Probability Foundations for Electrical Engineers
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Lecture - 14
Independence of Events

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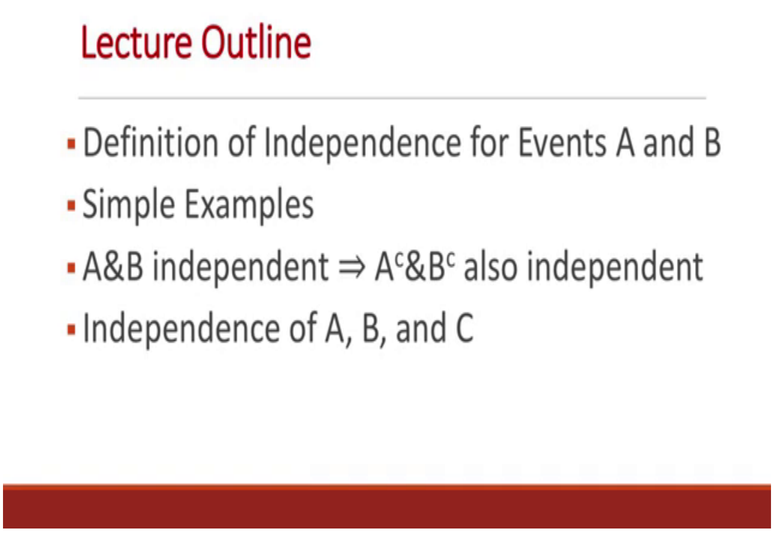

So, today we are going to move to this topic, which I have written on the board right. Independence is a very important again a very important concept which sort of flows from what we have already talked about its just basically conditional probability. In other in other words supposing you have 2 events A and B right.

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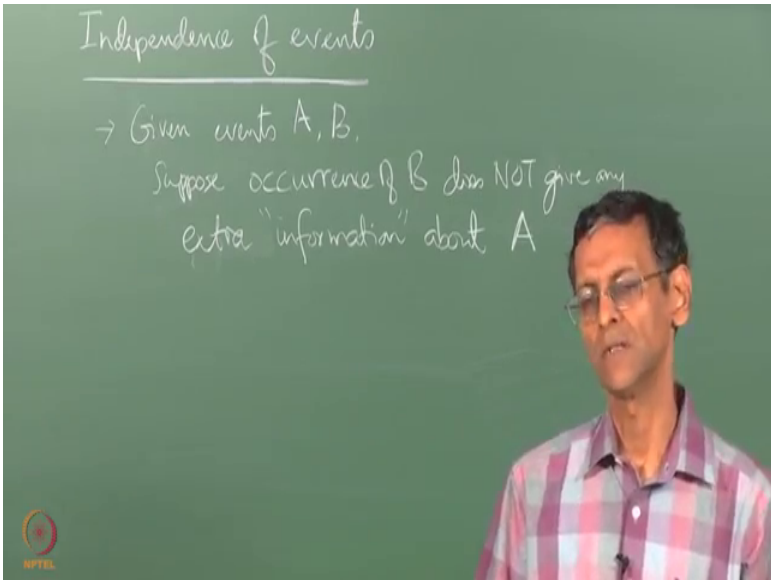

And the occurrence of one of these does not say anything more about the other right. Supposing you have A and B and which are let us say disjoint right then you know that it B has occur a cannot occur. So, that is P of A given B or P a B given a will be 0 even though P of A and B by themselves will not be it will be positive.

So, that is saying a hell of a lot right that or if A and B are highly related you know P of A given B can be very close to unity and things of that kind right, but supposing you have this situation I e.

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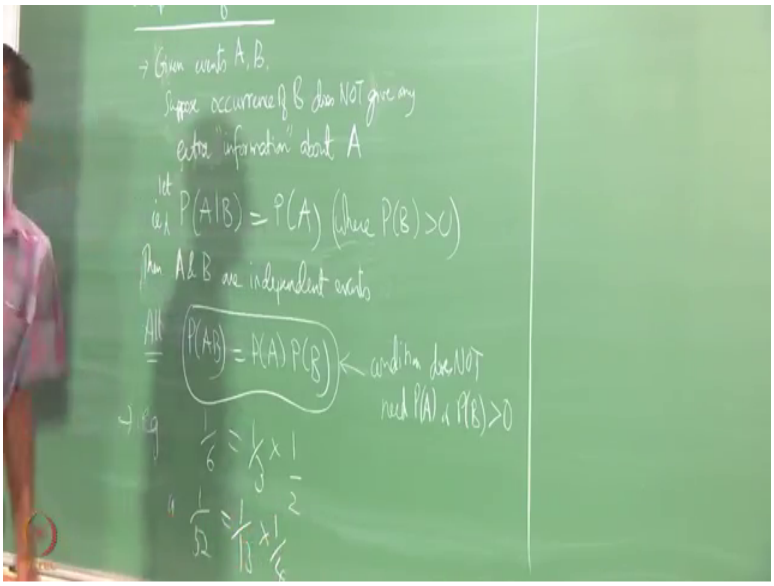

