Probability Foundations for Electrical Engineers
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Lecture - 09
Part 1

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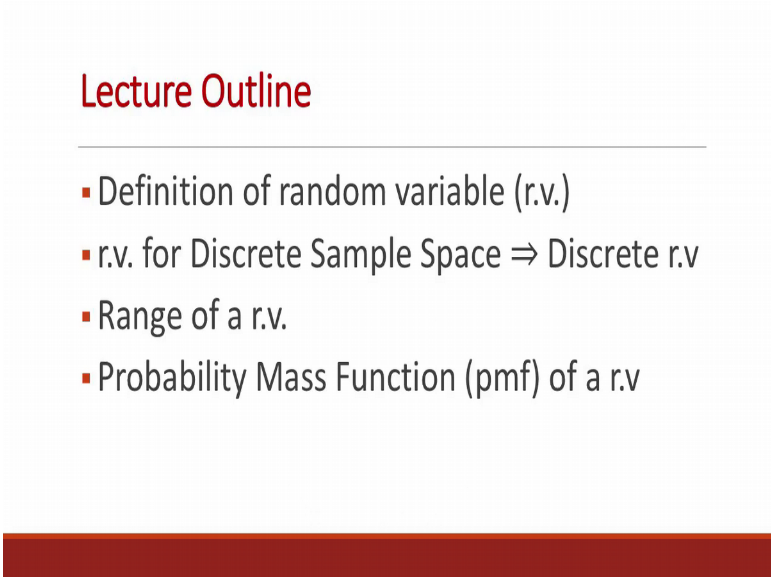

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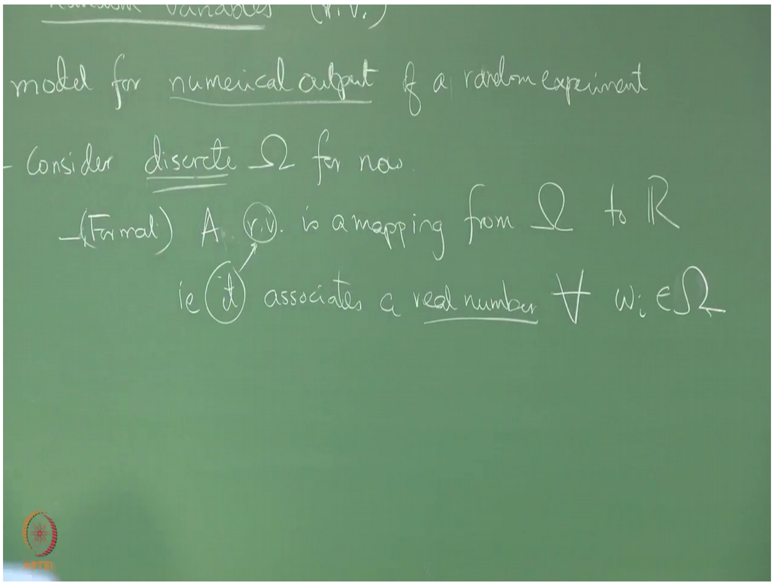

Start on: that so we are going to abbreviate random variables with by R v in lowercase small letters. So, this is a very important concept in through in probability, why did allows events to be viewed as a something it is speed out numbers right and how to manipulate these numbers so on.

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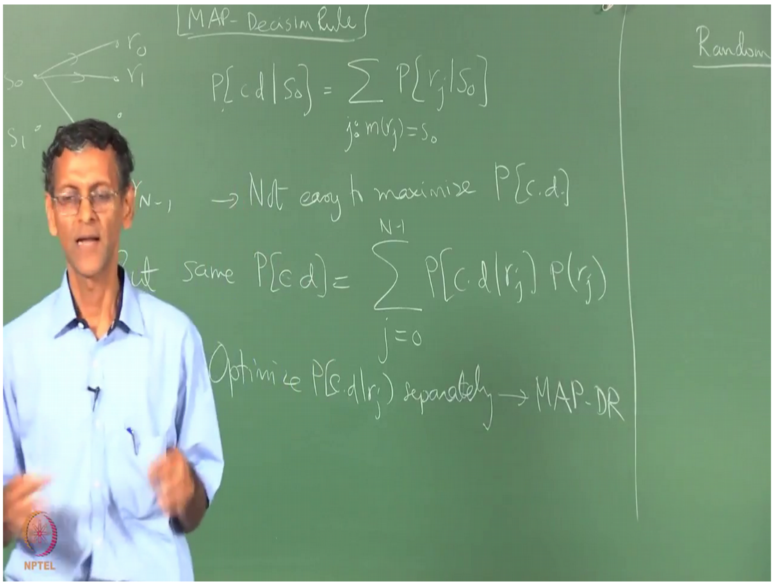

It is so important that right the whole theory you might say is built around this concept of random variables and how do you generalize from that right, how to build bigger things right. So, we start with the most fundamental thing first right and continue from there.

