Control Engineering

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Module - 12
Lecture - 04
Controllable Decomposition and Observability

So in this little lecture, we will first talk about Controllable Decomposition. So, why do we need to do this? So, in the previous lecture, we saw basics about controllability that the system is controllable, if the rank of a certain matrix is $\boldsymbol{n}$, and if that rank is $\boldsymbol{n}$, then I could actually do something called the pole placement to solve control problems in the state space ok.

So, now we did not answer the question what if the system is not completely controllable, what if the rank is not $\boldsymbol{n}$, but it is some number $\boldsymbol{m}$ which is less than $\boldsymbol{n}$ ok. So, let's quickly check what we can do with those kind of systems ok.

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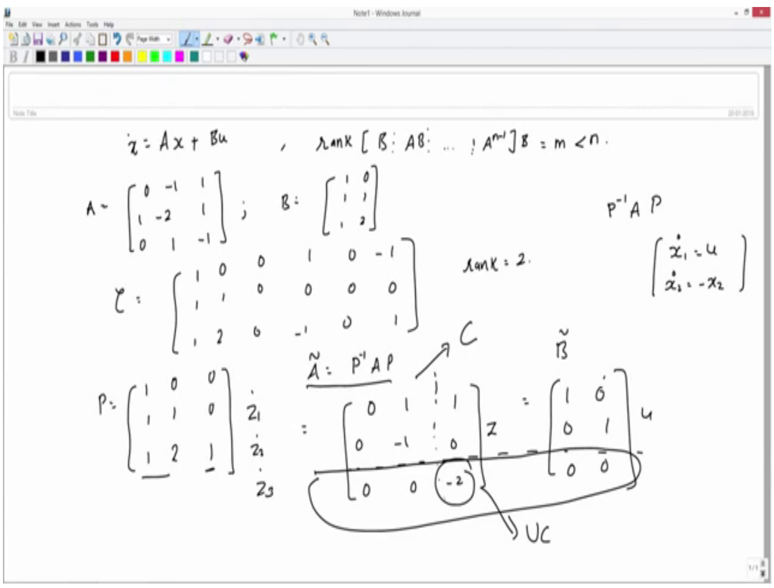

So, let's say that I start with the system $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$. And suppose that the rank of this matrix $[\boldsymbol{A}: \boldsymbol{AB}: \dots \dots\ \boldsymbol{A}^{\boldsymbol{n}-\boldsymbol{1}}\boldsymbol{B}]$ is some number is some number $\boldsymbol{m}$ which is less than $\boldsymbol{n}$ ok. Let's to keep it simple; let's do it with the help of a little example. Where $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$; $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ ok. So, the controllability matrix $\boldsymbol{C}$, which I call is $\boldsymbol{B}$, just be $\boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$; $\boldsymbol{AB}$ would be $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$, and $\boldsymbol{A^2B}$ would be $\begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ ok. And we can easily check here that the rank is $\mathbf{2}$ ok.

Now, what do we do when the rank is $\mathbf{2}$? I cannot completely control the system because what I learnt ok, even though we didn't do the proof is that for to place all the poles, we need the system to be completely controllable ok. So, let's do a bit of a magic here. Let's say I want to transform the system into something else right. And we knew that, so that $\boldsymbol{P^{-1}AP}$ is a good transform which transform the system from some $\boldsymbol{x}$ coordinates to certain $\boldsymbol{z}$ coordinates ok.

So, let's construct this $\boldsymbol{P}$ in a nice way now. Let say $\boldsymbol{P}$ is I take two independent columns from this controllability matrix. So, the obvious ones are directly from the $\boldsymbol{B}$ matrix. Now, can I add another column here such that the rank of this $\boldsymbol{P}$ is $\mathbf{3}$ right. So, I am giving you two independent vectors in $\boldsymbol{R^3}$, and I am asking you to construct something else which is again independent of the first two right. A simple choice would be something like this ok. Now, this is an invertible matrix this $\boldsymbol{P}$. So, I can always do this transformation.

