Control Engineering
Dr. Ramkrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

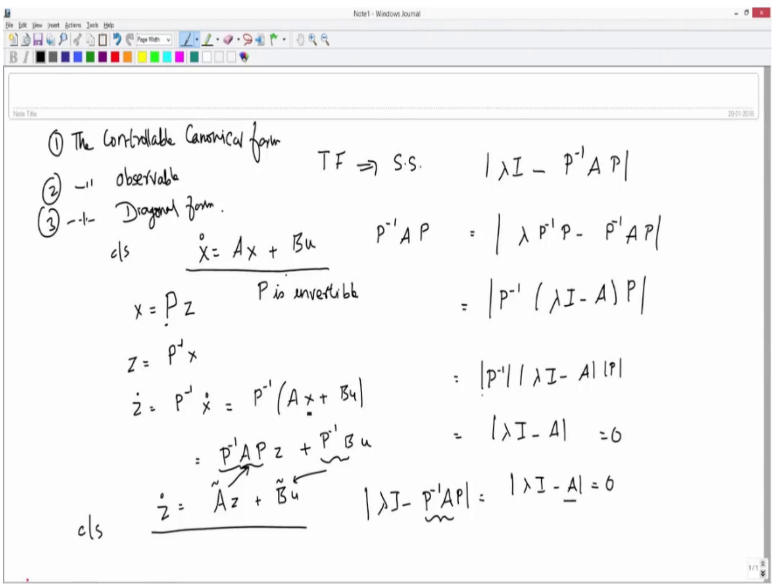
Module - 12
Lecture - 03
Controllability and pole placement

Hey guys. Welcome back to this lecture series on Control Engineering. And, so today we'll keep continuing with the state space analysis. So, far what we have seen is given a system how to write it down in the state space form, given a non-linear system how would you linearize that around an equilibrium point and analyze stability.

So, today we will do further or go a little further deep into this topic and essentially deal with what is called as a controllability, and what does controllability mean, is a system always controllable? what if it is not always controllable? and if it is controllable how do we control it? So far we have seen control methods or design methods using the root locus where we take the help of a root locus plot to achieve the desired closed-loop specification.

Similarly, we did some design problems with the help of bode plots. So, let's see what all that means, in the context of a state-space analysis, ok. So, this let us quickly write down what have we seen so far. So, far we have had state-space representations. And what we saw, yes in the other lectures was three different forms.

(Refer Slide Time: 01:31)



So, one was the controllable canonical form. Similarly, we had the observable form and also the diagonal form. And we also saw how to convert a given system into its diagonal form. There are several other details of the diagonal form in the terms of the Jordan form, but I will try to skip that, ok.

The first is now how to convert system from a controllable to observable or observable to a diagonal or whatever. So, if I say I have a transfer function and I convert it into a state-space, now is the state space representation unique? and how are these three related to each other? So, let's do it in a little abstract way.

Let's say or whenever I am converting. So, let's say I have a system which is like $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$ and I say how do I convert it from one form to the other. And we had earlier seen that we could actually do a transformation of $\boldsymbol{A}$ matrix to something which looks like $\boldsymbol{P^{-1}AP}$, ok. So, where does this come from? So, let's say I start with $\boldsymbol{x}$ I define a new variable $\boldsymbol{z}$ and say $\boldsymbol{x}$ is related to $\boldsymbol{z}$ via this matrix $\boldsymbol{P}$, ok. So, this matrix $\boldsymbol{P}$ is invertible. So, $\boldsymbol{P}$ is, ok. So, what happens now?

So, if I write down this in terms of the $\boldsymbol{z}$ coordinates, I will have $\boldsymbol{Z} = \boldsymbol{P^{-1}X}$ which means that $\dot{\boldsymbol{z}} = \boldsymbol{P^{-1}}\dot{\boldsymbol{x}}$. What is $\dot{\boldsymbol{x}}$? $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$. And what is $\boldsymbol{x}$? $\boldsymbol{x} = \boldsymbol{P^{-1}z}$. So, I have if I write it down again, so $\boldsymbol{x}$ in terms of $\boldsymbol{z}$ would be $(\boldsymbol{P^{-1}AP})\boldsymbol{z} + \boldsymbol{P^{-1}Bu}$.

So, I have a system in the new coordinate $\boldsymbol{z}$ which looks like let me call this say $\widetilde{\boldsymbol{A}}\boldsymbol{z} + \widetilde{\boldsymbol{B}}\boldsymbol{u}$, where this guy $\boldsymbol{P^{-1}AP}$ is the $\widetilde{\boldsymbol{A}}$ and $\boldsymbol{P^{-1}B}$ is the $\widetilde{\boldsymbol{B}}$, ok.

So, the invertibility is needed because well we need the matrix to be invertible here and you can always go back from the $\boldsymbol{x}$ to $\boldsymbol{z}$ and $\boldsymbol{z}$ to $\boldsymbol{x}$ and vice versa all the time, ok. So, given a system in a certain way say it's in the observable form I could go to the controllable form via some transformation $\boldsymbol{P}$, ok. What is this transformation is what we will see a little later, which kind of $\boldsymbol{P}$ to use to get from a standard form to a controllable form, standard form to an observable form, and so on, right, ok.

Now, what has changed here? Now, can I say that the system in the $\boldsymbol{x}$ coordinates is the same as the one in the $\boldsymbol{z}$ coordinates? Well, let's see. So, what we are interested typically is in the pole locations, ok. So, we'll compare the characteristic equation of this guy to the characteristic equation in $\boldsymbol{z}$, characteristic equation of the system written down in the $\boldsymbol{x}$

coordinates to the characteristic equation of the system written down in the $\boldsymbol{z}$ coordinates, ok.

So, let's see. So, if I write down the characteristic equation for the system in the $\boldsymbol{z}$ coordinates I'll have $|\boldsymbol{\lambda I} - \boldsymbol{P}^{-1}\boldsymbol{AP}|$. So, this will be I can write this down as $|\boldsymbol{\lambda P}^{-1}\boldsymbol{P} - \boldsymbol{P}^{-1}\boldsymbol{AP}|$, I am just multiplying by $\boldsymbol{P}$ and its inverse.

