Probability Foundations for Electrical Engineers
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Lecture – 11
Examples: Conditional Probability

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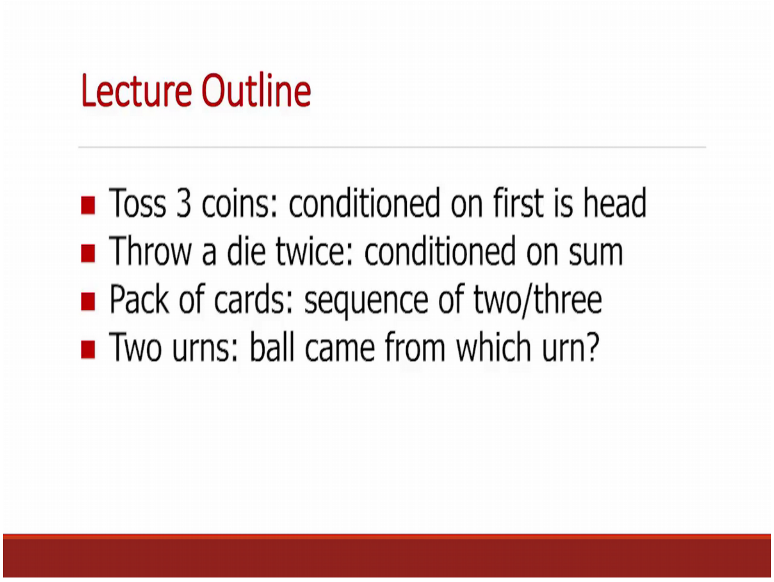

Welcome to this lecture on examples for conditional probability. Now, conditional probability is quite crucial and important for doing various probability calculations. In fact, in the previous lecture you would have seen that when we use the simple method of writing the sample space in terms of equally likely outcomes and used the only method of dividing the number of favorable outcomes divided by the number of total outcomes, often we ran into problems when the problem when the sample space became very large and when the even space became very large.

For instance, the simple question of having 3 cards in a sequence when you draw them one after another and looking at the overall sample space as equally likely outcomes and dividing with the events etcetera and got a little bit complicated. Now, if you look at what happens when you draw 3 cards in a sequence you drawing cards one after another. So, one card you draw the next card you draw again randomly. So, that is the kind of situation where conditional probability is extremely useful in doing computations. You can condition on what happened on the first draw and then think about how to do probability later on. So, you can break up a large event or a large sample space into a sequence of smaller sample space conditioned on what happened in the first time second time etcetera.

So, we will see a few simple examples in some examples you will see it does not make too much of a difference whether you are conditioned or not, but in some examples you will see things will become a little clearer. You can write down the answers more quickly and easily in nice intuitive fashion using this method. So, let us get started.

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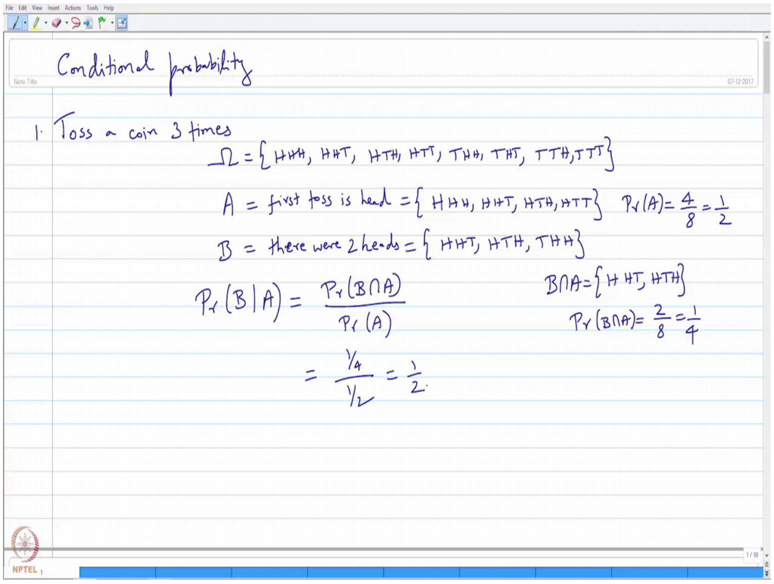

The first example will again look at toss a coin 3 times this is a relatively simple example where I will just illustrate the ideas of conditional probability. What happens here as you know is that the sample space is as has 8 possible outcomes, I have written this several times now so you must be getting used to this. There 8 possible outcomes you can take them as equally likely and one can define events etcetera.

