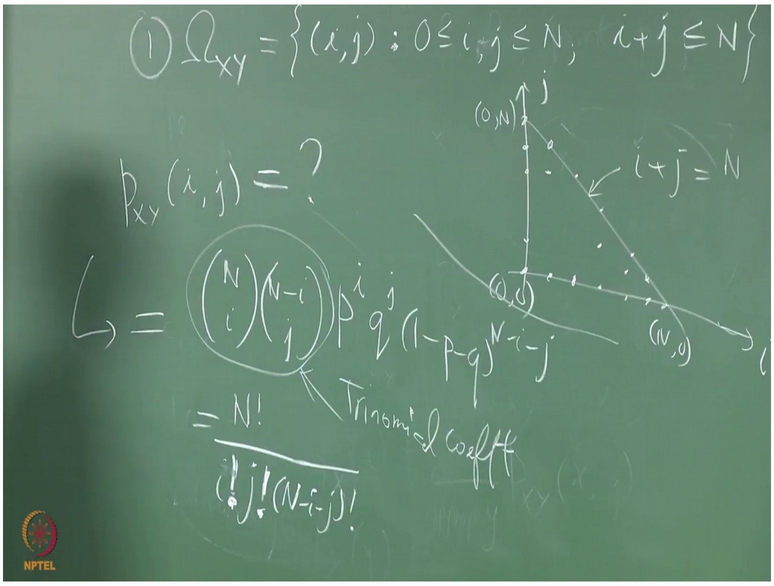

So, if I want to draw it like this I would get something like this. So, what would be this line this would be the line i plus j equal to N you can do it like this ij right and it is all these points on a nice grid like this it is just to give you an idea this is the point N 0 this is 0 N and. So, on and you have 0 0. So, this is a set of points omega xy. Now we want to ask again the same old question what is this joint probability px yij which is the joint probability X taking the value i and Y taking the value j on a single run of the experiment which gives you X and Y values right what how do you go about writing this remember X measures the count of event one which happens with probability p and. So, in this straightaway you are saying that X occurs that is a event one occurs i times even j event Y event 2 occurs j times.

That is what we are saying right and neither occurs N minus i minus j times. So, so what do we have here? So, this is going to be it is going to have this term p to the power of i q to the power of j 1 minus p minus q to the power of N minus i minus j that is going to be a term right this is go what is this a probability of this is a probability of a particular sequence of i times getting count 1 j times getting sorry; i times getting event one j times getting event 2 and N minus i minus j times you are getting neither of them a particular sequence how many such sequences are possible.

Student: N choose i N choose i.

The number of times such events are possible will be N choose i multiplied by N minus i choose j this is together called the trinomial coefficient and what is if you expand it out what do you get.

So, let me say here clearly one last one more time event; event one occurs with these are probabilities right of event events one and 2 occurring on any trial and you are running you are running experiment for N trials and this is the joint probability that you get i times event one and j times event 2 and that instead of writing all that in English we are writing it compactly like this. So, what was this trinomial coefficient you can see clearly see this would be equal to N factorial divided by i factorial j factorial into N minus i minus j factorial right it is another way of right N factorial by i factorial that N minus i will factorial will cancel and you will get exactly this this j factorial here N minus i minus j.

So, basically that is all we need. So, in the case of 1 and 6 getting one and 6 on die throws p will be equal to one by 6 q also will be equal to 1 by 6 right, but I do not have to have you 1 and 6; I can have 1; I can have 1, 4 and I can have a square and I can get whatever 5 square and 5; 5, 6 here or something right or that also will be equal or whatever I can have square here and only 6 here whatever I do not have to have p equals q; obviously, is it not. Now we have already said that the marginal pmfs.

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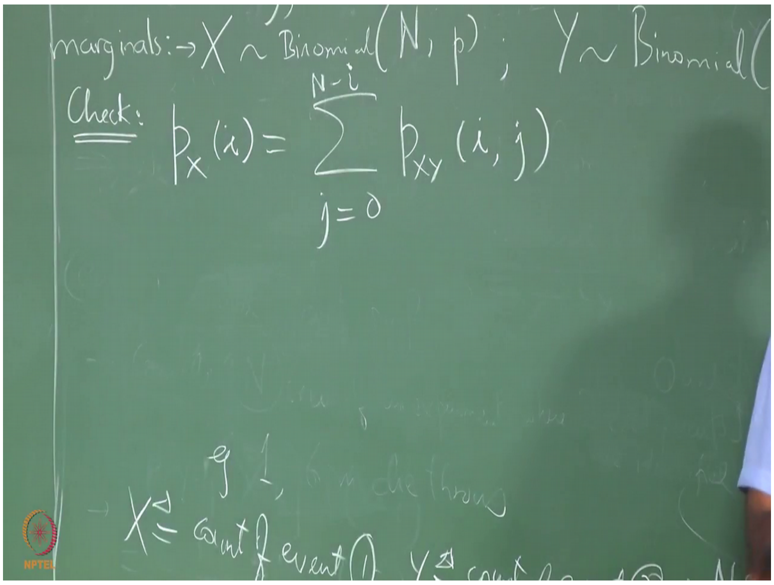