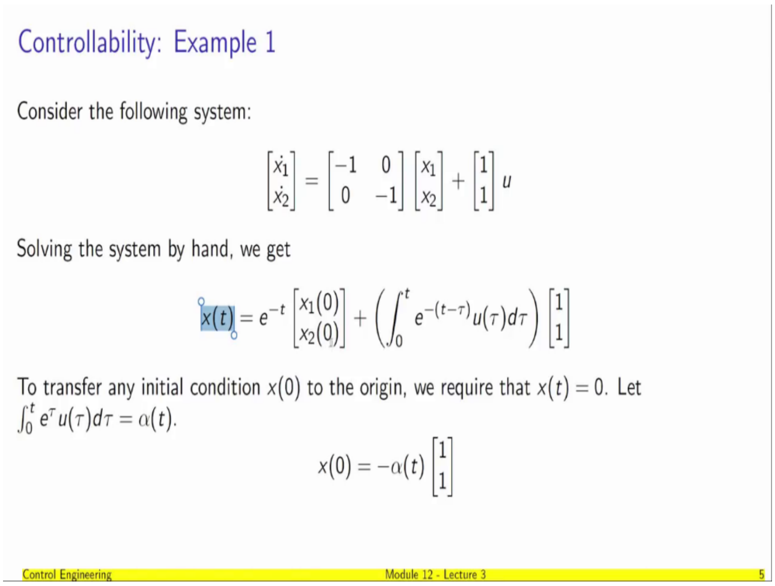
So, this will be $|\boldsymbol{P}^{-1}(\boldsymbol{\lambda I} - \boldsymbol{A})\boldsymbol{P}|$, where $\boldsymbol{I}$ is identity matrix times $\boldsymbol{P}$. I am just using some basic properties of matrix, the determinant just come out like this, $(\boldsymbol{\lambda I} - \boldsymbol{A})\boldsymbol{P}$. So, what am I left with say this guy cancels with this guy and I am just left with $|\boldsymbol{\lambda I} - \boldsymbol{A}|$.

So, if I equate this to 0, I get the poles. So, which means that if I write this down in summary $|\boldsymbol{\lambda I} - \boldsymbol{P}^{-1}\boldsymbol{AP}|$ is $|\boldsymbol{\lambda I} - \boldsymbol{A}| = \boldsymbol{0}$. So, the characteristic equations are the same which means that the poles here and the poles here would exactly be the same.

So, almost like nothing changes, right. So, I am exactly dealing with the same system. I am just doing a little bit of coordinate transformations. And we all would have done coordinate transformation in some math course- vector calculus, where we transform from a rectangular to a polar coordinate, cylindrical coordinates and nothing changes in the system. Something very similar is happening here, right.

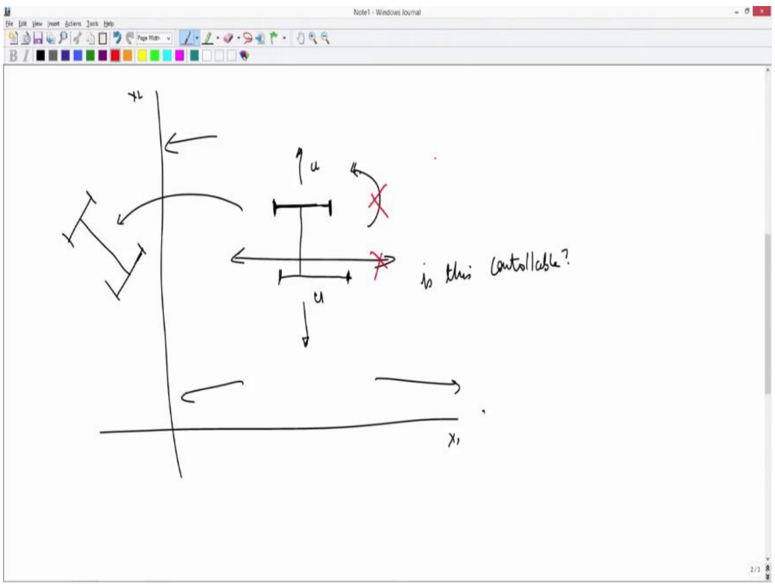
So, each and each of these forms the controllable, observable, and diagonal are useful in their own way. A bit of the analysis which we will do as we progress through this lecture, ok.

(Refer Slide Time: 07:27)



So, the next thing is controllability. So, let's again look at it with the help of an example, ok.

(Refer Slide Time: 07:41)



So, let's say I have say I am moving in a plane, just for simplicity, this is just for illustration and say I have a simplified model of a car and say all I could do is I can just control, but I can just move in this direction, right. I can just go forward I can go back, but steering is not allowed. So, this movement is not allowed, ok.

Now, with the application of input $\boldsymbol{u}$ in this direction I can move forward, right in this way, application of $\boldsymbol{u}$ here I can move backwards. Now, can I say that this simplified version of a car, the picture is not drawn very well, but can we see that this is actually controllable? Right. But I am just allowed to move this way or this way.

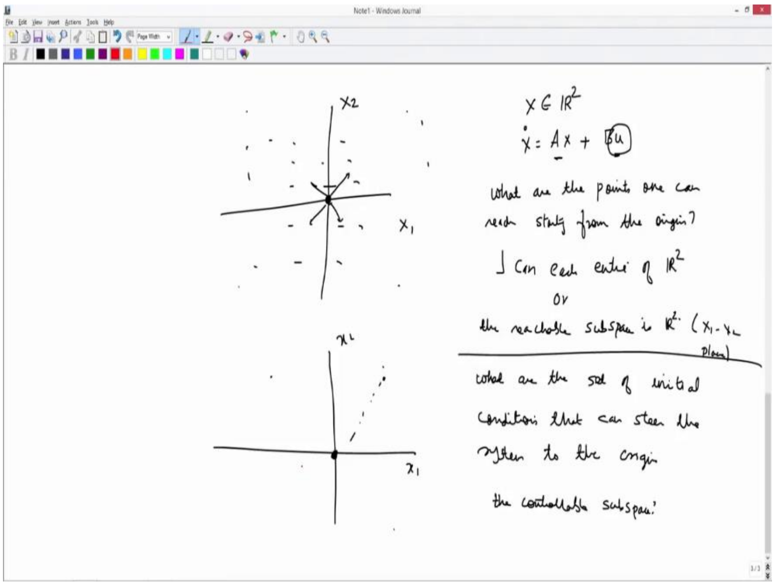
So, if you look at it from a normal car objective, whenever we drive a car what would we want is we want to drive across the entire domain that we have to go this way, you want to go, this way, this way, and so on, right. So, what should also be allowed is and I start from that position and I end up here, ok. But this control action u will not allow me to go to this position essentially because the steering is not allowed, ok.