So, to illustrate conditional probability I am going to even define 2 events I will define the event a as being first toss is head. So, I am conditioning on the first toss I am saying the first toss is head. So, this is an easy event to write down you have HHH, HHT, HTH and then HTT. So, these are the 4 elements of this event A. Then one can also define an event B that there were 2 heads. So, this is the event B. If you look at that event you have HHT HTH, THH. There are 3 different outcomes that are favorable to saying when there are 2 heads, HHT, HTH, THH.

So, you might want to ask what is the conditional probability of B given A. So, this is a way in which we would define the conditional probability in Professor Andrew that define conditional probability gave a definition for this probability of B given A you know is the same as probability of B intersect A divided by probability of A. So, this is a simple calculation that one needs to do find out the probability of A, find out the probability of B intersect A and you divide one by the other you get a conditional probability. So, this is a simple definition.

So, now a probability of A we know is easy to do this is 4 outcomes out of 8, so that is just half. So, if you notice this event A we could have easily said the probability of A was half just based on the definition, it says event a is first toss is head you do not have to look at the entire sample space of 3 tosses to figure out the probability that the first toss is head, first toss is head is going to have probability half right you do not have to do this outcomes and counting method and all that. So, eventually that is what is important in doing problems you have to figure out how to quickly find probabilities without doing this complicated thing of writing down the entire event and counting etcetera.

So, what about B intersect A? So, we have a being first tosses head B is there were 2 heads what will be B intersect A, B intersect A you can take these 2 sets and intersect them that is one way of getting there be HHT, HTH and from that you can figure out probability of B intersect A its 2 by 8 which is 1 by 4. So, this event is also easy to write down based on its definition, B intersect A is what? First toss should be head. So, first toss should be head and you should have 2 heads which means one of the other 2 tosses should also be head. So, you have HHT and HTH. So, easy to write down we get 1 by 4. So, this conditional probability becomes 1 by 4 divided by 1 by 2 which is 1 by 2.

So, probability that there will be 2 heads given that the first toss is head is actually half there is various other ways of doing this, but this is one way of computing the conditional probability. So, hopefully this example was simple enough. There are various other examples that you can do like this where you actually define an event A, another define B and then you condition B given A.

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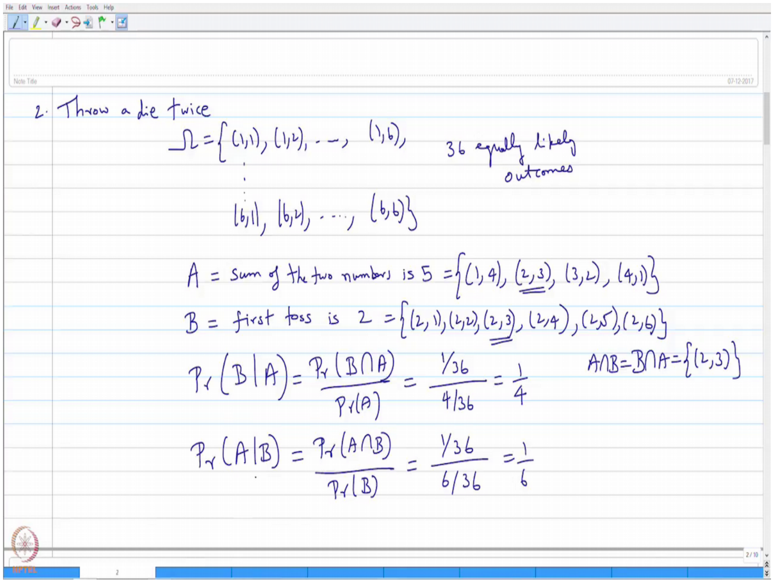

So, I am going to do one more example of such a case with throwing a die we will throw a 6 faced fair die twice throw a die twice you remember the sample space, this was 1 comma 1, 1 comma 2 so on till 1 comma 6 and all the way down to 6 comma 1, 6 comma 2 dot dot dot 6 comma 6.

Once again there are 36 equally likely outcomes here. So, I am going to define an event A which is sum of the 2 numbers that showed up when you through the die is let us say 5 so that is the event A. And the event B is first toss this 2. So, let us do this the hard way first and then I will point out a lot of an easy way to do this problem which will help you as we go along. So, what is this event A? Sum of the 2 numbers is 5 it could be 1 comma 4, 2 comma 3, 3 comma 2, 4 comma 1. First toss is 2 you actually have 2 comma 1, possible 2 comma 2 possible 2 comma 3, 2 comma 4, 2 comma 5, 2 comma 6. I am going to ask probability of B given A.