So, by elementary reasoning or whatever by prior reasoning or whatever you want to call it we have said what have we said X is binomial with parameters N and p Y is binomial with parameters N and q we want to now derive this these 2 marginals. So, these are the marginal just call them marginals marginal pmf how do we derive this from this joint it is actually easy to do all you have to do is make sure that you add the right add the you take a particular value of.

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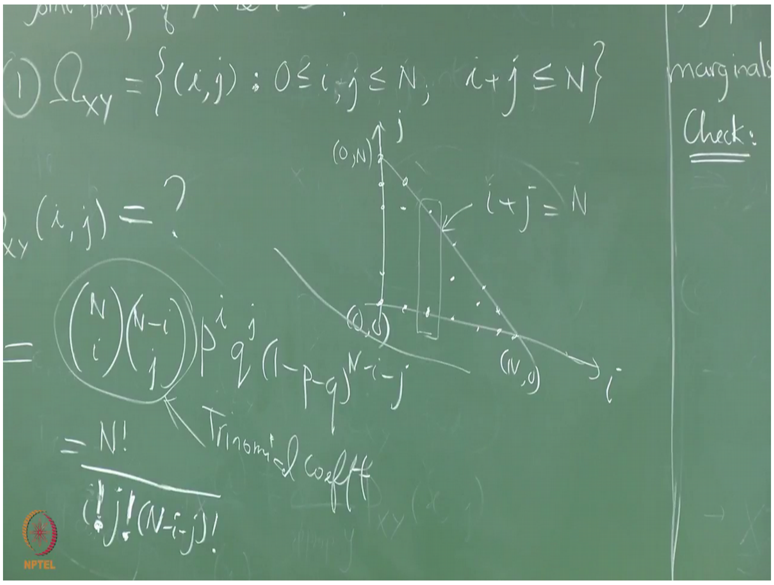

X; such as I some or whatever some any I right what would be the limits on Y it will go from 0 given that you are adding for this particular value of X you would go from 0 to N minus i. So, to get this answer; so, we check; in this case pxy of k i comma j what will be the limits of i and j; oh sorry, j j will be it going from 0 to I just now said.

Student: (Refer Time: 15:17).

You can do this and check for yourself that it will magically come out correct or not. So, magically maybe note that this is right the summation is only over j which means that the i can be sorry the j p power I can be pulled out this N choose i also. So, if I write it like this is N choose I can be pulled out. So, it is you know you can start you can start from this version with which the 2 different common combinatorics will terms pull out the N choose i leave the N choose j I choose j. So, let me just do this.

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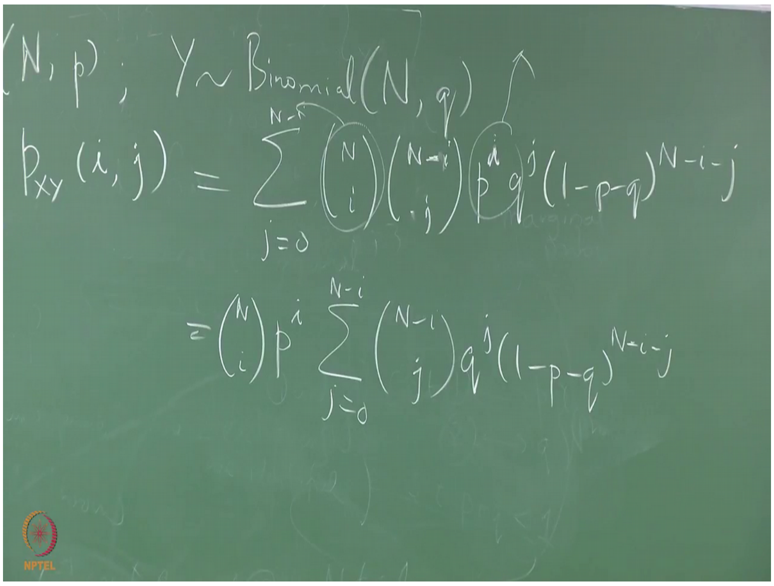

Just to I am just writing it inside. So, what do I do here this guy I can take out; this guy also I can take out. So, therefore, this is N choose i p power i, what do I have inside; I think this is easily summable; this are a very simple binomial summation; what do you get when you do this.