Now, how do we qualitatively analyze this or when is a system controllable? The system here, with just the two $\boldsymbol{u}$'s which either goes in this direction of $\boldsymbol{x_2}$-positive direction or in the negative direction of $\boldsymbol{x_2}$, is this controllable or for complete controllability I also need the steering, right. So, that I can go from this position to actually this position, ok.

So, let's first understand this with the help of an example, right. So, I have a system well in the this is a very beautifully written down in the diagonal form $\dot{\boldsymbol{x_1}} = -\boldsymbol{x_1} + \boldsymbol{u}$, $\dot{\boldsymbol{x_2}} = -\boldsymbol{x_2} + \boldsymbol{u}$, ok.

So, these are essentially differential equations if I just try to solve those I get something like this that $\boldsymbol{x}(\boldsymbol{t})$ starting from initial conditions we'll have a solution like this, which essentially means I will ask myself a question given initial states $\boldsymbol{x_1}(\boldsymbol{0})$ and $\boldsymbol{x_2}(\boldsymbol{0})$ what are the states $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, I can reach after time $\boldsymbol{t}$, ok.

(Refer Slide Time: 10:43)



So, if I again were to draw a graph of this, so let's say I am moving again in $\boldsymbol{R}^2$. Let's say $\boldsymbol{x_1}$ and say $\boldsymbol{x_2}$, ok. And say I am starting, for simplicity say I am starting from the origin, right. So, I have a system in $\boldsymbol{R}^2$ again of the form $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$.

Now, I would ask myself a question with any given u which is arbitrary unconstrained, but bounded what are the directions can I move? So, let's say I can move this direction, this direction, this direction, this direction, all $\boldsymbol{360°}$, right, right. So, this would mean that starting from origin I can reach any other point in the $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ plane.

So, if I ask a question what are the points one can reach starting from the origin? In the application with appropriate control. If I say that I can reach all of this space in the plane any point, then I can say that I can reach entire of $\boldsymbol{R}^2$ or the reachable subspace is $\boldsymbol{R}^2$, the entire $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ plane on both directions, ok.

Similarly, I can also say well starting from any other points or say what are the set of initial conditions that can steer the system to the origin. So, if I say if I start from here, so this is my $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ plane by the application of some control can I go back to the origin? Starting from here can I go back? Right and then, if I find the set of all points that will give me the controllable subspace, ok.

What is holistic about this origin? Nothing really. I can even say what are the set of initial conditions which I can reach this point or this point there is nothing really holistic about this or even this origin here.

So, the first thing reachability will tell me what are the points I can reach starting from a given point which in this case is the origin. Controllability would mean what are the points or what are the initial conditions that will steer me to the origin by application of proper control input. So, in the linear time-invariant case it turns out that these two are similar, they are exactly similar. Reachability would mean controllability and vice versa. If I could go from this point to this point, I can actually come back from this point back to the origin, ok.

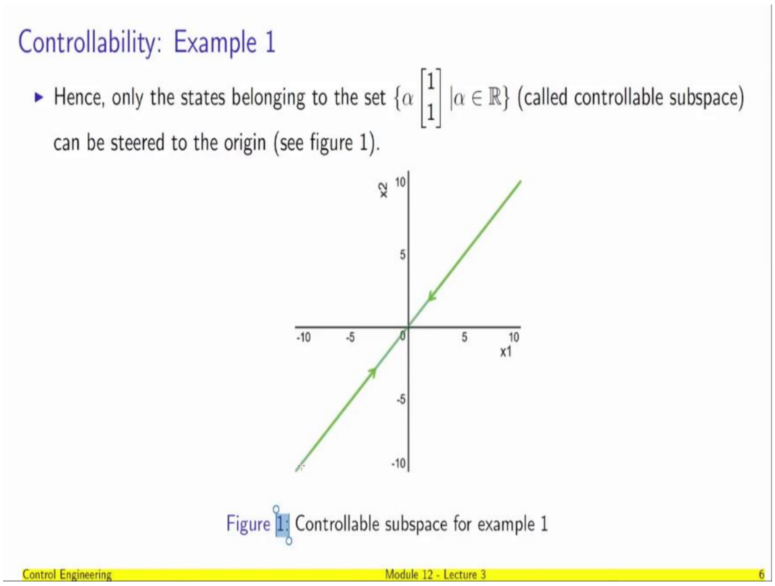
So, if I go back to the example here if I say well if I am not allowing the steering position the only directions I can move are this one that I can move in x_2 direction on both ways, but I cannot move in the x_1 direction. So, this movement is not allowed or it's not possible because I cannot steer. I cannot turn, it this way or even this way, ok. So, let's see more mathematically what these things mean.

So, if I say that a system like this starting from any arbitrary initial conditions what are the points it can reach in time t. Well, if I also additionally impose the condition that well I just want to reach the origin like I did here, right. In this example here, my control problem was to find out the set of points sorry the set of points which will steer me to the origin.

So, let's say here that $x(t)$ is actually the origin, ok. So, what will happen? So, I have a zero here $e^{-t}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ and this entire guy. So, it turns out if I just rearrange this terms properly that the set of initial conditions $x(0)$ is a set which looks like this $-\alpha(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is α? α is this thing, ok.

So, in this thing what is unknown? I know the initial condition $x_1(0)$, $x_2(0)$, I know the final condition which is the origin, I can compute these guys, only unknown is the input. So, does there exist an input, right which can steer the states to the origin, or what are the states if I just plot this down in on a piece of paper. So, it looks like this, right.