Let us say that this case I am saying the occurrence of B right. So, that is P of A given B. Supposing, this is exactly equal to P of A supposing you have let P of A given B equal to P of A right which is where of course, P of B is greater than 0 right. So, if this situation prevails right then you say that A and B are independent right, but instead of writing it in this form it is written in a slightly different form right.

So, the alternate then this is how basically independence relates to conditional probability, but the way in which it is written is slightly different as many if you already encountered right. So, the alternate formulation is to say that P of A B, this is a condition for independence. So, they are right in other in other words the joint probability is the product of the individual probabilities right and this condition works this condition is right it does not require P of A or P of B to be more than 0.

So, you condition does not right does not need P of A or P of B to be strictly positive. So, it works even if P of A or P of B is 0 why you cannot write this right. This is as long as this conditioning event has more as nonzero probability here right, but supposing P of you know you can all if this is an easier way to remember the same thing right. So, because and these 2 are equivalent when P of B is nonzero it is obvious because of their definition of P of A given B P of A, P by P of B right.

So, you do not have to remember this it is enough to remember this right. So, independence is right means exactly of A and B means exactly this. So, you can claim that right if A or B either of them is the impossible event phi right then clearly this equation will be satisfied you can say phi is independent of any event.

Any other event which is true or if A or B is omega then also this equation is going to be satisfied. But those are extreme cases in general right you have the situation where P of A is between 0 and one P of B, A B also bit is between one P of A B again between 0 and one such that P of A B becomes the product then right you say that the events are independent right.

Now, there is a common misconception that independence right is somehow the same as mutually exclusive or disjoint. For some reason this is this myth or this false conception is prevalent on in in many you know beginners. So, we have to dispel that right up front right that 2 are completely in general different I mean they are not they are those 2 concepts do not overlap at all right.

If A and B are mutually exclusive in general they cannot be independent right unless you take some well let us not even look at the exceptions right. So, remember that right independence I mutually exclusive are totally different concepts right. So, where does this happen in practice or what are some very simple examples. If this question is asked just to a bunch of students immediately the answer comes well if I toss and I twice they first throw is independent of the second throw or the events of that sort, but you do not need a repeat you know right repeated trial like that or a compound experiment right why you are actually combining essentially 2 different experiments you do not need that. You have another example where let us say I am just tossing a dye once right, and I still I can still define 2 independent events have you seen an example of that.

No supposing I give you the identity 1 by 6 equal to 1 by 3 into 1 by 2 right can you construct from this identity? We all know that 1 by 6 is exactly equal to 1 by 3 multiplied by 1 by 2 right nobody is going to complain that I time I am trying to pull a fast one new on that one here right. Now I am I am asking you right define A and B such that.

Student: (Refer Time: 06:33).

That left side is A B P of A B etcetera etcetera.

Student: (Refer Time: 06:38).

What will be an answer come on.

Student: (Refer Time: 06:43) multiple of 2 (Refer Time: 06:44).

I want the intersection to be one right. So, multiple of 2 and multiple of 3 there is only that is only one intersection is only one and that would fit the bill right.

So, this A could be the event that you got a multiple of 2 which is 2 4 6 or B could be I you know as per your example B could be a multiple of 3 which is 3 6 and clearly there is only one point in the intersection which is therefore, P of A B right. So, you do not require right a compound experiment to get independence it happens quite often right or for example, you have 1 by 52 equals 1 by 13 into 1 by 4. This is your right and even more strike you know obvious case right and I do not have to construct the full example I think you can do it by yourself right. So, the 1 by 52 would be a specific card in a 52 card deck and 1 by 13 would be that the suit that it represents and the 1 by 4 would be the face value of the card right whatever it is a 2 to 2 yes, they right.