So, once again this going by the basic definition is probability of B intersect A divided by probability of A and let me do it of right here B intersect A is what there is only one element in the intersection of B and A right its 2 comma 3 it is the only common element everything else is not there, so its 2 comma 3. So, probability of B intersect A is 1 by 36, probability of A is 4 by 36 and the answer ends up being and, so 1 by 36 divided by 4 by 36 the answer ends up being 1 by 4.

So, you can also calculate the other probability which is actually a bit nicer probability of A intersect A given b. So, given that the first toss is 2, what is the probability that the sum of 2 numbers is 5? So, here again you will need the probability of B intersect A or A intersect B if you will both of these are the same divided by probability of B probability of A intersect B and probability of B intersect A are the same right. So, these 2 are the same. It does not matter how you intersect them. So, the numerator you again have 1 by 36 denominator you have 6 by 36 which is 1 by 6.

So, now when you do things repeatedly like this when you throw a dice twice you do not have to necessarily write down this entire sample space and argue it like this. So, supposing particularly the second one is very easy to argue so if you look at probability of sum of 2 tosses is 5 given first toss is 2.

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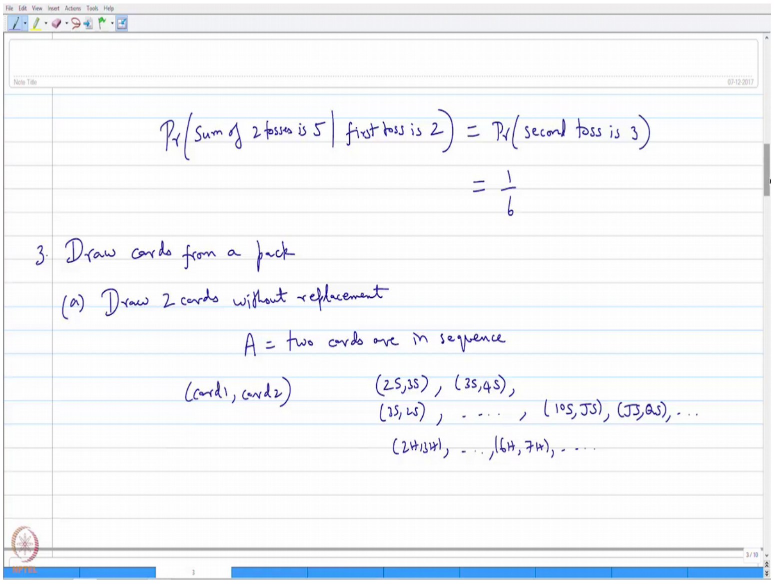

So, you have given already that the first toss has resulted in 2 that is already there you only have to throw the dice one more time, you throw it one more time and you want the sum of the 2 tosses to be 5, which means what? This is the same as the probability that the second toss is 3.

So, this is the argument or inference which is very important I think. If you look at given that the first toss is 2 you already finish the first toss now forget about the first toss it does not exist anymore for. You are only looking at the second toss no, but you want the total to be 5 so first already resulted in a 2. So, this is the same as the probability the second, second toss is 3 and that is simply 1 by 6, 1 out of 6 possibilities in the second toss. So, once you get enough experience with conditional probability this is the kind of calculation you should be doing. So, you can of course, write down the entire sample space and argue cetera, but you can easily argue just based on how the conditioning affects the sample space and particularly in repeated experiments like this. So, this kind of argument is extremely important.

So, for instance maybe I want to find the probability. So, hopefully this illustrates how you can argue about conditional probability without in an easy way just using English directly as opposed to going to the mathematics notation. Let me also go back to this picture hopefully you saw how this thing worked out.

Quite a simple example, we will do slightly more complicated examples and when we do them we will see again such arguments help you a lot. In general when you do things multiple times you can always condition on what happened the first time that tends to simplify all your calculations, you can condition on what happened the first time and then you will just repeat the calculation things become much much easier.

So, the next example we see is draw cards from a pack and first case I will consider is we draw 2 cards without replacement and I am interested in this event a which is the 2 cards are in sequence. What is the meaning of 2 cards are in sequence? Let us say we draw card one in card 2. So, this could be let us say 2 spades, 3 spades or 3 spades 4 spades or in fact, they can also be 3 spades 2 spades this is also actually in sequence even though comes in reverse sequence so on, you know I mean anything else also could be a sequence. So, 10 spades jack spades, jack spades, queen spades like that ok.