So, what will the new $\widetilde{\boldsymbol{A}}$ be? $\widetilde{\boldsymbol{A}} = \boldsymbol{P^{-1}AP}$ would turn out to be something like this, $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{bmatrix}$. And what is $\widetilde{\boldsymbol{B}}$? $\widetilde{\boldsymbol{B}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ ok. So, let's carefully look at this ok. So, let's say if I write the system like this right, $\begin{bmatrix} \dot{\boldsymbol{x}}_{\mathbf{1}} = \boldsymbol{u} \\ \dot{\boldsymbol{x}}_{\mathbf{2}} = -\boldsymbol{x}_{\mathbf{2}} \end{bmatrix}$, it can be easily checked that this system is not completely controllable. This system which means which because you know you can see there is no direct control which is entering the $\boldsymbol{x_2}$ coordinate ok.

So, here I can clearly see which of the two states is not controllable, you can easily say that $\boldsymbol{x_1}$ is controllable, and $\boldsymbol{x_2}$ is not ok. Now, what happens when I do a transformation like this? If I look at this thing here, if I say I have this new coordinates in the $\dot{\boldsymbol{Z}}_{\mathbf{1}}, \dot{\boldsymbol{Z}}_{\mathbf{2}}, \dot{\boldsymbol{Z}}_{\mathbf{3}}$ is this new $\boldsymbol{A}$ X $\boldsymbol{Z}$ X $\boldsymbol{u}$. That $\boldsymbol{Z_3}$ has no influence of the control, because the entries of the $\boldsymbol{B}$ are $\mathbf{0}$. Whereas, $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$ are actually controllable, because there is some, say some numbers coming in here.

So, what I have done with this transformation is I have split the system into a controllable part and an uncontrollable part. So, this is my controllable part, and this thing is my uncontrollable part ok. So, what did we do here? We, well, what was given to us is, were the system which was not completely controllable, in such a way that the rank of this controllability matrix was some number $\boldsymbol{m}$.

So, based on this $\boldsymbol{m}$, we pulled out $\boldsymbol{m}$ -independent vectors from the controllability matrix. I add remaining $\boldsymbol{n-m}$ vectors such that the rank becomes $\boldsymbol{n}$, and I do this kind of similarity transformation. So, while doing this, I have a system where I can clearly see which part is controllable and which part is uncontrollable. And this is what is referred to as the controllable decomposition ok.

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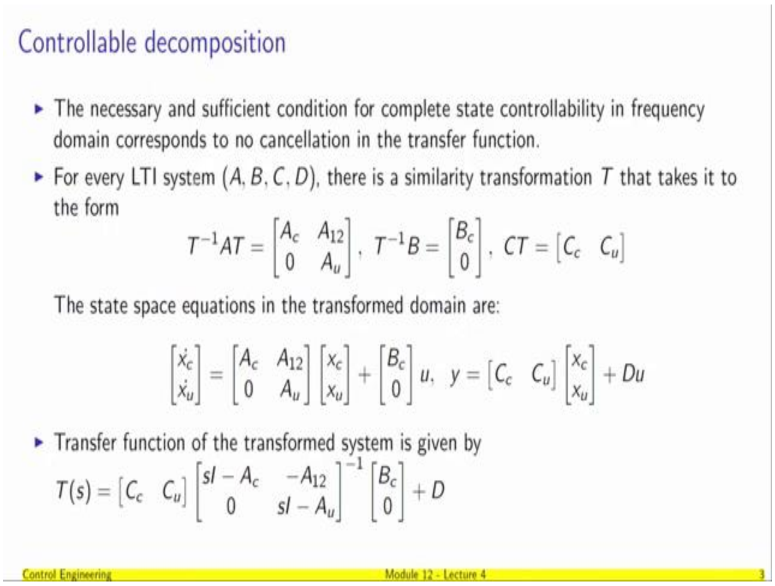

So, there exist a similarity transformation we exactly saw ok, I called it $\boldsymbol{P}$ there, but it's, $\boldsymbol{T}$ here, doesn't really matter. So, the $\boldsymbol{T^{-1}AT}$ is something like this. So, you have the controllable part; you have the uncontrollable part right. And even the $\boldsymbol{B}$ splits very nicely into the controllable and uncontrollable part, $\boldsymbol{C}$ will have some numbers; this is not really important ok.