Student: 1 minus p to the binomial (Refer Time: 17:01).

So, it is 1 minus p minus q plus q whole power N minus i right.

So, by a simple binomial summation is it not. So, q plus 1 minus p minus q whole to the power N minus i; that is all it is for all I does not matter.

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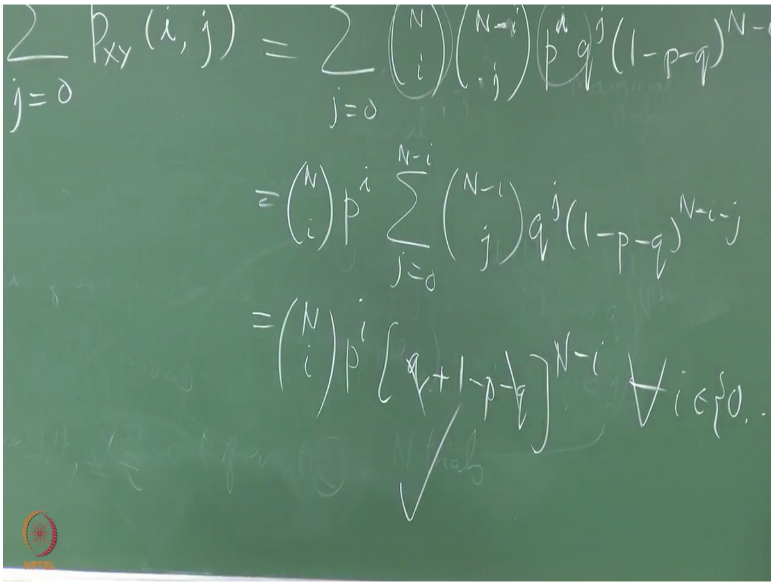

When i equals 0 also, it is true. So, then i equal to N also it is true any N sorry, any i it is the same summation holds. So, what is this binomial? So, it works where they put a tick mark here. So, particularly simple well by simple by the same time it is right it also shows how the combinatorics exactly falls in line that is what we have to keep in mind it takes combinatorics always should fall in line if you do not get this then something is wrong. So, right similarly I mean there is no the no point in doing for you can; I mean all you have to do for py of j is just switch things around and it should come automatically alright. So, this that is why you have to understand clearly the region omega xy where the non0 probabilities are concentrated right and it is not a area it is basically a set of points and discrete case.

So, if X and Y are jointly discrete they must be individually discrete they cannot suddenly become continuous as we will see continuous random variables will come to later. So, X and Y are jointly discrete implies that X they; right.

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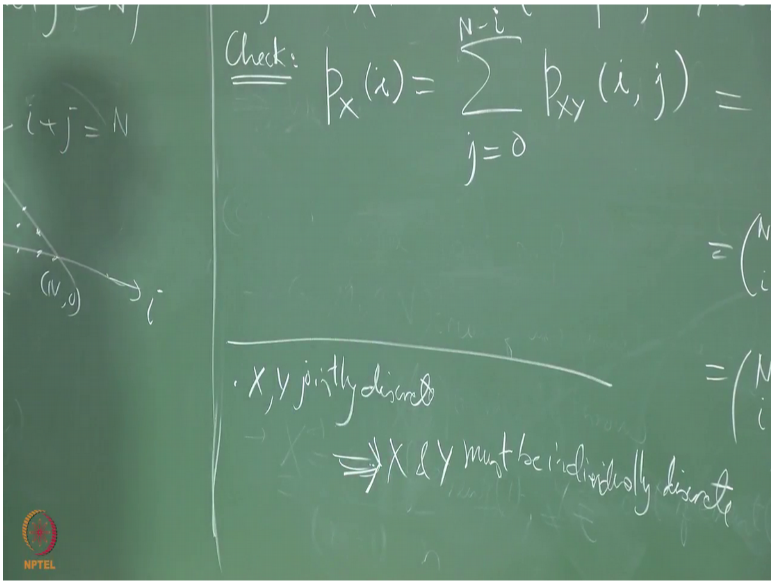

X and Y jointly discrete; that means, in other words they are described in terms of a joint pmf if they are jointly discrete they must be individually free in other words a set of values that individually they can take is also countable . So, there is all these discrete random variables therefore, fit into a nice pattern of points in a few minutes that are left today I will I will look at another application of this trinomial pmf right another situation where it happens supposing you are throwing N capital N balls into some m bins.