So, let me begin this by asking you people what some of you seen this before, what is your def what do you say what is your idea? I am not even going to call it a definition what do you what do you understand by the concept random variable.

Student: Mapping of sample space onto the real line.

The mapping of a sample space onto real line fine that is 1 very mathematical or somewhat mathematical definition, but what does it physically mean.

Student: Who come up a random experiment?

Instead of saying outcome well say output right, because outcome can be any of the omega I points which are more general than just number right. So, out come with that that word or that term we reserved right for what we see in the random experiment which can be more general right which need not be a number, but the numerical output which is somehow right measured and they can be more than 1 numerical output also.

So, 1 numerical output right that we can measure or we get somehow from a random experiment is called a random variable. So, that is the English right what is this is these are models for numerical output. So having said this now we will go to the formal definition which some of you have obviously, seen having taken earlier subjects in this earlier courses in this topic. So, we will for now of course, we going to restrict our omega to be discrete, all our considerations until I make a departure from this is only going to focus on discrete omega. But the idea of course, is more than just is where is applicable to more complicating, I mean the continuous are in uncountable case also.

So, but let me just state the definition in terms of the discrete omega for now. So, a random formal definition and I A random variable is a mapping what does this mean? So, that is a random variable is a function right, it map I it refers 2 random variable it associates a number a real number for all omega i in r.

Now I can say omega I because I have right I have said sorry all omega I element of a capital omega. So, I can use omega subscript I because I have said them space has to be the discrete right, I cannot use remember and I to index a sample point if the number if the space is not discrete right if it is continuous or uncountable. So, it is a mapping so it has to be applied to every omega i.

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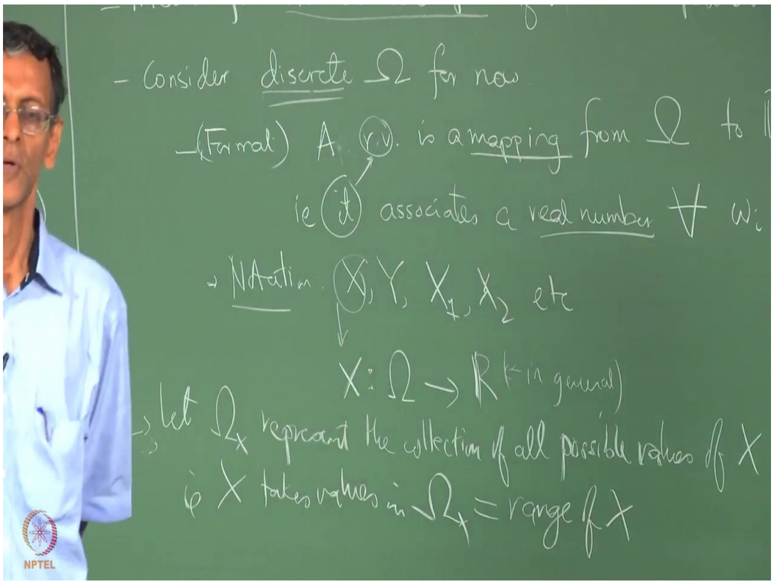

So, mathematically what we have notation for random variables, let me say that also upfront we use capital uppercase letters right and of course we will need subscripts right like 1 or 2 if you have many of them. Remember you can define more than 1 random variable on an experiment it is it should be required obvious. It could be I mean simple examples where the you know omega is more complicated than just some numbers would be picking up individuals at random from a fixed population, where you have let us say each individual corresponding to a 1 of the omega is.

How many numbers can you associate with an individual as many as you feel like right? You doing as many as many as they there that makes sense or are useful right, you can measure height you can measure heights are less controversial than weights can turn out to be somewhat more controversial because of other problems, but many way then you have blood pressure, but blood pressure at a given time which varies right. So, we are saying that it is just a number right, it is not going to be a function as of now right or anything more complicated than just then.