So, there should be the same suit. So, or you could have 2 hearts, 3 hearts, 6 hearts, 7 hearts like that. So, both card should be in the same suit and the number should be next to each other. So, that is called a sequence and we have card 1 and card 2 and we want these 2 guys in the sequence. So, I want to figure o1ut this event A which is 2 cards are in a sequence and I will do this cleverly by conditioning on the first card. So, now, if you notice depending on what card 1 is you may or may not have a sequence in the number of choices you have a card 2 will be different.

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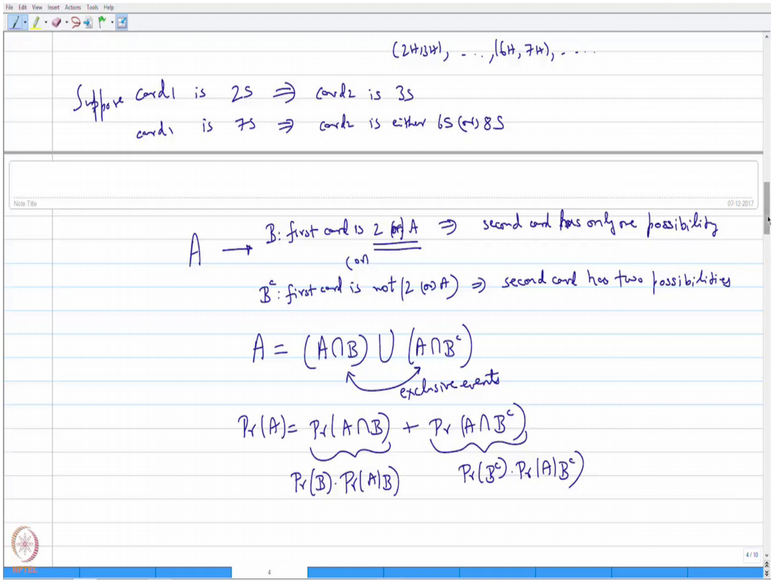

So, for instance suppose card 1 is, suppose card 1 is say 2 spades then if you want a sequence. So, this will imply card 2 is 3 spades only then you will have a sequence otherwise you will not have sequence. Now, if you suppose card 1 is 7 spades this will imply card 2 is either 6 spades or 8 spades. So, if you think about it very carefully if the first card is either 2 or as of any suit then the second card has only one possibility, if it is the first card is 2 of any sort its 2, 2 spades or 2 hearts or 2 clubs or 2 diamonds the next card has to be 3, 3 of the same suit. There is only one possibility. Same way with ace, if the first card was ace of spades or ace of diamonds or ace of clubs or ace of hearts the next card has only one possibility for a d base sequence it should be the king of the same suit, if it is ace should also have kings back. No other possibility exists for the 2 cards to be in a sequence.

On the other hand for any other card in the middle if the first card is anything else, let us say 3 spades or 4 spades or 10 hearts or jack of clubs or anything you have 2 possibilities for the second card and still you will get a sequence. So, that is what one needs to do and that is what one needs to use in the computation. So, you are going to look at this event of 2 cards in a sequence and first break it down on what happens in the first card the first card could be any 1 of the 52 possibilities, but you do not have to look at all the 52 possibilities in detail. You can first look at the one case in which you either had a 2 or an ace and the second case in which you had anything else.

So, you can define this event A which is 2 cards are in a sequence. So, you can consider the case where first card is 2 or ace which will imply second card this, this has only one possibility or first card is not 2 or ace its neither 2 nor ace which will imply second card has 2 possibilities. So, the first card you drew from a full pack of 52. So, what is the probability? That is either 2 or ace how many 2s are there; there are 4 2s, 2 spades, 2 hearts, 2 diamonds, 2 clubs and there are 4 aces same ways all 4 suits. So, you have 8 favorable outcomes. So, the probability that the first card is 2 or ace is 8 divided by 52.

Now, if you condition on that, if you condition on that then you have 51 cards left once the first card is drawn you have 51 cards left, out of this 51 after the first card has been taken and you know its 2 or ace you only have one possibility left only, 1 by 51 is the probability that you will successfully have a sequence. So, the way you have to think of this is you can define an event B which is first card is 2 or ace and this is actually B complement is not it first card is not 2 or ace.