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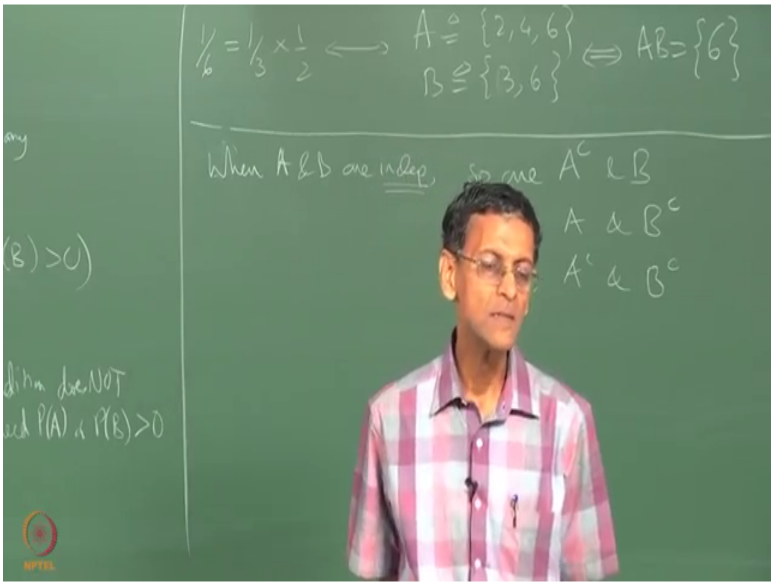

So, for the sake of the camera however I may have to write a few more things. So, this a is right. So, so that whatever we spoke has been right put there all right. So, clearly right if you if this equality prevails then and bow and everything is nonzero you can obviously, divide by either pa or P B and get both of the conditional probabilities B equal to the unconditional probability this P if A given B will be P of A, and P of B given A also will be P of B both of them will hold yes.

Student: What if we define B to B probability of (Refer Time: 09:13).

Obviously it will be exclusive right if you say B is 1 3 5 sorry 1 3 5 you are saying B is 1 3 5 then clearly a B is 5 so obviously, there will be exclusive and not independent. So, the question is right if this is I am not sure you know what was a point of the I mean this is you wanted to explain you know look at independence and explain usually exclusiveness in the same example right. If that is the point of the question yes right 2 4 6 and 1 3 5 are exclusive, 2 4 6 and 3 6 are independent right similarly for example, if I took a to be perfect square and B to be an odd number then also you get independence one 4 and 1 3 5 are independent.

Student: Is it possible that you know probability of a and probability of B are non-zero, but they do not have any intersection and it can still be independent.

That is only how it cannot be happen if you take some extreme example or trivial example like a being no a being A or B being 5, only then that equality is going to be satisfied it is not going to be satisfied if A and B both have nonzero probability right. This equality is satisfied only you know if you if A and B are exclusive only if one of them has to be the empty set otherwise it is not going to be satisfied right. Definitely any other those are late comers here did do you have got you are right.

So, let us move on. So, A and B are the exclusive very important result is that a and sorry A and B are independent, they important result is that A B complement right A complement B, A complement B complement all of them are independent events right. So, A and B independent. So, are A compliment and B, A and B this is a very intuitive right result because you do not want right if A and B are independent you know B complement to somehow tell you something more about a then B that B could not tell you.

So, to prove or to understand this little more let us focus on this k on this example of a complement and B complement, let us take that now others you can do by yourselves right so.

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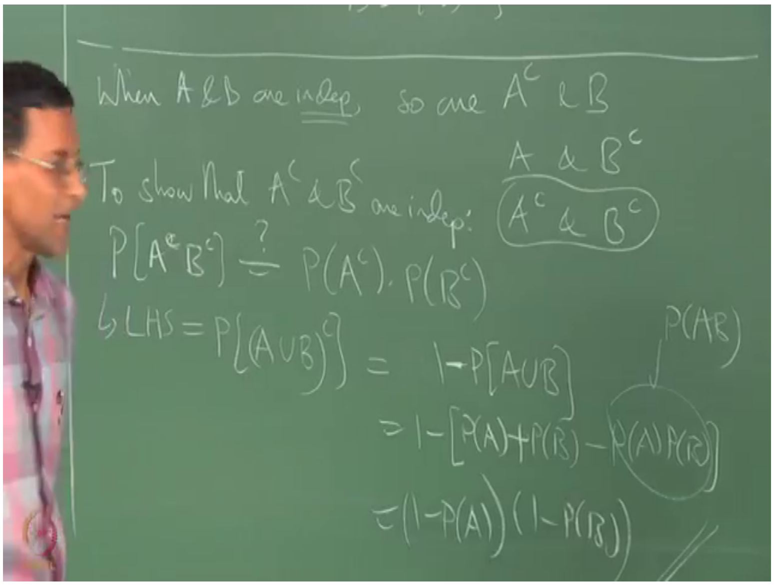