So, at the beginning we just are looking at numerical outcomes or sorry or numerical output. So, they are out the random experiment itself throws at you and outcome right and you map that outcome to an to number to a numerical output in more than 1 way right. So, 1 consistent way of mapping is 1 random variable, like the set of all the heights right or all the values that the height random variable takes in an experiment right thing like that.

So, I do not think I mean it is a fairly intuitive concept and I hope that I do not have to spend too much time verbalizing it right, this is ok so far we are just mapping so this notation we will use. So, supposing I take X is a mapping from omega to R in a more general way, but we can we can tighten this range R is a set of real numbers.

So, this is the R in general we will right R because we are not going to right as of now, we do not know what else to say you get, but you might say why should it be R why cannot it be a smaller? Set well there is some merit in that we will see. Especially for a discrete space this turns out that this R is a little big, you do not definitely cover all of R in any for any discrete omega for by any random variable, but this notation this mapping notation is very important to take that do not the argument of the random variable is always the omega for that experiment, the sample space of the experiment.

So, you should look at the output of the random variable is a number. So, the random variable is just a name that says it is you know it refers to any of any of possible outputs, some said that has to be defined carefully based on the all the mappings that you have. For example, if I say if I toss 2 dice and say that right then the output I am looking for is the sum of the numbers in the 2 dice or obviously do not write expect to get some crazy real number like pi or e from that experiment. So, so we have actually the collection of outputs especially for a discrete space is highly limited by circumstance right. You do not have all of R or anything like that in fact, you can obviously see that if omega is discrete the collection of possible outputs is also countable it cannot be uncountable.

So, what is the notation I am going to use, I need 1 more notation to keep things straight. Instead of R I am going to say well let omega X represent right the collection of for all possible outcomes or outputs, that I am going to write as all possible values of X the mapping X takes values in this set omega X.

So, we say that X takes values in omega X, that is going to be that terminology we will use throughout I any random variable will take values in it is range space. So, this omega X is the range space of the random variable X that is the range of x. So, instead of saying that the range is R you can think of X being a mapping from omega to omega X. And since I want means something like omega on the right side here because it is a universal set as far as X is concerned, X cannot take values outside of omega X. Whereas, this omega can be some abstract space this omega X is always a subset of R is that clear this omega X is not only that for any discrete space this omega X, must be a countable subset of R cannot be suddenly become uncountable.

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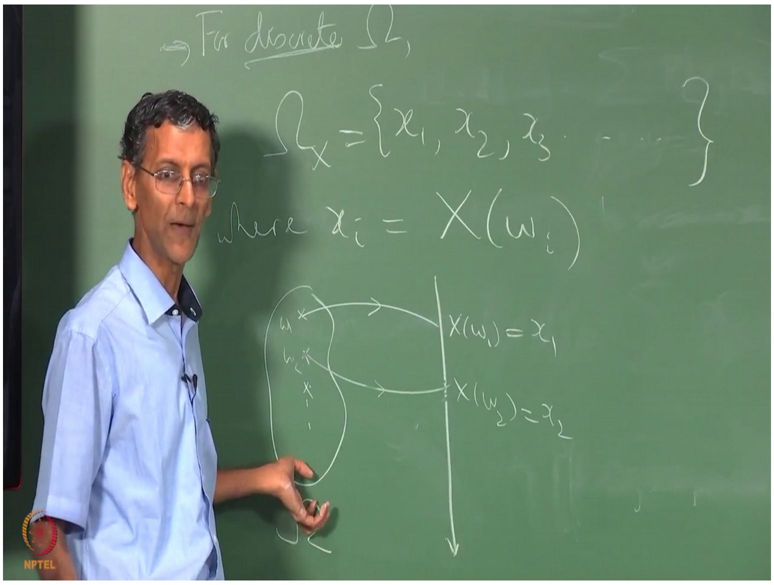

So, omega X will be of this form right where X i can be capital X of 0 just to cut a long story short I can write my omega X like this right and of course this is this is valid right this kind of statement is only valid for discrete omega not continuous or not uncountable omega.

So, that I do not draw any more pictures I can just write it like this, but if you want to visualize this picture you can draw this capital omega on 1 side and you draw the real axis here and do some mapping, but maybe I will do it since I am being recorded. So, whatever so people draw the right real axis vertically rather than horizontally just to make it easy to show the mapping. So, you have omega 1 omega 2 whatever note of course, that this you can have a countable infinite collection of points here, there is nothing to stop you from doing that.