So, what is interesting here right? So, I have a system now which is explicitly written in terms of $\boldsymbol{x_c}$ which is the controllable states, $\boldsymbol{x_u}$ which are the uncontrollable states ok. What I want is the transfer function of this, well, I can simply write down the formula for the state space to the transfer function form $[\boldsymbol{C}][\boldsymbol{sI-A}]^{-1}[\boldsymbol{B}]+[\boldsymbol{D}]$ ok. So, I will skip these computations.

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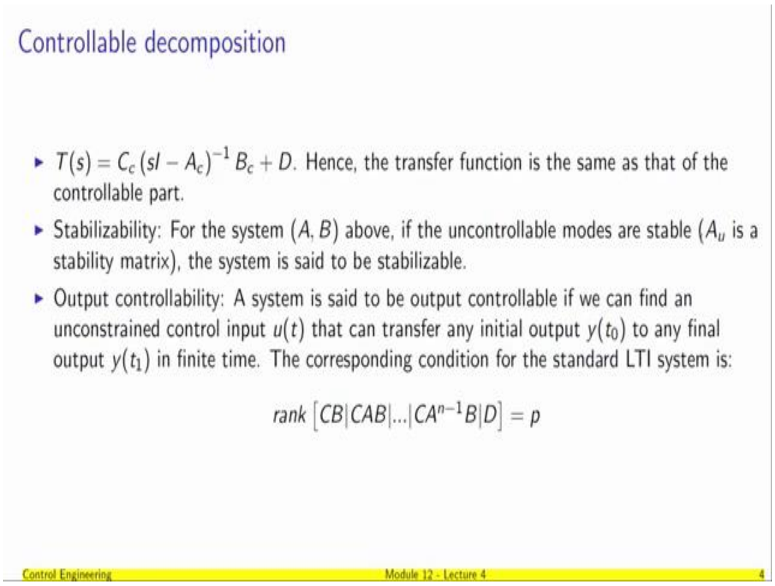

But what is interesting to see is that the transfer function will have numbers or the entries which are only corresponding to the controllable part $\boldsymbol{A_c}$ is the one which is corresponding to the controllable part; $\boldsymbol{B_c}$ is also the controllable part; $\boldsymbol{C_c}$ is also the controllable part.

So, what is important here is that the transfer function is only the transfer function of the controllable part. Therefore, whenever I give you any transfer function, it should, I could do anything with that transfer function if and only if I know that this is a transfer function of a completely controllable system. Otherwise, you see that some poles of the system go missing, or there is some kind of inherent pole-zero cancellation whenever the system is not completely controllable ok. So, what can we do with this kind of systems right?

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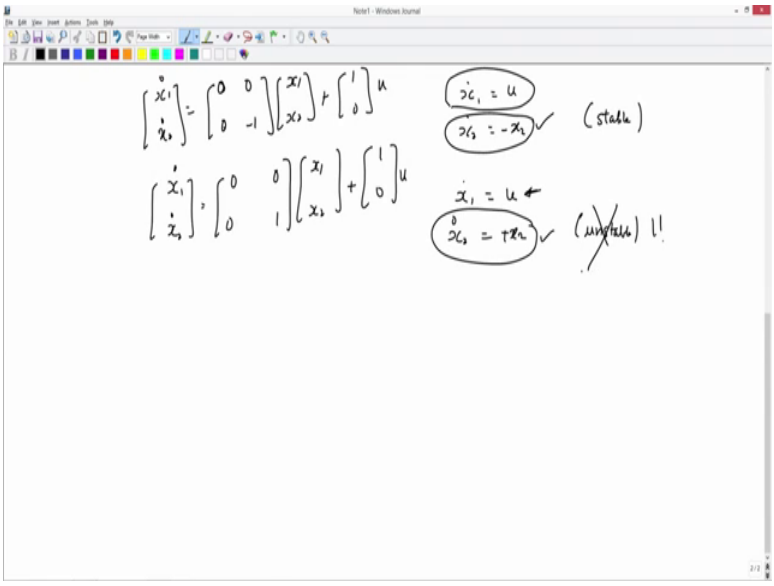

So, say I will just take the other example which where $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ ok, which can be written again as $\dot{x}_1 = u$, $\dot{x}_2 = -x_2$ ok. What can I do with this system? Well, you can see that I can actually control the first state provided that the second state is stable.