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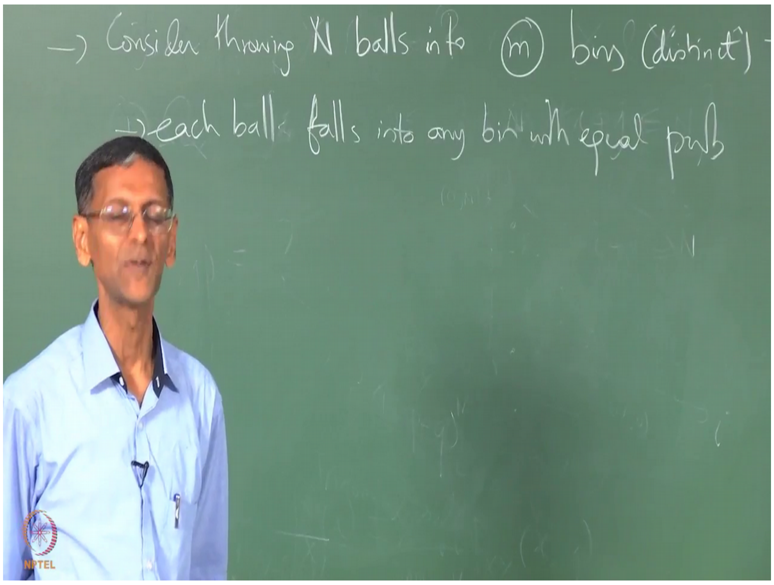

This N unfortunately or whatever I have to keep it right the same as capital n. So, right this happens when I write notes in 2 different places and one place I we used capital N small n, but I cannot do it on the on the board right. So, I have to synchronize. So, if I have to right throw I mean these m bins are distinct in other words they are numbered one to m you know exactly right which bin a ball has fallen into bin number one as supposed to be number m. So, the; what is this standard assumption you make here each ball can fall into any other bins with the same probability right.

So, any bin with equal probability right oh and if you want you can have them the balls is numbered or not it does not really matter because ultimately what are we going to do we are going to count the number of balls that are fallen to any one of these bins right just compare this with the number of getting number ones and sixes by throwing a die N times you basically created how many bins you have created 6 bins that is all here I am generalizing to m bins and I am just saying ball falling into any one of the m bins with the same probability which is one by m here and each throw is independent of any other throw rows are independent which means probably this this basically multiply out.

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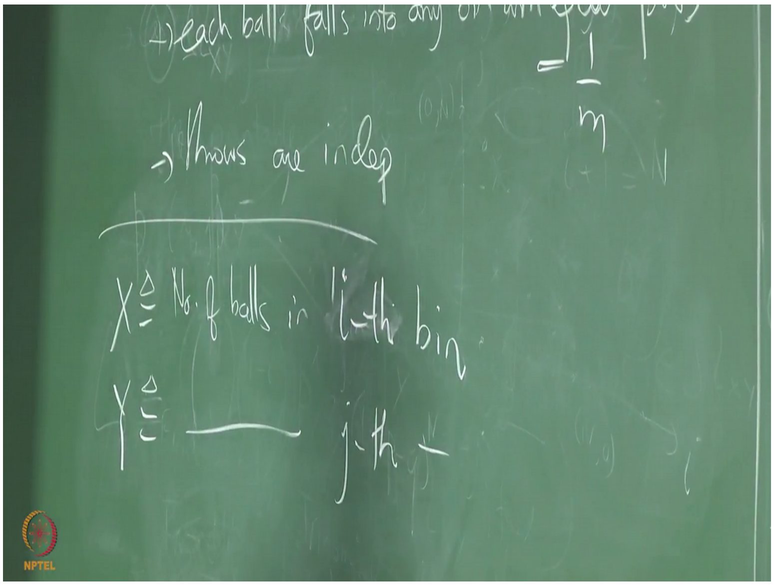

So, how do I define X; X for example, by definition is a number of balls in some bin i the ith bin or let me write it like this ith i dash th bin right and Y the jth j; obviously, is not the same as I do you do you do you not get back exactly the same situation that we had earlier except that you do not have p equal to q you have sorry you do not have p and no as q is different.