So, you have this map to X of omega 1 is some x1 and there is again right, no only rule is that each of these points must be mapped right to a unique point, I mean sorry must be mapped to some point out here. So, you cannot map this to 2 points here for given 1 random variable there is no way that by mathematically you are prohibited from saying X omega to it is both X 2 and X 3 or something a like that makes no sense. But you can have multiple points out here omega I mapped to the same X that is perfectly. In fact, that that is what happens when you say that for example, that the X is for a pair of dice is the sum of the values. So, the 2 dice right you had 4 3 or 3 4 in both cases you going to map it to 7.

So, you may have thirty 6 values here sorry 36 points here the cardinality of omega this capital omega is thirty 6 in that case that out rolling a pair of dice, but this is much smaller you can go count the number of points that you can get right, you can get only 2 to 12 is not it integers from 2 to 12. So, anyway and these of these points do not have to be integers they can be some real numbers. So, no reason insist that they must fact that is not at all, I can divide by square root of 12 or something if I wanted to do something, I can say sum divided by square root of twelve or something or it is a or pi times the sum of the numbers right. So, I can get real I mean I well what can I say god given real numbers on this axis there is no reason to restrict it any further.

So, I have omega X is being this collection of points right and whenever the outcome omega I occurs the random variable X splits out the value X of omega X omega i, which is written is small X i write and I index is only whether it index is you know basically either something here or something her but we have to be careful that X i note that some X i can be some equal to some X j.

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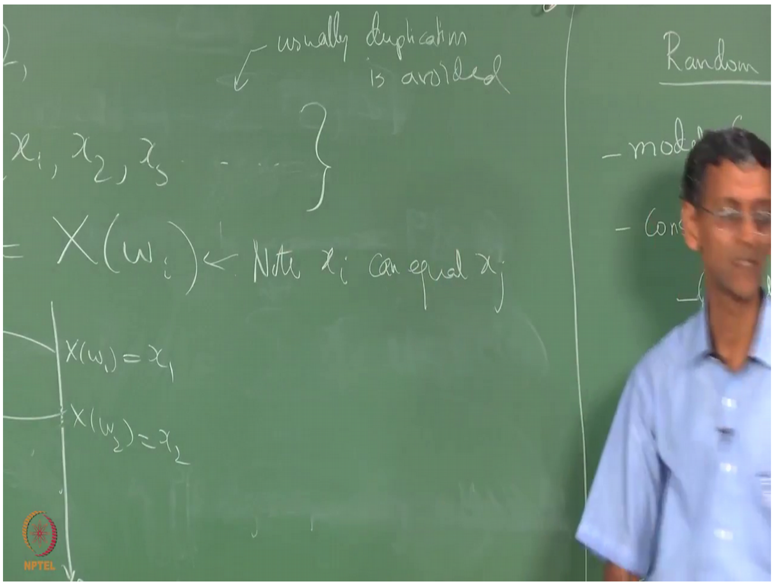

If you write it like this it is possible you know if you write it like this it sort of implies a 1 to 1 mapping which need not be the case. So, if I write it like this omega X, but here we will avoid duplication. So, there is a certain difference between these 2, here there is no point in repeating the X i twice you do not realize it once is it.

So, there is a small difference between these 2, but it is I mean we cannot be in just writing it in some simple fashion like this, I cannot be too mathematically and say when I want to write make sure that I can write I want to avoid all of this here and so on right. As long as you get the point right we will go ahead with this, we are now going to keep using this notation too much. In fact, I am probably not coming back to this anymore.

We are just going to. In fact, suppress this argument when we use the X the random variable that is the way it is always being written, you say the height of somebody or the number you know the you say this sum of the pair of dice without you it. So, when you write it you just write it as X, when you refer to that random variable you do not put capital X of omega anything right. So, it is like saying f is a function instead of saying f of X is a function.

So, this notation omega X is something which you again we will not we do not have to keep on reminding ourselves about what did all the time, as long as we know what it is some in the beginning you can write, you do not have do it does not you do not have to keep coming back to it right, that whether this X y 4 is equal to X 1 or not right its. So, here usually you do not know usually we do not know why right now duplication is not done just for say you I am saying this right nothing else.

So, let me let us look at the most important concept or the most important property induced by this mapping this X. So, how what is called the probability mass function right?