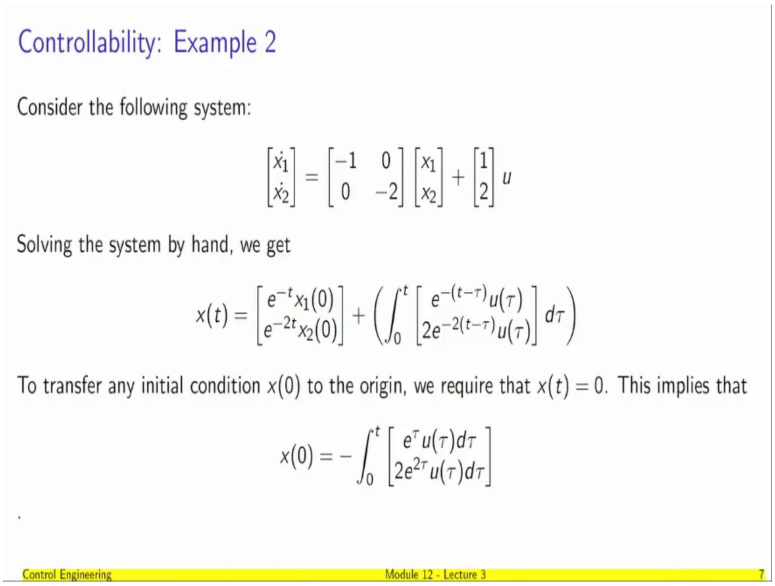
(Refer Slide Time: 16:29)



The points which can take me to the origin are only in the green line. For example, if I have start at a point say in the fourth quadrant or the second quadrant or any other point which is outside this green line I will not be able to reach the origin, ok.

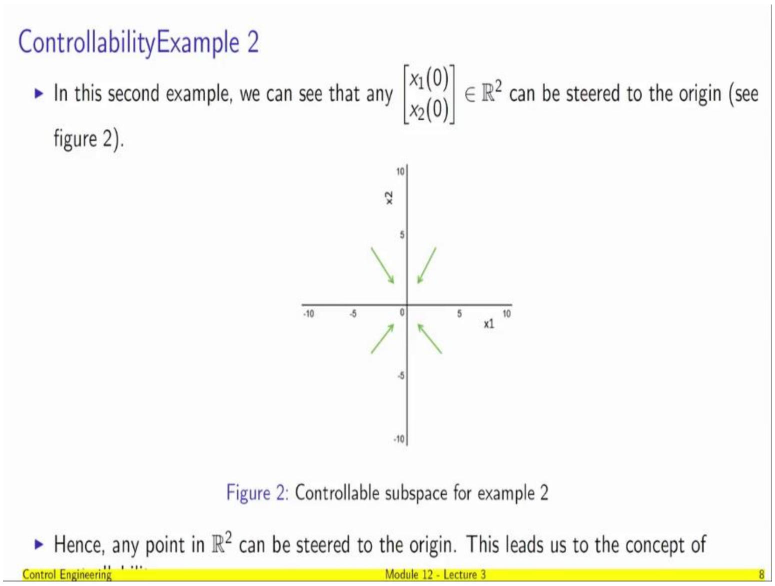
Now, let's see some other example or in this example, the controls controllable subspace is of dimension one. So, I guess cannot travel all points in the state space it is something similar to what was happening here, right. I was not allowing steering of the wheels, so I can only move in the positive or negative $\boldsymbol{x_2}$ directions, ok.

(Refer Slide Time: 17:12)



So, let’s see some other example. I just have again the system in the diagonal form with entries $\begin{bmatrix}\mathbf{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{-2}\end{bmatrix}$, control as $\begin{bmatrix}\mathbf{1}\\ \mathbf{2}\end{bmatrix}\boldsymbol{u}$, ok. Now, if I just write down again the solution of this say given certain initial conditions that I want to go to the origin, the equation would transform to something like this.

(Refer Slide Time: 17:35)

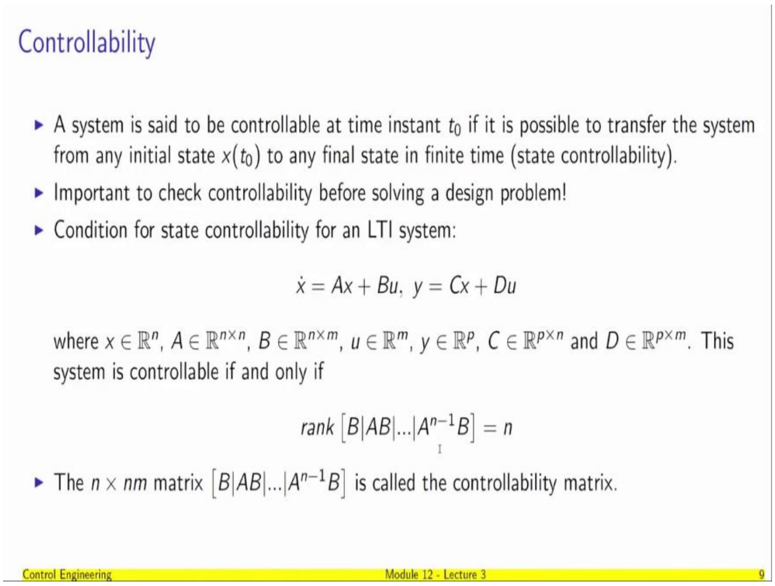


And if I plot what is the controllable subspace or what are the points starting which I can reach the origin it turns out to be the entire subspace. So, I can start any point in the first quadrant, second, third, or fourth, all will steer me to the origin or steer the system to the origin, right. So, any point in $\boldsymbol{R^2}$ can be steered to the origin, ok. So, this leads us nicely to define the concept of controllability.

Now, given any system would I keep computing these solutions all the time right; if I have a system which is of dimension 5, it may really be difficult for me to compute all the solutions, right. So, what we will see is given the properties of system which is $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$. The properties of the system are the $\boldsymbol{A}$ matrix and the $\boldsymbol{B}$ matrix. So, given this information on the $\boldsymbol{A}$ matrix and the $\boldsymbol{B}$ matrix can I say if the system is controllable or not?.

So, what is the definition of controllability first? So, based on what we had argued so far.

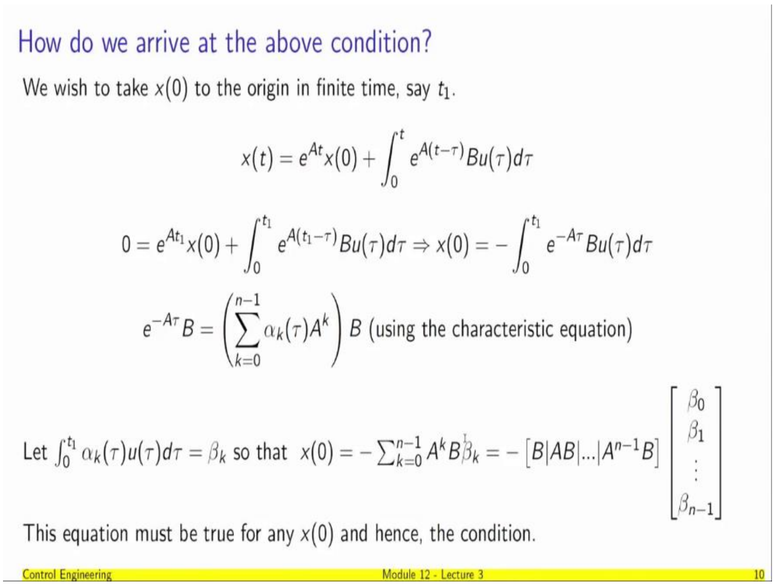
(Refer Slide Time: 18:37)