So, this probability of, what we have to show you show that a complement and B complement are independent what do you want to do need to show is it true that P of A complement B complement is I will put a question mark here about the equality sign which is which is what right we are trying to show you I want to show that, this is the same as P of A complement multiplied by B of B complement right this is what we have to show. So, to proceed this left hand side we use De-Morgan's law right.

So, which tells us that this LHS is exactly what P of A union B the whole complement right by d De-Morgan's law, since a complement B complement is the same as a union B the whole complement right and this is 1 minus B of 1 minus P of A union B. Now it is interesting here right incidentally P of A union B itself now can be written in terms of just pa and pb because we know how to write pa B is not it. So, what is P of A union B in this case it is P of A B plus P of B, I will just skip the step and directly write P of A P of B here, because I know that this is right P of A B I know that A and B are independent. So, this is P of A B right see. So, look how nicely algebra comes to our risk to our aid here right. So, what happens?

Student: (Refer Time: 14:07).

You get a very interesting factorization of this quantity out here, which is basically 1 minus P A multiplied by 1 minus P B and that is exactly this. So, similarly the others right can all be shown with all the tools that you have I am not going to write spend the valuable time out here doing everything every small little thing right. So, again the takeaway is very important is that if you have 2 independent events right you will have more than the just the 2, you have all of these as well right and neither A nor A complement change the situation as far as B is concerned and vice versa.

Now what we want to do is extend this to 3 events right.

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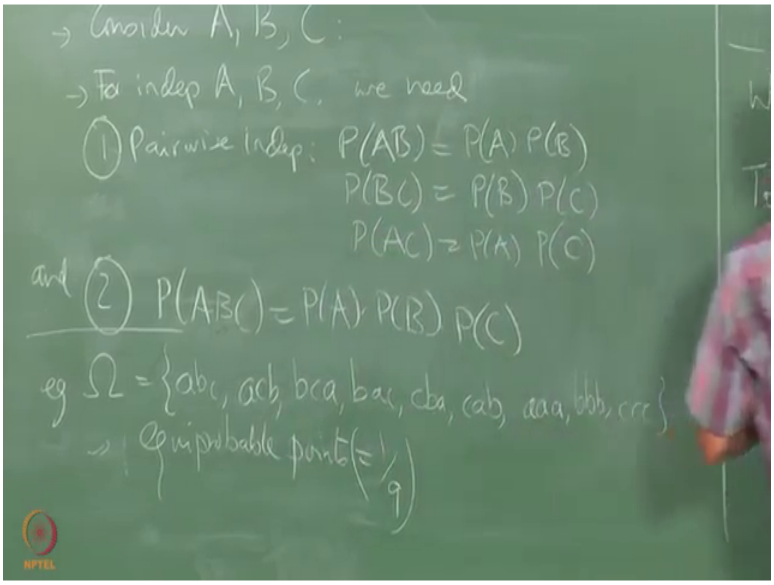

Supposing I look at 3 events when do I say 3 the 3 events are independent. Now you know I have more things that I can do I can consider A and B C or I can consider A and B union C and things; so that kind of right. So, I want not just pairwise independence I also want something beyond pairwise independence.

So, first thing of course, pairwise independence, we must have for independence of A B and C we need for independent ABC we first need pairwise independence what is pairwise independence? P of A any pair should write the joint probability should be product all 3 pairs right it should the independence should hold. On there on top of this you want one more what is that one more thing you want? We also want a P of ABC to be the product of BA PBC we need one and we need 2 P of ABC to be equal to PA, PB, PC.