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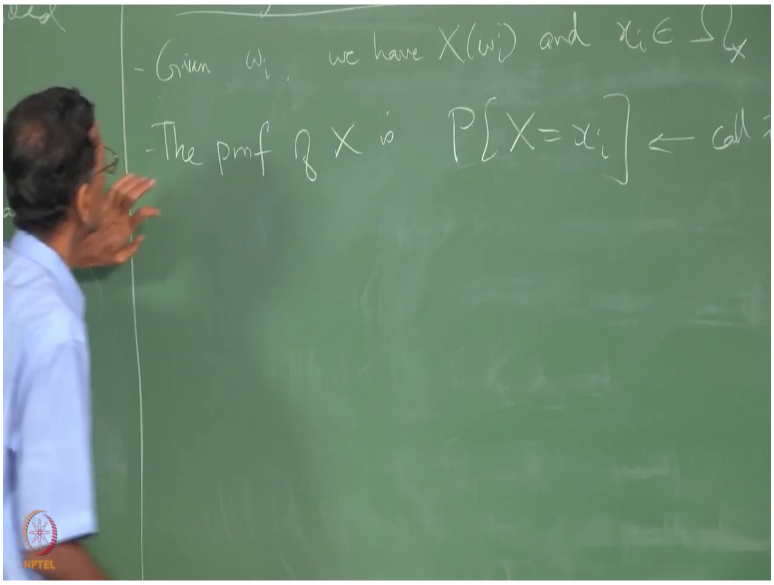

What do you understand by? This is now going to say how we can you know manipulate just only a random variable X without worrying about this underlying space from on which you defined X.

So, this is going to be given omega I we have X and X i element of omega x. So, I cannot talk with a probability that this X equals 1 of these points in the set right. So, that is a problem, when I do it systematically few systematically for all the points in omega X I get what is called the probability mass function for X. It is nothing more or less than just this is P of X equal to; let us X I that is what is a probability that X takes the value X I.

So, you can call this as sum pi right and note that I am not going to put right any regress or whatever right limits on what I should be right or anything I am just going to say I can be any positive integer typically. So, that this probability why does it make sense because, so given some if you have omega I right and it has a probability obviously this event has a probability.

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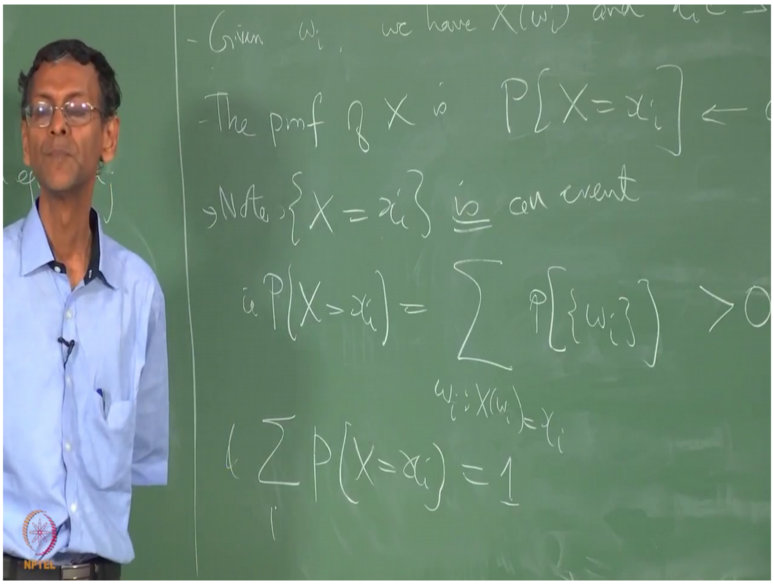

So X equal to X i is an event, note that it is an event why because our event space for the discrete omega is a power set. So, any such thing you say on omega or any combination of omega is any union of this omega I s this is an event.

So, this can this has to be in 1 of those events right, I can have at the maximum some numbers here mapping to that is more the number x i. So, what does this mean this X equal to Xi means I got either omega 1 or omega 10 assuming that omega ten also mapped to that at X I or omega whatever right, if I had some 2 or 3 points mapping or more than that all those anyone with those points will give me this event and so what is the probability of this. So, I be the probability of X equal to X I will be will be a sum of what P of this x. Remember we start out for a discrete space by assigning probabilities to all the sample points omega i.

So, you going to again it is very similar to some earlier notation we wrote, we are going to add it over all those points such that omega I such that on X of omega I equal to X i. So, everything here all the X is now a numbers right, earlier we had more abstract events we wrote some such thing like this right, except here now instead of saying calling this some event a we are explicitly writing the event like this X equal to X i, otherwise is no difference between this and any other event that we saw earlier.