A system is said to be controllable at t naught, right, if it is possible to transfer the system from any initial state $\boldsymbol{x(t_0)}$ to any initial state in finite amount of time. So, the key here is I should be able to reach from point $\boldsymbol{A}$ to point $\boldsymbol{B}$ in some finite amount of time. Doesn't really help me much if I can say that I could travel from Bombay to Delhi in like an infinite amount of time, right. So, you want to travel or your car should be designed in such a way that you travel in finite amount of time, ok.

So, before I solve any control problem it is important for me to check if the system is controllable or not, ok. So, let's see how we do that. So, I am given a system in the state space form $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$, $\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$, ok. So, it will turn out that we will just shortly derive this result quickly that the system is controllable only if this matrix $[\boldsymbol{B}|\boldsymbol{AB}| \ldots . |\boldsymbol{A}^{\boldsymbol{n}-\mathbf{1}}\boldsymbol{B}|]$ it will be of dimension $\boldsymbol{n}\, X\, \boldsymbol{nm}$, if this rank is $\boldsymbol{n}$ then the system is controllable, right and this is both a necessary and sufficient condition, ok.

(Refer Slide Time: 19:56)



So, how do we derive this? We will quickly do this we won't really try to memorize a derivation we will just, but we will try to make use of this relation, the rank condition for controllability, ok.

So, I start with this equation $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$, the outputs really do not matter towards at the moment. So, we can just forget about this for a moment. So, $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$, the solution would look something like this. So, by now you also know how to compute $\boldsymbol{e^{At}}$, ok.

So, given any initial condition assume that I want to reach the origin in some time $\boldsymbol{t_1}$, I will ask when does the solution exist. What does it mean by the solution? Again, if I look at this expression I know the initial condition $\boldsymbol{x}(\mathbf{0})$, I know the final condition $\boldsymbol{t}(\mathbf{0})$, I just want to find out, I also know the matrix $\boldsymbol{A}$, I also know the matrix $\boldsymbol{B}$, all I want to find out is can I reach the origin starting from any arbitrary initial condition by application of some control. So, this is the unknown.

So, we will just use some properties of how to compute the $\boldsymbol{e^{At}}$, you can refer to the earlier lectures which we had on this. So, $\boldsymbol{e^{At}}$ can be written as some series in powers of $\boldsymbol{A}$ in this way, ok. So, what I do is I just plug this in over here. Again, in this expression I am just finding out if there exists a $\boldsymbol{u}$, the only unknown in this equation is the $\boldsymbol{u}$.

So, if I substitute for $\boldsymbol{e^{-A\tau}}$ over here what I end up is equation which looks like this $\boldsymbol{x}(\boldsymbol{0})$ is summation of this $\boldsymbol{x}$ of this summation term in the powers of $\boldsymbol{A}$ times some numbers $\boldsymbol{\beta_k}$. What is this $\boldsymbol{\beta_k}$? So, whatever is in the $\int_0^{t_1} \boldsymbol{\alpha_k(\tau)u(\tau)d\tau}$, I just called them $\boldsymbol{\beta_k}$. I can always compute this, right this is always some number, ok.

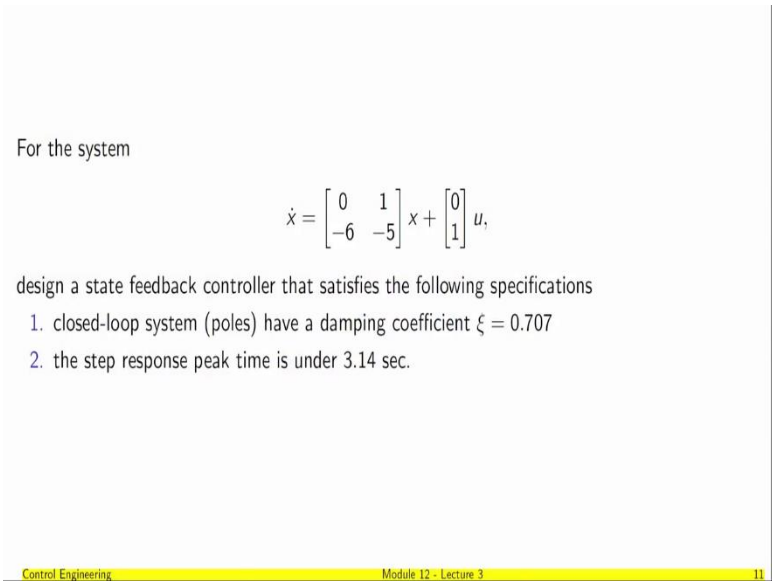
So, it will turn out that if I, you know I can also write this as $\boldsymbol{x}(\boldsymbol{0}) = \boldsymbol{B\beta_0}$. If I just take for $\boldsymbol{k = 0}$ what I will have is $\boldsymbol{A^0}$then which is identity. So, it will be $\boldsymbol{B\beta_0}$ here. For $\boldsymbol{k = 1}$ I will have $\boldsymbol{A^1B\beta_1}$, this is $\boldsymbol{AB\beta_1}$, and so on, until I reach $\boldsymbol{n - 1}$, ok.

Now, where does the unknown sit in here? The unknown sits in here in this $\boldsymbol{\beta}$s, right. The $\boldsymbol{u}$'s appear in $\boldsymbol{\beta}$. So, when does the solution exists? A solution will exist if and only if this matrix is invertible, right. $[\boldsymbol{B}|\boldsymbol{AB}| \ldots . |\boldsymbol{A^{n-1}B}|]$ is invertible, ok. So, this is exactly the rank condition which we are trying to derive here that the system is controllable if and only if this matrix is invertible, sorry, not invertible, if this matrix is of rank $\boldsymbol{n}$, ok. Similarly, a sorry not about the invertibility, but this matrix should be of rank $\boldsymbol{n}$, ok.

So, this is the condition which we will use. We may not necessarily you know remember this proof all the time, but what is important is to remember the condition that the rank of this matrix should be $\boldsymbol{n}$, ok.