Via both via is all of this. So, basically were 4 conditions now 1 2 3 4 why do why do we need this one this all 4 conditions are needed if you want a to be independent of the union c a to B in independent of say for example, from this you can clearly see that A must be independent of B and BC, because B P of B into P of C is P of B C by this condition here right. So, clearly a is independent of B C if you put all 4 likewise you can show that P A will be independent of B union C also right. So, to get any possibility I know any common any such combination right to come out to be independent you need all 4 conditions right. So, it is very easy to come up with examples where you have pairwise independence, but you do not have real fully independence right.

So, if I consider a simple example like this which have far I can take, it from some book if I take if I have let us say some strings of letters that right let me make sure this is have to do it very carefully. So, I need to take this paper and make sure I do not miss anything if I have some strings of characters and I am going to give them all equal probabilities. So, I have basically these are now not events ABC are just some 3 characters I am writing down on a piece of paper or something and right I am going to pick any one of these strings, but with equal probability; how many strings I have nine strings right. A abc ac and the order is very important abc, bca ba, abc, acb, bca, bac, cba c a b. So, I have 6 of these right 6 permutations of a b and c all 6 permutations are here right.

then I have 3 more a a a, b b b, c cc in other words let us say that I have written these string I character strings or nine pieces of paper right e and then I juggle them jumble them up right and pick one any one of these right just a piece of paper or our hat right. So, I will I attach a probability 1 by 9 each of these strings right. So, equal probable points. So, these points in omega repeat problem with probability 1 by 9 now how do I define a b and c right I am going to define capital a the event a as the event that that later a comes up first.

And so on with B and c oh no sorry no no no a will be let me write it down. So, of continuing this example let a B the event that a comes first and b.

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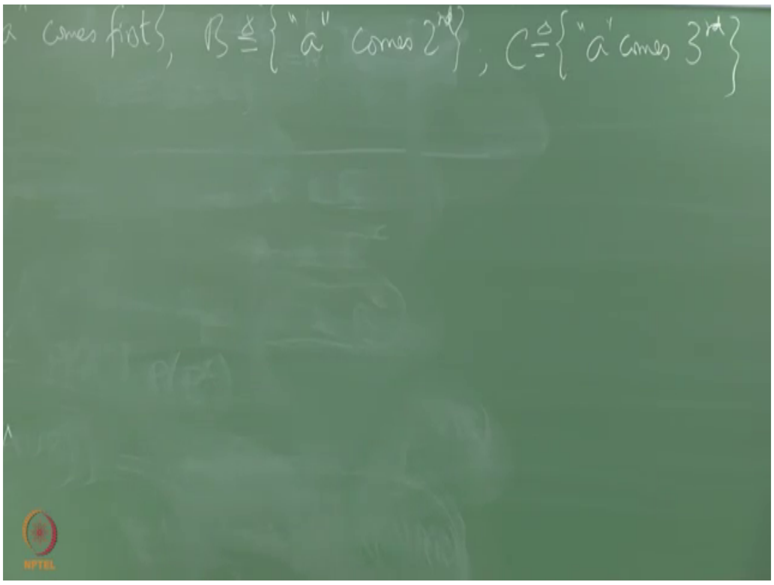

The event by definition the event that a come second right, again I have to make sure I do not mistake this and c the event that a comes third now you can check for yourselves that a b and c are pairwise independent right.

So, for example, if I pick this right this string then my event a has happened because a is come first right. If I pick if I pick this then the event b has happened right because the a has come second, but what is the probability of the event a right its clearly 1 2 3.

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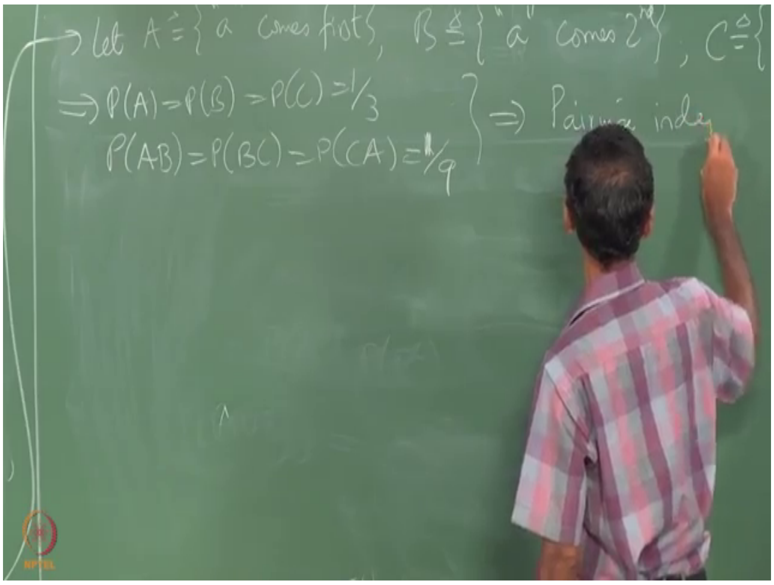

