Probability Foundations for Electrical Engineers
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Lecture – 29
Example: Conditioning on an Event, Indicator Random Variables

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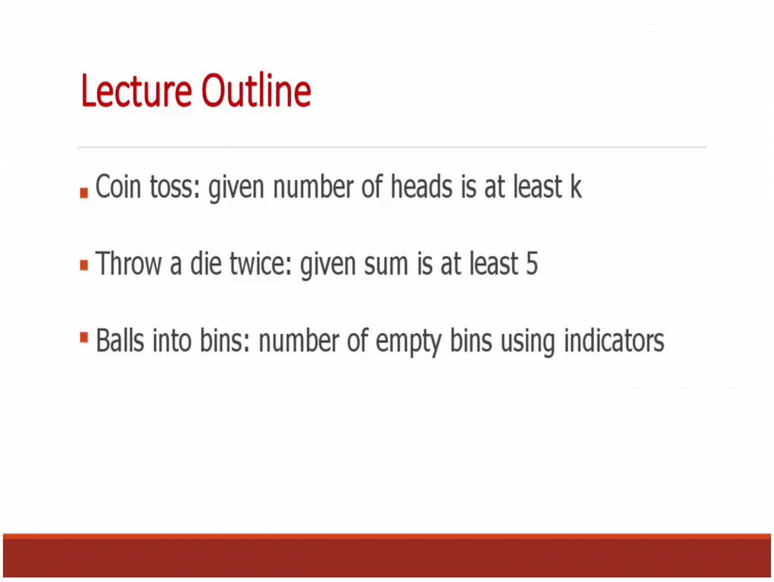

So, in this lecture we are going to see examples of conditioning by events. So, you have a random variable it has a certain distribution and then I want to condition the random variable on an event based on the random variable. And based on that I want to do some computations you saw Professor Aravind's lectures which describe all these things in nice detail and he is also given some nice examples. So, I am going to take some simple examples and illustrate how that happens.

And also another thing we will look at in this lecture is indicator random variables. So, an indicator random variables are extremely useful in many practical cases when you want to model something or when you want to write out an event carefully using random variables. So, I will use that example. So, let us begin with condition.

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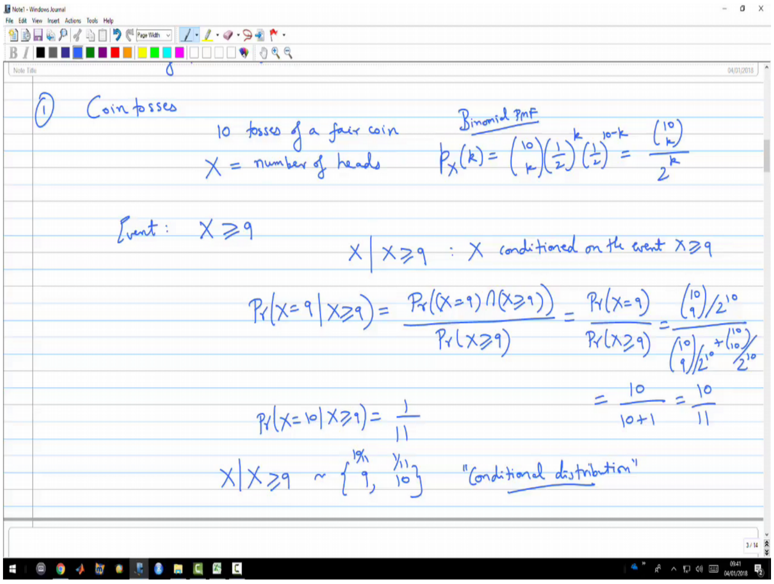

The first example I will take is take coin tosses. Let us say we have 10 tosses of a fair coin and my random variable X is number of heads. So, we have seen the distribution here probability that X k is 10 choose k half power k times half power 10 minus k. So, actually if you look at it very closely you will see that this is half power k half power 10 minus k. So, you just have half power 10. So, if you have a fair coin the binomial random variable simply boils down to 10 choose k. So, this is the PMF, this is the binomial distribution, is very familiar.

So, now one event that you might be interested in is let us say X is greater than or equal to 9. So, this is an event I am interested in. What is the meaning of X being greater than or equal to 9? The number of heads I had is either 9 or 10 somebody told me that I got that information from somewhere.

So, given that the number of heads they had is either 9 or 10 I might be interested in the PMF of X, what is the distribution of X. So, this usually people write it in this fashion X conditioned on X greater than or equal to 9, this notation for X conditioned on the event X greater than or equal to 9. So, I might be interested in computing a probability that X equals 9. So, now, once I said X is greater than or equal to 9 there are only 2 possibilities here right. So, X could be equal to 9 or X could be equal to 10. So, given that X is greater than or equal to 9 what is the probability that X is equal what is the probability that X is equal to 10. So, that is the calculation I want to do.