So, I have to think of the event A as A intersect B union with A intersect B complement. So, this is said from mathematical description of what I am doing I am taking the event A and splitting it into 2 possibilities, either the first card was 2 or ace or it was not and then I am doing a union. Now, notice these 2 things are exclusive, exclusive events what do I mean by exclusive events either B happen to B did not happen if B happened B complement could also not happen. So, if you look at it this way probability of A is actually probability of A intersect B plus probability of A intersect B complement.

So, this is the kind of the first step in most of this conditioning methods for funding for finding probabilities. So, you first take this event A and you kind of look at the world under 2 different circumstances you define another even B and say whether B happened or B did not happen and you intersect A with that be an A would that be complement. Just like I did here first card is 2 or ace first card is not 2 or ace. Now, this defining this event B you need to be a little bit clever there as in you need to see what is that event B that matters. So, again if you think about it very logically and carefully you will see this is the event B which is critical here. If its 2 are ace I am doing the same thing for the second card second card gets fixed to 1 possibility if it is not 2 or ace second card has 2 possibility. So, you have to think on what happens in that sequence and then decide what this even here B will be.

Now, to evaluate these probabilities one can use conditional probability. So, what is probability of A given B? It is probability of B times probability of A given B what is this? This probability of B complement times probability of A given B complement right. So, this is where conditional probability is useful. So, know what is probability of B? Probability of B is first card is 2 or ace that is 8 by 52.

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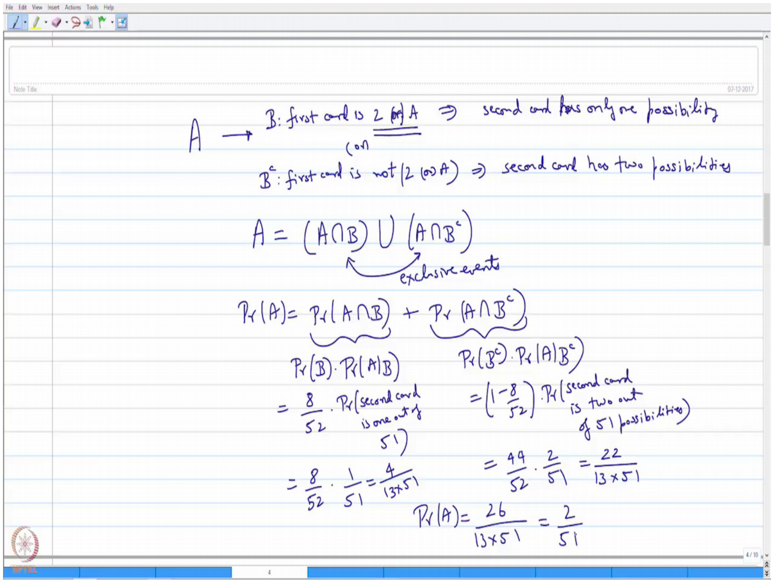

And the first card is gone I have 51 cards left and whatever I drew is the first card I only have one possibility in the remaining draw to have successful outcome of a sequence and so probability of A that I have a sequence is simply probability that second card this one out of 51. So, this is 8 by 52 times 1 by 51. So, what about here? Probability of B complement is 1 minus 8 by 52 right. So, 1 minus 8 by 52 times here what probability of second card is 2 out of 51 possibilities right. So, what is 1 minus 8 by 52? 52 minus 8 is 44, 44 by 52 times 2 by 51.

So, if you do these 2 things finally, you can put it together and write down probability of A we actually write it down here. 8 by 52 if you want you can simplify as 4 by 13. So, here also you will have 11 by 13 and 11 into 2 is 22, 4 26. So, correct 4 by I am just doing a quick calculation here. So, 4 by 13 into 51, this is 22 by 13 into 51 ok. So, 4 plus 22 is 26, 26 by 13 into 51 this cancels that is 2 by 51. So, the total probability that you have a sequence is actually 2 by 51.

So, this is how you compute the probability of a sequence. If you remember when we were doing it with equally likely outcomes we had to battle so many different possibilities and write down the whole set in a complicated way if you use conditional probability and you condition on what happens in the first card and split it into 2 events very cleverly then write A as A intersect B union A intersect B complement you get this wonderful simplification and the answer comes out very very easily. So, this is the power of conditional probability in calculations. Hopefully you can contrast and see this ok.