So, here we are looking at those points omega I such that X of omega I is X I, so In fact, this is you can say this you know you always you know go back to this right, that for every omega I there is always some X i 1 unique X I for every omega I right and of course, keep in mind that you can have this all this kind of thing also that is all. Once you are ready with this then this does not any right this is only written for convenience in some sense, you do not really need to worry about oh my god, what is my omega X for this experiment, as long as you have done this mapping correctly. This will also almost follow suit by itself you do not have to worry so much about it, is just nice to not have to worry about all of i.

This is the only reason I am doing this notation omega X I to tell you that you do not have to for a discrete omega, I there is no reason to worry about all of our there is no way you can get that. So, this is I mean even conceptually so this is it is nice to know that you write you never going to go outside of some countable set omega X that is all, otherwise right you are only interested in this is this or you have a question? Here what did we when we started the whole thing we said only once outcome can occur at any time in any trial of an experiment. So obviously, the omega is have to be disjoint. This is there if you go back to the first or second lecture I said they are mutually exclusive collect collectively exhaustive, that is how you think of capital omega and omega small omega is right. So obviously, I mean omega is have to be the disjoint, if you get 3 4 on a roll of die you do not get 4 3 you only get 3 4; remember right the ordering is important.

So, in other words you know especially in the discrete case it turns out that all these P Xi is are typically positive numbers between 0 and 1.

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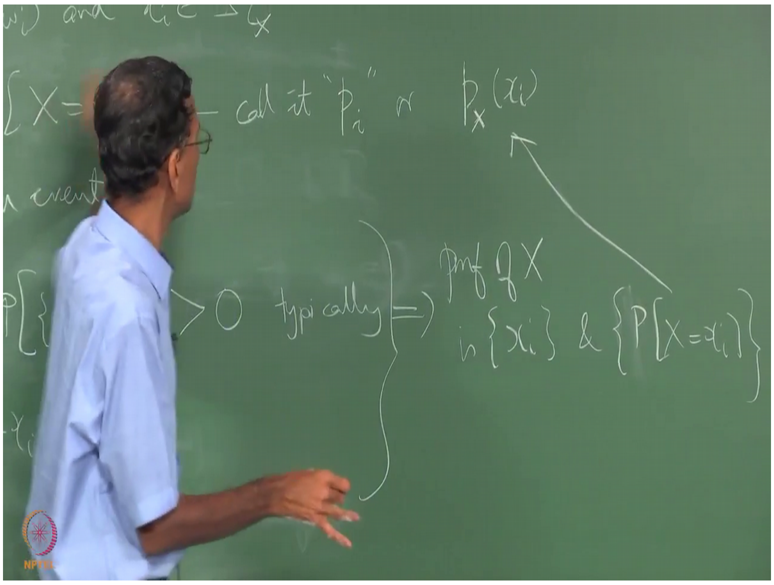

So, this is usually greater than 0 typically. So, that is the significance of all these numbers here, that they all have under the mapping X, all these numbers have positive probabilities right and not only that the sum of all those probabilities has to be 1.

So, how did I p of X equal to X I overall i. So, here again I said that the definition of X are of here, that is why I am not saying I going from 1 to 5 are anything. I am simply putting this means adding up all I this must be equal to 1, you go if you add all these probabilities here you are adding only it right you may not be adding anything here. If there is a 1 to 1 mapping between omega I and X I said I this there may not be any addition at all, but here you are adding right. So, what is PMF? Then a PMF is basically the right it is not just the p of X equal to X is also the X I, so the X i the collection of points X I and the probabilities are collect are together called the p m f of X.

Now, this is a bit cumbersome right. So, we will write it we have said we call it p I sometimes or better more common way of writing it is to is to say you write it like this. put a p here to distinguish it from this P right because we like to use small a lowercase letters also occasionally right, but uppercase letter is always used for the random variable right and these random variable will come as a subscript because, you want p X if X and y are 2 different random variables you find out the same experiment right, there PMF are going to be different.

So, you have to have some notational difference between them and we get that notational difference by subscript right. You put p y for the PMF of y and p X will be a prop PMF of X right and so this probability you write it well sometimes you may write it as pi for a short, but if you want to be more explicit about the role of X you must write it like this.

So, in otherwise you can think of a PMF for splitting again this plot you take of volume unity, but instead of saying I want to divide it among some right abstract points omega. I now I am going to pick some bunch about collect a countable set of numbers in the real line and give it only to those real numbers that is all I am doing. So, I need to keep in mind both the values of the random variable can take and the probabilities in each of those points, so that is how I generate PMF.