So, that is actually not really that hard if you think about it, it is again the use of, so the use of the conditional probability definition. So, you will have to do probability of X equal to 9 intersect X greater than or equal to 9 divided by probability that X is greater than or equal to 9. So, X equal to 9 intersect probability that X greater than or equal to 9 hopefully you agree with me simply probability that X equal to 9 divided by probability that X is greater than or equal to 9.

So, you get a very simple sort of answer here 10 choose 9 divided by 2 power 10 whole thing divided by 10 choose 9 divided by 2 power 10 plus choose 10 divided by 2 power 10. So, this 2 power 10 and all that will cancel and if you think about it and choose 9 is simply the same thing as 10 choose 1, that is just 10. So, it is 10 by 10 plus 1. So, that is just 10 by 11 ok. So, that is the probability that X equals 9 given X greater than equal to 9. And same way you can compute probability that X equals 10 given X greater than or equal to 9 and that will simply work out to 1 by 11.

So, if you want to think of the PMF, you want to think of the PMF of X given X greater than or equal to 9 this guy has a distribution it takes just two values 9 and 10 it takes 9 with probability 10 by 11, 10 with probability 1 by 11. So, this is the conditional distribution that we have derived. So, this is the kind of calculation you have to do.

Now, we can repeat it from various other similar situations and scenarios you in again and again end up in similar cases. So, let us look at the next example.

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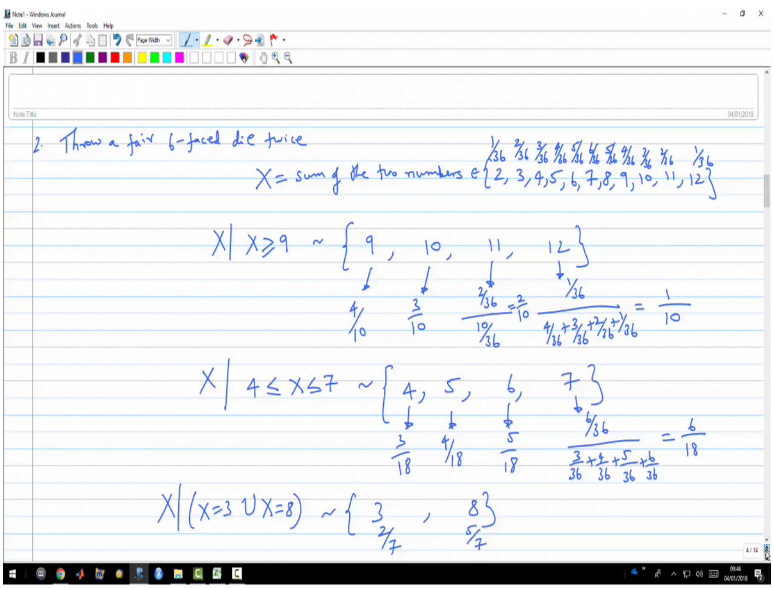

The second example I want to do is let us say we throw a die fair 6 face die, twice and the random variable I am interested in is in the sum of the 2 numbers. So, you might remember this distribution.

So, this X in the sum of 2 numbers takes values from 2 3 4 5 6 7 8 9 10 11 12, probability that it takes value 2 is 1 by 36, probability it takes value 12 is also 1 by 36, probability it takes value 3 you could have favorable cases are 1 2, 2 1 that is 2 by 36. For 4 it is 3 by 36 so on it goes. So, you have 4 by 36, 5 by 36 and for 7 you will have 6 by 36, so you will have 6 by 36 and then it will start decreasing allow again 5 by 36 for 8 you have to start with 2 right, so 2 plus 6, 3 plus 5, 4 plus 4, 5 plus 3, 6 plus 2. So, you only have 5 possibilities there. Again 4 by 36, 3 by 36, 2 by 36 and 1 by 36, so that is the PMF for this distribution yes we have done this earlier in an example.

Now, if you start conditioning supposing I want to condition X given X is greater than or equal to let us say 9. So, this I am going to do this a little bit quicker maybe I will show you 1 or 2 calculations. This would take 4 values 9 10 11 12 and what is the probability takes value 12? Conditioned on greater than or equal to 9 that is going to be 1 by 36 divided by 4 by 36 plus 3 by 36 plus 2 by 36 plus 1, yes that is just 1 by 10. So, what about this guy if you do the same calculation you will get 2 by 36 divided by the same denominator as here right. So, you will have here this is just 10 by 36 same denominators here. So, that ends up being 2 by 10. So, what about 10 here? You have 3 by 10, here you will have 4 by 10. So, that is the conditional distribution for X. So, these are relatively simple calculations to do. I just wanted to show you a few simple examples before you can look at more generic examples.