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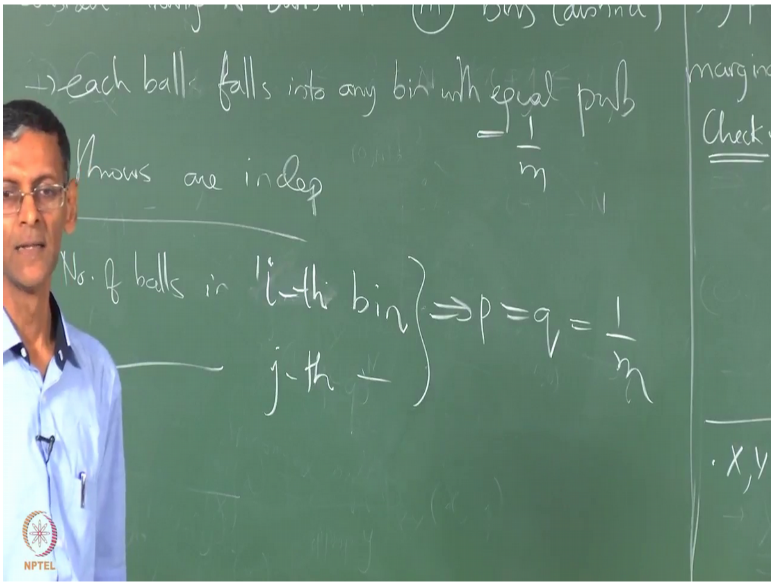

So, p and q are both equal to one by m otherwise this is identical to that not only that I can actually look at more bins also like we could do for the die rolling you can have a larger collection of random variables, but since we are restricting this discussion to 2, I am just looking at 2 bins 2 different bins right.

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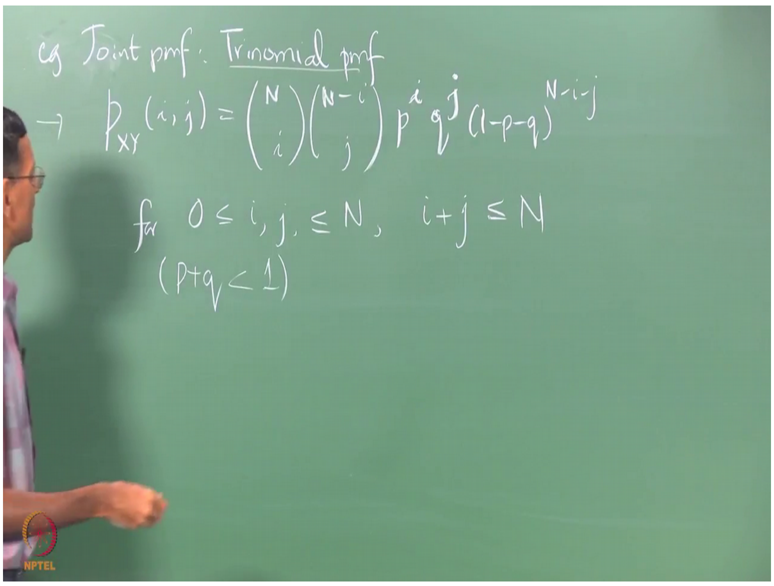

We have we have got to start now we are already some minutes late. So, then you press the record button. So, let us quickly recap this the trinomial pmf that we looked at the end of yesterday's lecture, right, let me see me we may restarted, ok. So, we did this we looked at this yesterday the p and q are the events of are the probabilities of the 2 events which are exclusive and p plus q must be also smaller than one right it turns out if you make p plus q equal to 1; it becomes trivially binomial I will leave you to think about that right I am not going to look at that special case here alright it turns out that this will reduce to that is Y will become N minus X or something; you can go and think about what happens if p plus q equals 1. So, so we are assuming p plus q is strictly smaller than 1.