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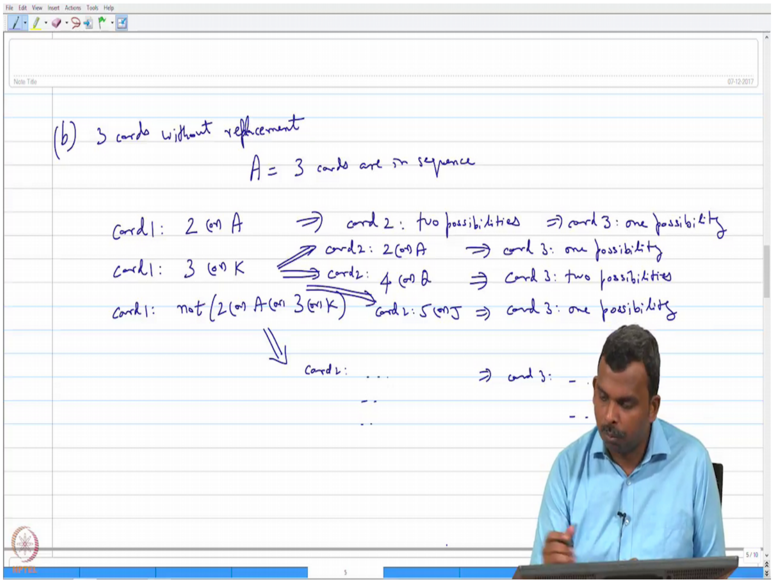

So, the second part next part is a little bit more interesting. Let us say we draw 3 cards without replacement and then we ask the same question what about this event A that 3 cards are in sequence. So, now, we have to condition a little bit more carefully. It is not as easy as before you have to look at more cases. So, you left it you will have to figure out what happens, what is important about the first one, how many cases it will result in the second one and again the argument is not too hard I am not going to write down all the details I will just sketch the arguments and I will let you complete it. Once again now only when you do these problems on your own and you are happy about it you really learn. So, do not take this as something that I am not doing fully this is encouraging you to complete this problem.

So, what are the various things that can happen in the first card that affects you? So, if your first card is 2 or ace. So, if you look at card one if its 2 or ace something will happen and then if your first card is 3 or king also it impacts any other way and then card 1 is not 2 or ace or 3 or king then something will happen think about it. So, if you have 2 or ace the next card has only 2 possibilities. So, this will imply card 2, 2 possibilities. So, alright think about that the next card is 2 possibilities and the next card will have only one possibility.

So, if I draw a 2 of spades my next card if I want the 3 to be in a sequence the next card has to be either 3 or 4 right, those are the 2 possibilities. And once I have drawn the second card it is either 3 or 4 the third card is fixed if I drew a 3 in the second card next card has to be 4, if we drew a 4 in the second card next card has to be 3 that is fixed. So, that is what happens. If it is 3 or king what happens? If it is 3 or king you have you have a few more things that can happen when one needs to think about it very very carefully. In fact, you will have to consider 2 cases here card 2 this is 2 or ace card 2 it is not 2 or ace ok.

So, if card 3, card 1 is 3 or king you can have situations where card 2 could be 2 or ace. So, so in case of the card 2 is 2 then or ace then you have card 3 being only one possibility right. So, one needs to do this bit carefully as in if you had a 3 you have to only consider 2, I mean it is if you had a 3 and then you cannot erase. And if you had a king you have to look at ace so that case has to be properly handled am I am doing it a bit roughly here I am just sketching it out for you do not be very careful. If it is not 2 or ace, if you had 4 or queen for instance, so it should be careful here that is not 2 or ace it is better to write 4 or queen then you will have card 3 having 2 possibilities ok.

In fact, there is one more possibility here card 2 can be 5 or jack in which case also card 3 has just one possibility. So, just enumerate all these possibilities is a bit more complicated and then once again if it is not 2 or ace or 3 or king again you have to do what are the various possibilities for card 2. There will be 1 or 2 possibilities here and then that will imply different possibilities for card 3 and then you have to do this. So, the 2 cards in a sequence was slightly simpler we could get that say easily 3 cards in a sequence more complicated things are going to happen and what happens in the first card impacts what happens in the second card and together they impact what happens in the third card and you have to understand this tree of possibilities very clearly and put conditional probabilities on each of these and then find the total probability. It is not a very easy exercise I will make it an exercise for you in the assignments to do you should try this problem and get to the answer it is a very it is kind of exciting to do these kind of problems when you want when you win conditional probability.