What does it mean when I say this? Take instead a system which is like this $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, which means this is the same system $\dot{x}_1 = u$, $\dot{x}_2 = +x_2$.

And let's see how the uncontrollable part of the system behaves here and here. The uncontrollable part here is actually stable, because I can just directly solve for $\dot{x}_2 = x_2$. This is unstable, so this is bad news right. So, I can do anything or I can there is some hope when the system is not completely controllable if and only if the uncontrollable part is stable by itself.

Here I could do nothing with the system, because whatever smart kind of input I choose whatever expensive, whatever you know the best kind of u, my x_2 is still unstable, which means that the overall system is unstable. So, there is nothing I could do with these kind of systems right.

And therefore, we have now a weaker notion of controllability that is stabilizability. For the system A B, if the uncontrollable modes are stable, then the system is said to be stabilizable. I can do something with the system only if the uncontrollable modes are stable ok. And then as a very small extension to the definition of controllability. So, far we have talked about the state going from point $\boldsymbol{A}$ to point $\boldsymbol{B}$ in some finite amount of time, and we just relaxing at the point $\boldsymbol{B}$ is actually the origin ok.

So, what do I say, well, I want the output to be controllable not the states, because typically I have $\dot{\boldsymbol{x}_1} = \boldsymbol{Ax} + \boldsymbol{Bu}$, why is some $\boldsymbol{Cx}$ sometimes some $\boldsymbol{Du}$. So, the definition translates can I transfer the output with some initial value $\boldsymbol{y(t_0)}$ to some value $\boldsymbol{y(t_1)}$in finite time and with application of some control.

This again I will just leave for you to derive that the corresponding rank condition would just be in terms of the output matrix also $\left[|\boldsymbol{CB}||\ \boldsymbol{CAB}|\ \dots.\left|\boldsymbol{CA^{n-1}B}\right|\ \boldsymbol{D}\ \right]$should be $\boldsymbol{p}$. And this is $\boldsymbol{p}$ is like that the number of outputs here ok.

So, what have we learned here is, what do we do if the system is not completely controllable. If the system is completely controllable, I can just use $\boldsymbol{u} = -\boldsymbol{Kx}$ to achieve the desired performance in terms of placing the poles at the appropriate locations.

Even in root locus, what we were doing, we are just shifting the poles right, via the root locus shifting the dominant poles to the left depending on if you wanted a faster response right, and also poles corresponding to when we were doing the lag compensation right. So, essentially playing around with the poles, this also is something similar ok.

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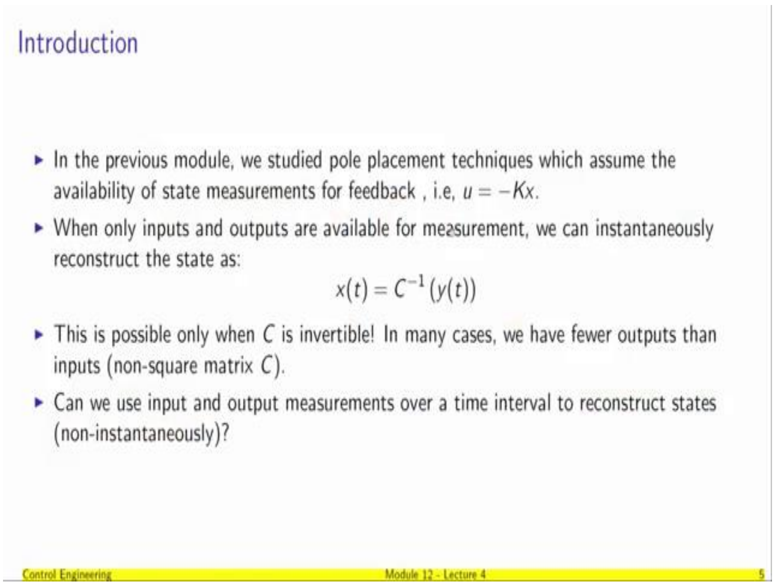

So, what do we assume when we say $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$, we always assume that $\boldsymbol{x}$ is measurable that somebody is actually giving me $\boldsymbol{x}$. Sometimes now what I am; what am I; what am I measuring here? I'm measuring the outputs $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$, let's just ignore $\boldsymbol{D}$ for the moment right.