So, once I know that the system is controllable, what do I do with it? How do I then control the system? So, when I was doing the design with the root locus or with the bode plot I always knew that well I can use either a lead compensator a lag compensator, or even more simply the gain adjustment. If the gain adjustment doesn't work, then I go to one of these lead or lag compensators depending upon the specifications of the problem. So, how does that translate in this case?

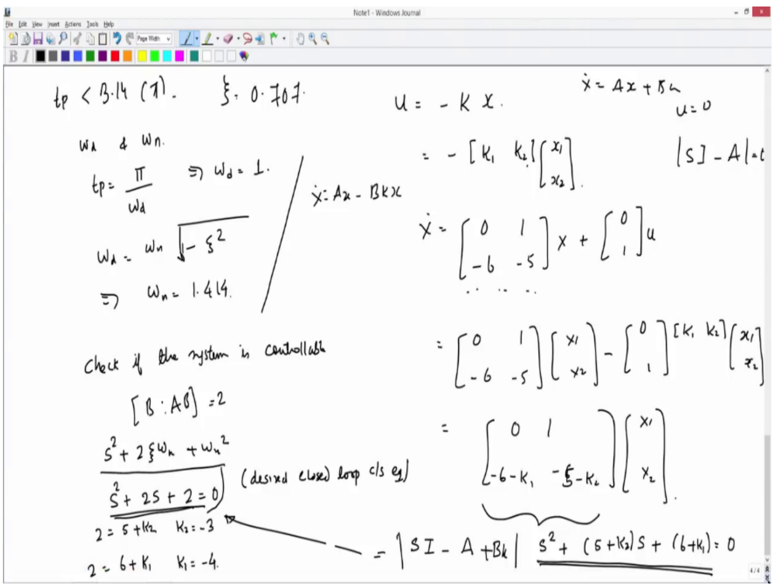
(Refer Slide Time: 24:00)



So, let's come to a problem. Now, let's say that for a given system which is of this form $\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-6} & \mathbf{-5} \end{bmatrix}$, can we design a controller $\boldsymbol{u}$ which I call as a state feedback controller which has the following specifications that the damping coefficient is 0.707 and the response the peak response time is under 3.14 seconds, ok.

So, there are a couple of methods which we could use in this. So, let's just write try to write down what the problem statement is for us.

(Refer Slide Time: 24:51)



So, first is the peak time is given to us the t_p, peak time should be less than 3.14 or π, ok. And it's also said that the ζ the damping coefficient should be 0.707, ok.

So, now based on this I need to compute what is ω_d and what is ω_n. So, if you remember the formulas for the peak time, so I think $t_p = \frac{\pi}{\omega_d}$ and this turns out that $\omega_d = 1$. And ω_d was related to ω_n; why are the damping coefficient with this formula, $1 - \zeta^2$ which will give us that $\omega_n = 1.414$, ok.

Now, what does this mean? Right. So, first is well, it says design a state feedback controller. So, a state feedback controller typically is of the form $u = -Kx$, or in this case well, so the system is of dimension 2, I will have a $K_1\ K_2$; x_1 and x_2, ok. Now, how does; what is the original system?

Original system is of the form $\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$; ok. So, I could just quickly check what would be the ζ and ω_n for the open-loop system, right. When $u = 0$ I just would say check it with the help of this characteristic equation $|sI - A| = 0$, right. And it is easy to check that the open-loop specifications are much further away from what is a desired closed-loop specification and therefore, we need this control, ok.

So, a first exercise which you could do is to check if the system is controllable. It means just quickly find the rank of B and AB, and this should definitely be equal to 2 and you can easily find out that the system is controllable. Also, it's easier to check because it is exactly in the controllable canonical form, ok.

So, let's try solving this problem, right. So, what happens? So, if I have a u of this form my closed-loop system looks like this $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $u = -Kx$. So, this thing here will be a $-[K_1 \quad K_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ which will look something like this, $\begin{bmatrix} 0 & 1 \\ -6 - K_1 & -5 - K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, ok.

So, now the closed-loop characteristic equation should be such that these poles or the eigenvalues should satisfy this condition, ok. Now, when does a second-order system satisfy this condition? Right. So, how should which mean, ok; this is what is desired in the closed-loop case, ok. So, how does $s^2 + 2\zeta\omega_n + {\omega_n}^2$ terms look like?

So, based on these two specifications of $\boldsymbol{\omega_n = 1.4}$, $\boldsymbol{\zeta = 0.7}$, this should look something like this $\boldsymbol{s^2 + 2s + 2 = 0}$, ok. Exactly I am just putting in the values of $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$ as desired, ok. Now, this is the desired closed-loop characteristic equation which will ensure that the closed-loop has a peak time of 3.14 or less and $\boldsymbol{\zeta}$ the damping coefficient of 0.707.

Now, what would I do? If this is desired now I compute this guy, $|\boldsymbol{sI - A + Bk}|$ as I would call it, where does this come from if I just say in a standard system $\boldsymbol{\dot{x} = Ax + Bu}$ with $\boldsymbol{u = -Kx}$. I can rewrite this as $\boldsymbol{\dot{x} = Ax - BKx}$. So, the characteristic equation of the closed-loop system would look like this. So, I would compare this $|\boldsymbol{sI - A + Bk}|$ with this guy, ok.

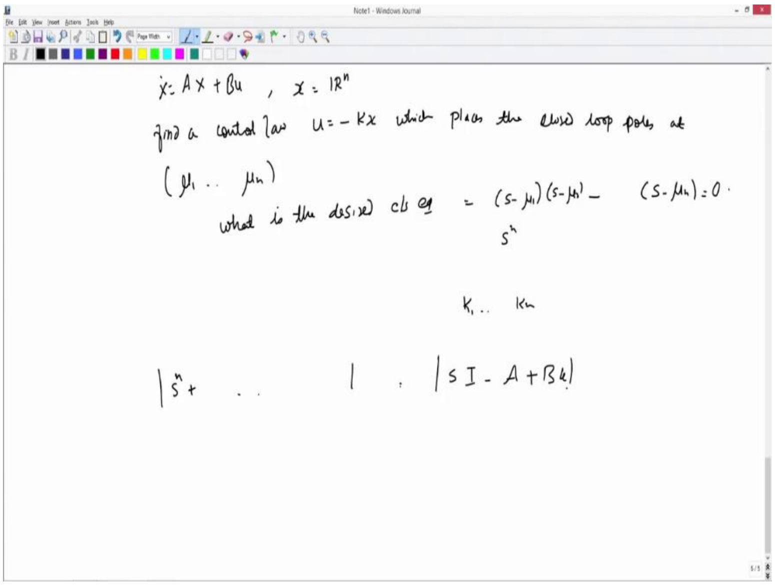
So, now I have an equation with two unknowns, ok. So, let's quickly write down what is the characteristic equation. $\boldsymbol{s^2 + (5 + K_2)s + (6 + K_1) = 0}$. So, I have to solve for $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$. How should this look like? Well, this should exactly look like this, ok.