So, this is all you can do. So, you can also do other conditioning. So, for instance I might want to condition X given, X is between 4 and if you look at this event then X can take values 4 5 6 7 and what is going to be the probability here. So, I will do 7 in good detail 6 by 36 divided by 4 to 7. So, you have 3 by 36 plus 4 by 36 plus 5 by 36 plus 6 by 36. So, that is just 6 by 18 like. So, likewise you will have 6 being 5 by 18, 5 being 4 by 18, 4 in 3 by 18 it. So, likewise you can do any other event I mean you do not have to have and you have even a continuous range you can even have something like let us say X conditioned on X equals 3 union X equals 8 it is. So, X equals 3 or 8 that is what it means. So, X given that 2 will only have 2 values 3 and 8. What is 3 and 8? 2 by 36 5 by 36, so this is going to be 5 by 7, this is going to be 2 by 7. So, this is a kind of probability that will work what if you condition around a minute given that random variable itself is involved in this.

So, the next example I want to give is to show indicator random variables.

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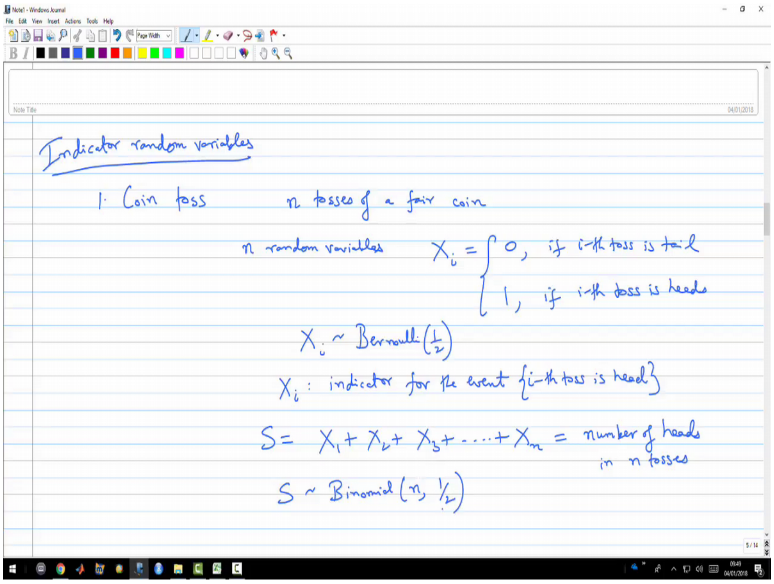

So, these like I mentioned they are quite crucial and you can use it in so many different ways. So, for instance let me show you one very little very important example which is maybe Professor Aravind also alluded to it that is very important to know that. So, let us say we have n tosses of a fair coin. So, what one can do is you can define n random variables as follows X i equals 0 if ith toss is tail and X i is 1 if ith toss is heads. So, this is a simple random variable.

So, I have n random variables here based on the n tosses of a fair coin X i is a 0 or 1 X i is clear a Bernoulli and the probability is half. Now, X i is also an indicator random variable right. So, X i indicator for the event ith toss is head. So, this is probably the simplest example that one can think of. And what is interesting about indicator random variables is you can combine them in different ways to create other very many interesting random variables. So, for instance I might want to refer define a random variable S it is just X 1 plus X 2 plus X 3 so on till X n. Looks initially a bit complicated. I am taking n indicator random variables and adding them together. But if you actually think about it you do n coin tosses define all these evens X 1 X 2 X 3 and add them up together what will you actually get? This will actually be the number of heads in n tosses right. That is quite easy to see, it is the number of heads and n tosses and clearly we can see that this S based on our prior knowledge will be binomial with parameters n and half. So, it is a binomial PMF binomial distribution.

So, this is a very basic way in which you can use indicator random variables and we will see later on this kind of defining random variables and adding them up together, combining them in interesting ways, creating new functions of random variables is a very powerful way of generating other random variables and powerful way in modeling because that is what happens in physical models and reality right. So, you have a few random variables the rucker and then they start interacting adding up, subtracting coming part of the function. So, you really need to understand how this works.

So, in this simple case was quite important quite easy to see, but you can have more complicated cases. So, for instance you can have a case where so, you have seen from our previous example that this balls on bins ends up being a very very notorious example for creating dependent random variables, dependent events and all that. So, we will see more on this later on, but let me just show you one simple case.