So, $\boldsymbol{x}$, so given the measurements $\boldsymbol{y}$, when can I get all of $\boldsymbol{x}$? This is possible if and only if the output matrix $\boldsymbol{C}$ is invertible ok, which means that the number of states and the number of outputs are the same, or this matrix $\boldsymbol{C}$ should be invertible. But this is not possible all the time, because sometimes we will have maybe only 5 outputs measured when the complete total number of states is 10 ok.

So, in that case, what do we do, is there again somehow? So, can we use the input we know the $\boldsymbol{u}$ all the time and the output measurements over a time interval to reconstruct the states, now reconstruct $\boldsymbol{x}(\boldsymbol{0})$ for example.

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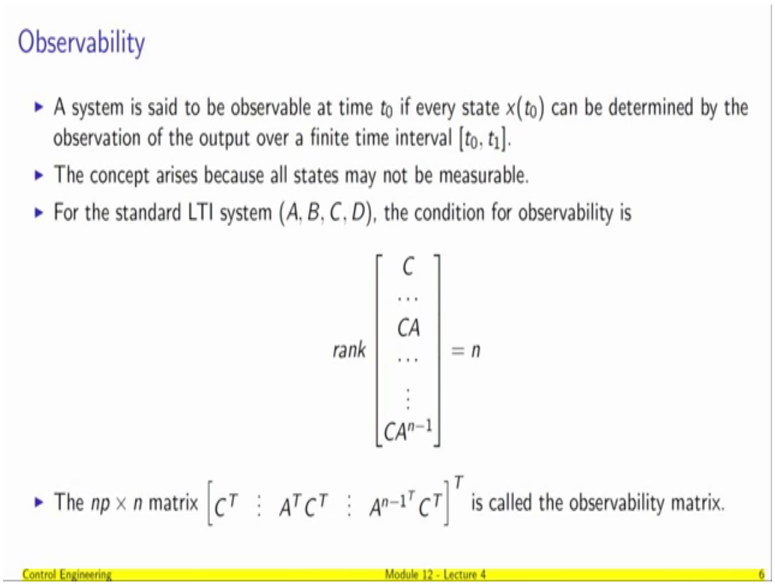

So, this is what leads to the definition of observability. I'll not go into the details of this, but we will just understand what it means. So, I will just stop by just giving a definition and how to check the conditions of on observability ok.

So, what is a definition? The definition says that the system is observable at time $\boldsymbol{t_0}$ if this $\boldsymbol{x}(\boldsymbol{t_0})$ can be determined by the observation of the outputs over the time interval $\boldsymbol{t_0}$ to $\boldsymbol{t_1}$ ok. Again these are important because all states may not be measurable. So, a simple computation like earlier would suggest that a system is completely observable if and only if the rank of this observability matrix is $\boldsymbol{n}$ ok. So, it is of dimension whatever $\boldsymbol{n_p}\ X\ \boldsymbol{n}$ matrix which is this one, this is called the observability matrix.

So, again given the properties of the system which essentially come from $\boldsymbol{C}$ and $\boldsymbol{A}$, we can determine if the system is completely observable or not. And also note that the matrix $\boldsymbol{B}$ does not really play any important role in this ok. So, this is where we will end. We will try to post some problems related to you know the various concepts which we have derived in state space analysis. And also possibly related to the observer or the observability matrix.

And this is actually this covers a you know good amount of stuff related to the state space analysis. And now I will not go deeper into how to design observers and so on, or how to design a observers and controllers simultaneously for a system that we will leave for some advanced course in control. But for the moment this is all what I thought would be useful to you to give you some insights on the theory on state space.

So, this is the last lecture of the entire course. And I hope you have you enjoyed the course, and you had a good time, and we actually had a good time preparing the course content, interacting with you while we were doing through the discussion forums. And well, good luck for your final exam if you are taking those.

And thanks very much.