So, this equation, this characteristic equation should be equal to this characteristic equation, right. So, therefore, I can just compare the coefficients, right which is $2 = \boldsymbol{5 + K_2}$ and $2 = \boldsymbol{6 + K_1}$ which means $\boldsymbol{K_2 = -3}$ and $\boldsymbol{K_1 = -4}$, ok. Let's assume. So, how do we do this, take or just check what are the closed-loop specifications, ok?

This will translate to an appropriate characteristic equation of the closed-loop; this is a desired one. On the right-hand side, I need to find out what are the gains $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ that will achieve this desired characteristic equation here, ok. And then I just write this down, I compare the characteristic equation on the right to the desired characteristic equation on the left, I just equate the coefficients, ok.

So, in some problems it's also the problem specifications could also be not in terms of $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$, but directly in terms of the desired closed-loop poles, ok.

(Refer Slide Time: 32:41)



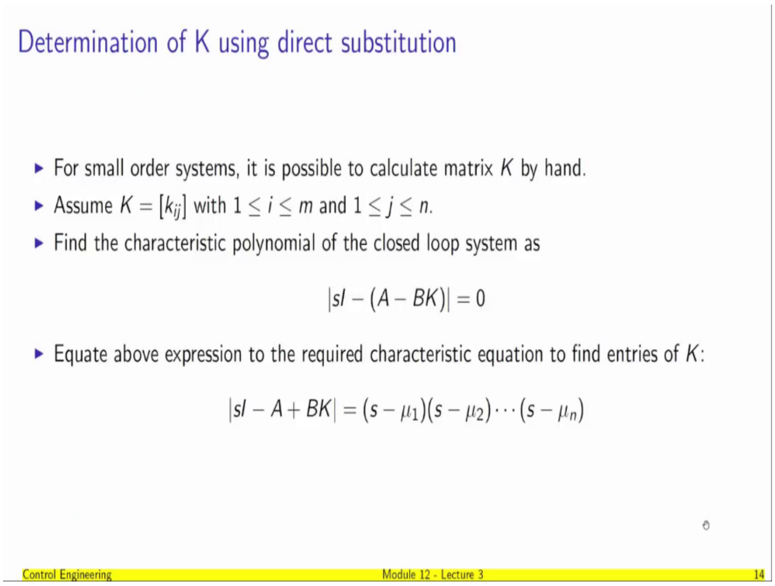
So, if you say that, given a system, $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$, this could be $\boldsymbol{x}$ is say in some $\boldsymbol{R^n}$, ok. Now, the problem could be of find a control law again $\boldsymbol{u} = -\boldsymbol{Kx}$ which places the closed-loop poles at say some values $\boldsymbol{\mu_1}$ till $\boldsymbol{\mu_n}$.

So, earlier we had the specification in terms of some system performance as a peak time, it also be in a term of times, in terms of the settling time or the rise time, damping coefficient and several other things. So, here I have just have specifications directly in terms of where the poles of the closed-loop system be, ok.

So, in this case, what I will do? Well, what is a desired characteristic equation? The desired characteristic equation would be I just look at this $\boldsymbol{s - \mu_1}, \boldsymbol{s - \mu_2} \ldots .\ \boldsymbol{s - \mu_n} = \boldsymbol{0}$, ok. This is I know all these numbers, right, I know all the $\boldsymbol{\mu}$ ones, so I have a polynomial in $\boldsymbol{s^n}$ with all the coefficients which are known to me. On the right-hand side, I would have unknowns in the terms of $\boldsymbol{K_1}$ till $\boldsymbol{K_n}$.

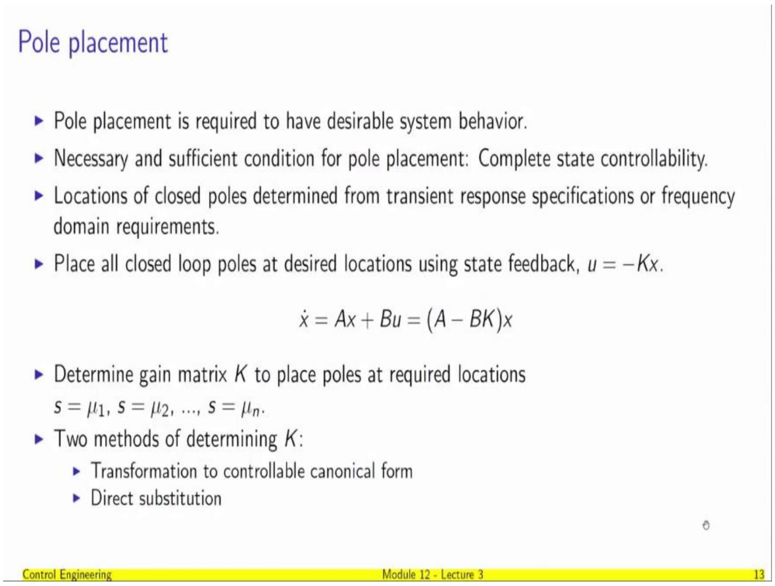
So, I just equate the characteristic equation on this one which is in powers of $\boldsymbol{s^n}$ plus, ok. We will see how to write this. And on the right-hand side, I should just equate this to the characteristic equation which I get by $|\boldsymbol{sI} - \boldsymbol{A} + \boldsymbol{Bk}|$, ok. Let's write this down little more formally now, ok.

(Refer Slide Time: 34:50)



So, this is really what. So, I have the desired poles which are $\boldsymbol{\mu_1}$ till $\boldsymbol{\mu_n}$ and I will have the characteristic equation as a polynomial in $\boldsymbol{s^n}$. On the right-hand side, I know $\boldsymbol{s}$, I know $\boldsymbol{A}$, I know $\boldsymbol{B}$, I just need to find out $\boldsymbol{K}$. So, I will have again a polynomial here of power $\boldsymbol{n}$ I just equate these two, as I did earlier, ok. So, well this is just straightforward, right, this is exactly what we did here.