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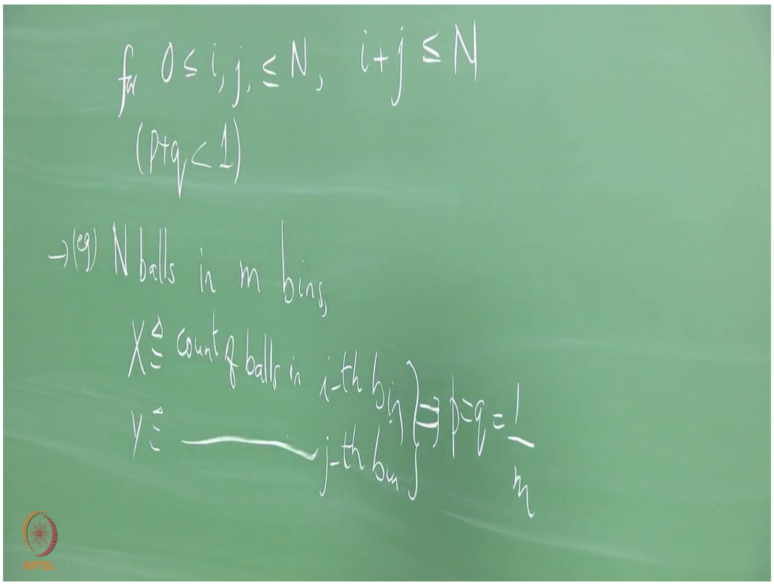

So what; where did we see this in action yesterday the eg; I will come back to this example which is occupancy right an example of occupancy which is N; N balls did I call them bins or cells.

Student: bins.

Then Y of course, the jeth. So, this implies that p will be equal to q will be equal to one by m. So, this is by itself an interesting probabilistic situation right you may have studied this in statistics somewhere does anyone recall what kind of statistics is refers to let me say some Maxwell Boltzmann some other you know this is one of those kinds of things right where any particular bin can take can take any number of balls right any of the bins can take potentially infinite number of box, but we do not have that many we have only capital N balls anyway.

So, but there is a different type of a statistic where an each bin can take maximum of one ball or something so that, all these come under those types of situations right. So, that is why I wanted to bring that and here of course, we are specifically looking at this situation of X and Y being these 2 random variables then you get exactly the trinomial pmf this is more general than this because here I am allowing p and q to be 2 different numbers, they are not they do not have to be the same if you for example, club some of the bins out here instead of saying I have been right how do I make this p and q different even here I can club the bins right I can say X can be the count of the balls in their sum let us say kth and the lth bin or something and then Y can be the ball and it can stay as it is. So, in that case p can become 2 ym and q will become will stay at 1 by m. So, it is even here it is possible to have different p and q depending on how you define X and Y anyway. So, let us keep moving on right ok.

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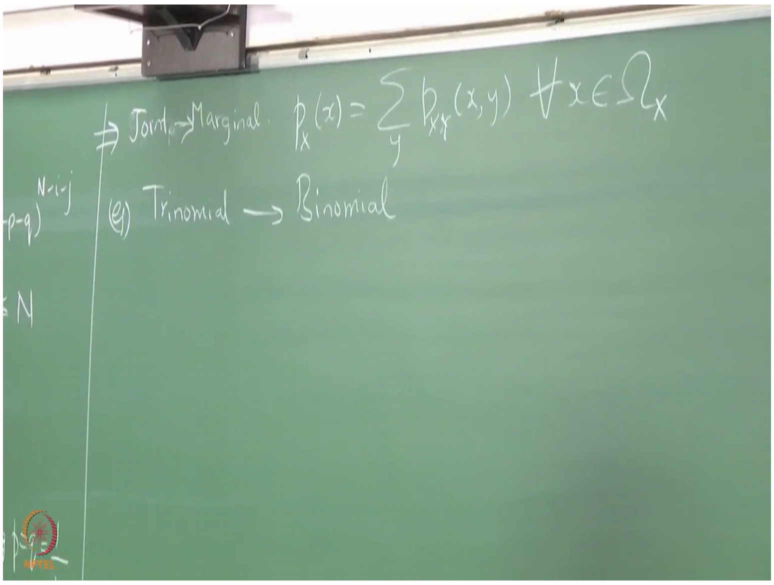

So, instead of spending more time on this example we will come back to it later and, but last thing I wanted to again recap was that the a merge I joined to marginal alright this is all obtained by this summation px of X if you do this for the trinomial case we checked it yesterday what did we get. So, trinomial gives you binomial, right. So, this calculation has to be repeated separately for all X and omega X in in that in this particular thing it was easy to do because the sum you had to do any one summation and this the I did not matter right whatever I you had the same summation helped, but in general you may have to sum differently, but does not matter it is still as doable.