So, so this will say that P of A will be equal to P of B will be equal to P of c is equal to 1 by 3. So, what is P of A B now? AB will be a coming first and k and a coming second. So, it is only happening here right. So, you can clearly see pairwise independence in action right 1 by 3 into 1 by 3 is 1 by 9.

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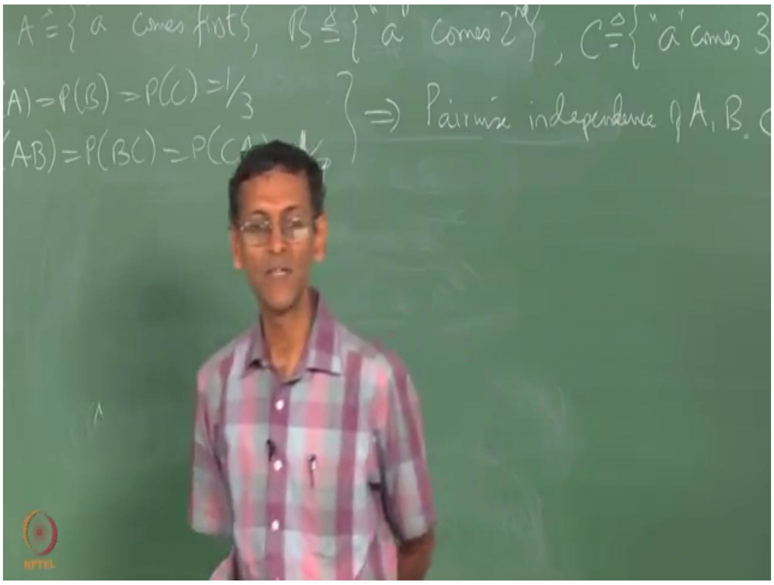

So, pairwise independence right, but do we have real independence no what is probability of abc? Still 1 by 9 which is not 1 by 27 right it happens only if you pick this ABC there is a point here right.

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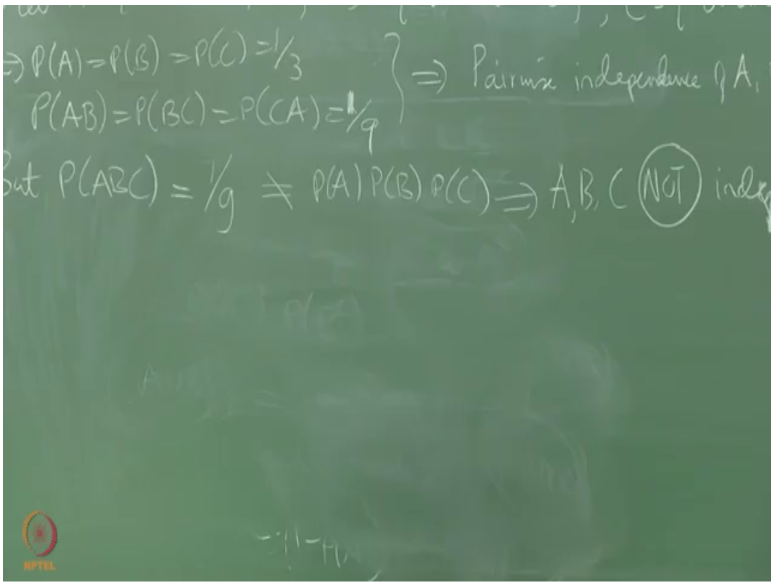

Where, a comes first a comes second and a comes third right; so which is not equal to ABC. Likewise right you can come up with many examples some of which we will put on the homework also right the first homework set will problem set will contain things of this kind right, but for the sake of the not independent right. So, you can extend this to more than 3 obviously, right if you extend you have to make sure that right you have to start from pairwise then go to any combination of 3 events then go to 4 and so on.

So obviously, as you keep increasing the number of events in the in the family that you looking for you know overall independence, in the they conditions will keep on growing the number of conditions you are satisfied will keep growing right.