So, we have seen 3 examples so far tossing a coin when in which case conditional probability was very very easy to think about and then we looked at throwing a die which made the thing a little bit more complicated, but still we could deal with it and then we saw the real power of conditional probability when we were drawing cards from a pack. So, when you could condition on what happens in the first card and then figure out what is going to happen in the second card etcetera. So, complicated calculations became really very simple. And then we saw a slightly more complicated conditional probability situation where we have to write down the possibilities in a sequence very carefully and that it is not very not impossible, but one needs to exercise a lot of care and caution in doing these calculations correctly and you will get to the correct answer ok.

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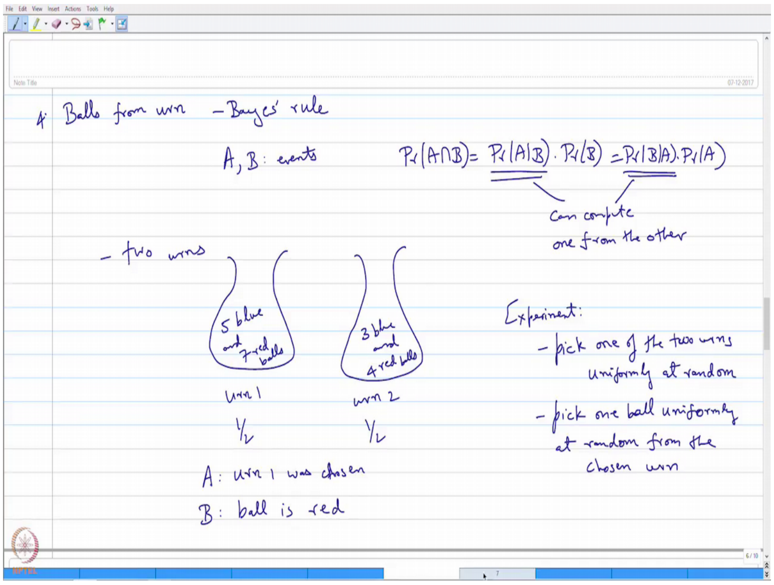

So, the last example I am going to do is drawing balls from urn this is a very very classic example of conditional probability. And this is also an example of Baye's rule this is an example of Baye's rule. So, what does I mean I think, I am sure you would have looked at Professor Aravind's lectures on bases rule you know what the actual rule is if you have 2 events A B, the rule basically equates 2 way of calculating A intersect B which is probability of A given B times probability of B and it is also probability of B given A times probability of A ok.

So, quite often it will happen it will turn out that one of these 2 probabilities, probability of A given B might be easy to compute or probability of B given A might be easy to compute. Once you have such a situation you can compute the other quantity using Baye’s rule. So, if you find probability of A given B is easy to compute probability of B given A can be computed using Baye’s rule. So, this is the basic rule and one uses this in various clever ways to get to the answer. So, in many cases one of these might be easy to compute, can compute the other one from can compute one from the other ok.

So, I will give you a nice example a very very standard example. Let us say we have 2 urns, the first urn has let us say 5 blue and 7 red balls it is a very classic illustration of Baye’s rule. The second one has 3 blue and 4 red balls. Now, the experiment goes like this there are 2 urns I am going to pick any one urn at random. So, this is urn 1. So, what is my experiment? There are 2 steps in the experiment you pick one of the 2 urns uniformly at random. What does that mean? I pick one of these 2 urns uniformly at random.

So, I pick urn 1 with probability half urn 2 with probability half if you one write that down here. After you have chosen the urn you pick one ball at random uniformly at random from the chosen urn. So, this is what happens you pick one urn at random uniformly half of each and then after you have chosen the urn I am going to pick a ball from that urn. So, I am interested in a couple of events let us say A is urn 1 was chosen and even B is the ball is red just to illustrate this thing of how probability of A given B might be easy to compute the probability of B given A might be more difficult to compute and how does one go about doing that. So, that is the main point of this illustration.