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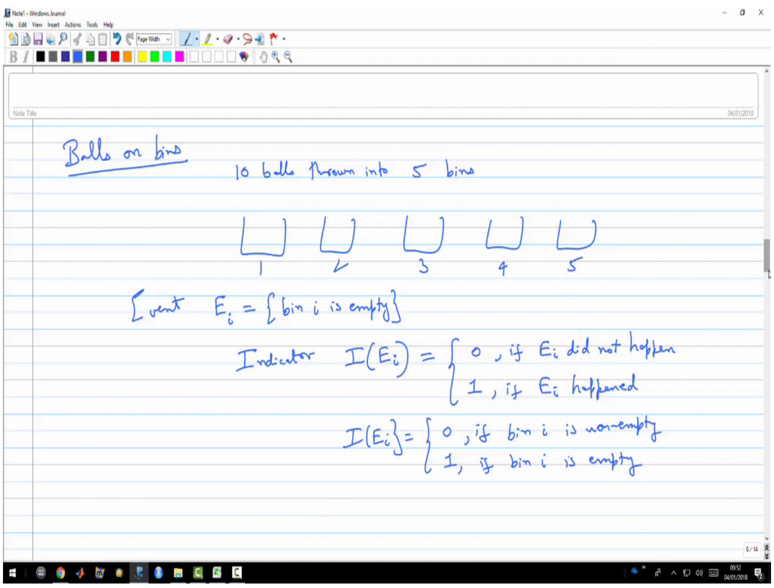

So, let us say we have 10 balls thrown into 5 bins ok. So, I have bin 1, bin 2, bin 3, bin 4, bin 5. So, after you throw you can define a lot of random variables here. I am going to define some indicator random variables. What is the indicator random variable? I might be interested in the event E i which is bin i is empty. Based on this I can define an indicator usually this is also a very good way of having an indicator that is a good notation you can put I of the event. This is going to be 0 if E i did not happen and 1 if E i happened. So, this is the indicator random variable for the event E i.

So, I of E i is going to be 0 if bin 1 is not empty and I of E 1 and I of E 1 is going to be 0 if bin 1 is not empty and I of E 1 is going to be 1 if p 1 is empty. So, that is the way to think about it. So, equivalently one can also write I of E i equals 0 if bin 1 bin i is non empty just 1 bin i is empty. So, you might want to look at for instance quite often we are interested in the number of empty bins. That ends up being given by I of E 1 plus I of E 2 plus I of E 3 plus I of E 4 plus I of E 5. So, this is the random variable which you can compose using indicator random variable. So, very short way of writing it if you do not like it I of E i's.

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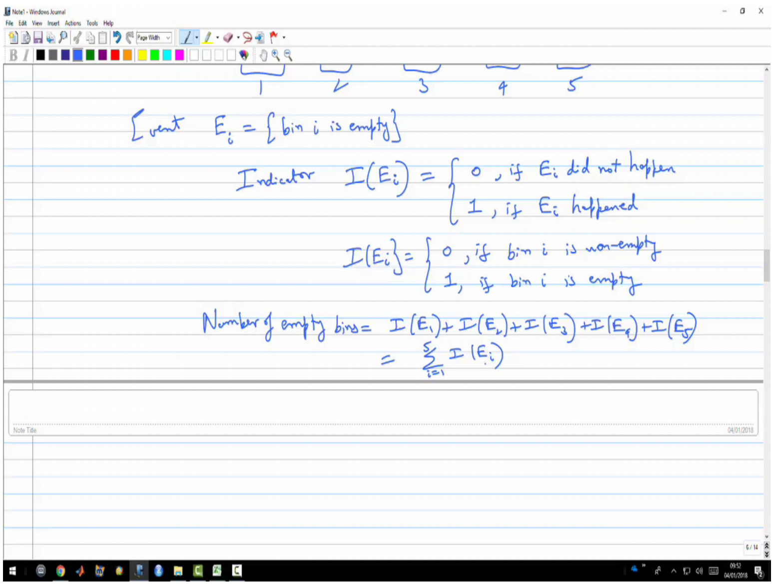

So, we will see later on this way of expressing random variables as sums of indicator random variables can be quite useful and you do some simplification. Maybe later on we will see a nice example. Maybe even exactly this example itself of how to do some calculations with random variable (Refer Time: 16:25). But be very careful here these events are not independent E 1 and E 2 and E 3 are not independent. So, these random variables are generated from dependent events. So, that creates some confusion in computations would be very careful, but this is something that we can do.

So, stop here for the lecture, pick up from here.