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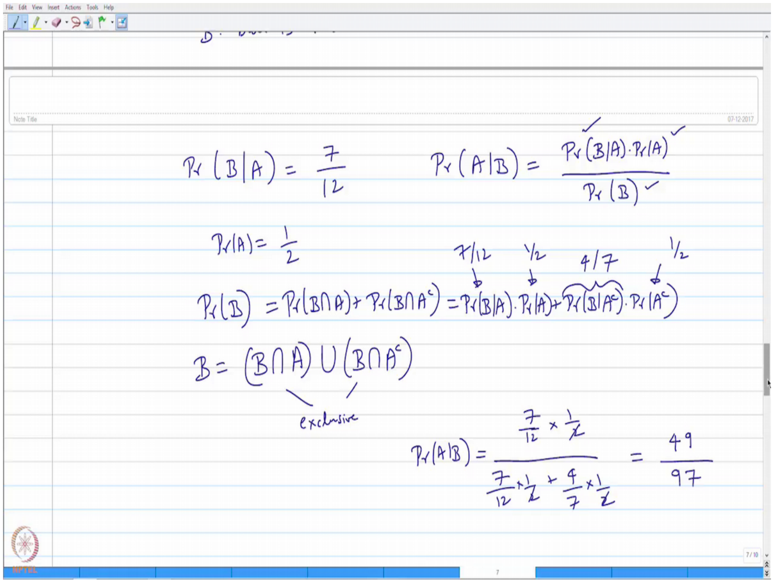

So, if you look at probability of B given A. What is probability of B given A, I have chosen the urn 1, urn 1 has been chosen it has 5 blue and 7 red balls what is the probability that ball is red given that the urn is the first turn its 7 out of 12. So, we have seen this question easily before it is just an favorable outcome divided by total number of outcomes, so you get 7 by 12. That is it alright hopefully you see that.

So, now, the question is what is the probability of A given B. What is A given B now? I am given that the ball came out to be red, but I do not tell you whether I took it from urn 1 or urn 2 the ball came out to be red and you have to find out the probability that the urn one was actually chosen, what is the probability, what is the chance that urn 1 was chosen given that the ball came out to be red ok.

So, it is an interesting question in some sense and Baye's rule can be used to answer this question. So, how do you use Baye's rule here? If you look at base rule this is probability of B given A times probability of A divided by probability of B, yes I have used Baye's rule here you can see probability of B given A is probability of A given B times probability of B divided by probability of A or you just push it to the other side you get, this probability right probability of A given B its probability of B given A into probability of A divided by probability of B.

So, now the numerator we already computed what is probability of A urn 1 being chosen is 1 by 2 what about probability of B alright. So, for probability of B one needs to do this the following conditioning. So, you have the condition on the event on which urn was chosen in the first row. So, you can write B as B intersect urn 1 was chosen A and then your union it would B intersect A complement. So, this is the first thing here. So, A is urn 1 was chosen A complement is what, urn 1 was not chosen which means urn 2 chosen. So, this is what we write in these 2 and these are exclusive events. So, probability of B simply becomes probability of B intersect A plus probability of B intersect A complement.

So, this is quite an important step. So, you break down the event B which is actually what happens after the second step, depending you break it down into different possibilities depending on what happens in the first step. In the first step you could have chosen urn 1 or you could have chosen urn 2 B is something that happened in the second step. So, you split it into multiple possibilities based on what happened in the first. So, that is. So, A write B as B intersect A union B intersect A complement and I get this.

Now, each of these terms one can use conditional probability. So, you have probability of B given A times probability of A plus probability of B given A complement times probability of A complement. So, this B given A we have already seen probability of A we have already seen. What is probability of B given A complement, what is the probability that the ball is red given that urn 2 was chosen. So, that you can see is 4 by 7, 4 by 7 this guy is half a probability that urn 2 thousand is chosen as half probability that urn one is chosen is half and this one we already saw 7 by 12. So, this calculation works out as follows. So, you know probability of B you know probability of A we know probability of B given A we can find probability of A given B. So, probability of A given B is 7 by 12 times 1 by 2 plus 7 by 12 times 1 by 2 plus 4 by 7 times 1 by 2.

So, one can simplify this you will get. So, if multiply this 2 will cancel throughout and then you can multiply by 12 into 7 and you will get 49 in the numerator divided by 49 plus 48, 49, plus 48 is 47. So, that is the probability that urn 1 was chosen given that the ball was red. So, you can put down a precise number for this calculation like I said in this problem when you pick 2 urns at random and you pick one ball from each urn probability that the ball is red given a particular urn was chosen is easy to calculate, but the other implication probability that the urn 1 was chosen given the ball is red its more difficult to compute. And you can use Baye’s rule and not just Baye’s rule also this total probability law what is the total probability law you want to find a probability of A complex event B you split it up into multiple events which are exclusive then you add up the individual probabilities using both those rules one can come up with a precise answer for this question.

So, this brings us to the end of the lectures on conditional probability. Hopefully this was interesting to you, hopefully you saw how a complicated event such as cards being in a sequence can be split up nicely using conditional ability and computer.

Thank you very much.