(Refer Slide Time: 35:21)



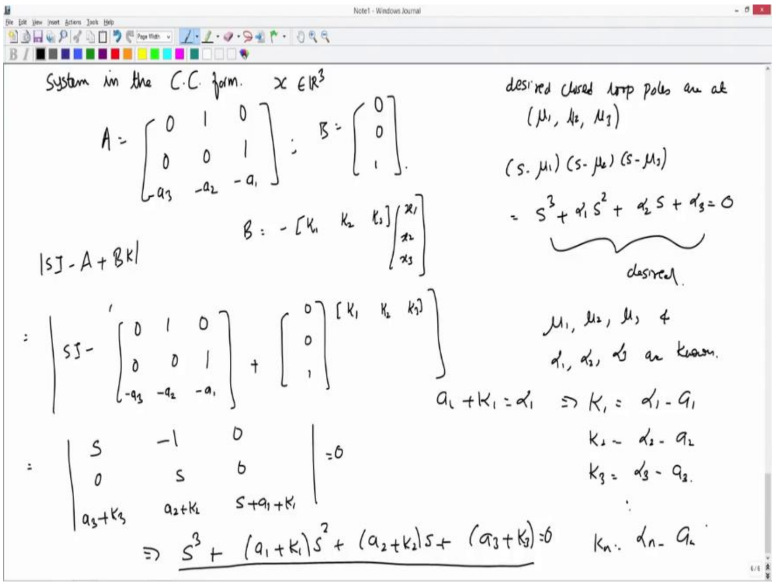
So, I will just quickly look at this. So, pole placement, so this exact this procedure which we derived or which we or where we design the controller, a state feedback controller

$\boldsymbol{u} = -\boldsymbol{Kx}$ to achieve certain performance specifications is called the pole placement technique, ok.

So, this is required to have the system desired behavior a necessary and sufficient condition is that the system should be completely controllable, ok. So, what do I do? Well, I know this $\boldsymbol{\mu_1}$ till $\boldsymbol{\mu_n}$, and I just equate these two I compare the coefficients and I and I arrive at the desired control law.

There is also other method from the controllable canonical form, ok. Let's first write down and then and then see what it means, ok.

(Refer Slide Time: 36:15)



So, let's say I have the system in the controller canonical form, ok. I am just being a little lazy to write. So, let's for the purpose of derivation let just say that $\boldsymbol{x}$ is in $\boldsymbol{R^3}$, ok. So, if in $\boldsymbol{R^3}$ my $\boldsymbol{A}$ matrix would look like this, it will be $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\boldsymbol{a_3} & -\boldsymbol{a_2} & -\boldsymbol{a_1} \end{bmatrix}$. The $\boldsymbol{B}$ matrix will have entries $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$, ok, ok.

So, let's first say that the desired closed-loop poles are at $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$, and $\boldsymbol{\mu_3}$, ok. So, my characteristic equation the desired one would be like this. So, let just write this down as say $\boldsymbol{s^3}$, and let's denote the coefficient in terms of some $\boldsymbol{\alpha}$, say this is like

$s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$. So, this is the desired in such a way that μ_1, μ_2, μ_3, and in turn α_1, α_2, α_3 are known.

So, this is this system is completely controllable, ok. So, just because it is just in the controllable canonical form. Now, I say I want to appropriately place the poles of the closed-loop system at μ_1, μ_2, and μ_3. So, what is the control law $B = -\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} [K_1 \quad K_2 \quad K_3]$ that will achieve this configuration in the closed-loop. So, the unknowns here are K_1, K_2, and K_3, ok.

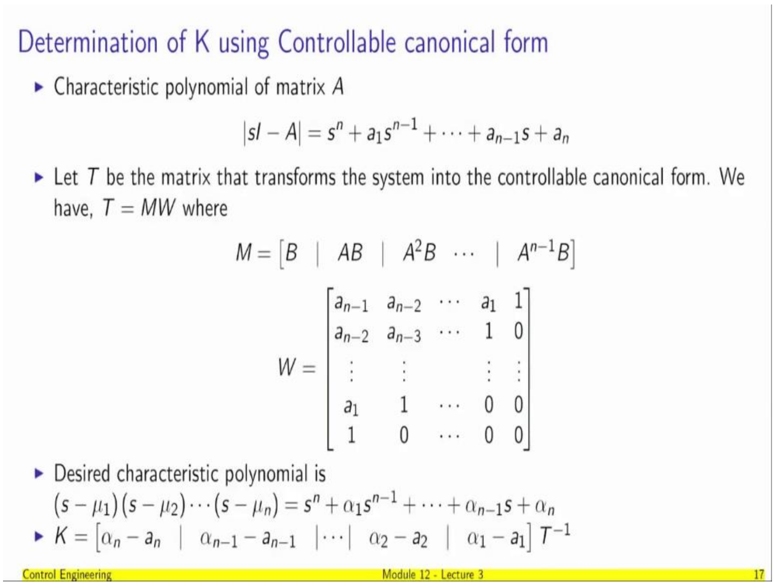
So, what is a characteristic equation? Right. The characteristic equation is $|sI - A + Bk|$. So, this would simply be sI minus, ok. What is A? $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [K_1 \quad K_2 \quad K_3]$; and this minus will show up here, ok.

So, if I just write this down I will get the following sI, ok, I will just write it completely in the matrix form. So, this matrix would be like this. We will have a s here. So, this is the entire, A, ok. So, this should actually be plus because I am looking at $|sI - A + Bk|$. So, this minus and minus will cancel out, ok.

So, I have $\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_3 + K_1 & a_2 + K_2 & s + a_1 + K_3 \end{bmatrix} = 0$. Or in other words, this means that the characteristic equation now is $s^3 + (a_1 + K_3)s^2 + (a_2 + K_2)s + (a_3 + K_1) = 0$.

Now, I want to compare this with this. Which essentially means that $a_1 + K_3 = \alpha_1$ or in other words, the unknown K_3 can be determined as $\alpha_1 - a_1$. This the a_1 is known to us. α_1 is also known to us. a_1 is from the given system matrix. α_1 comes as a result of the desired closed-loop poles. Similarly, then I can write $K_2 = \alpha_2 - a_2$, $K_1 = \alpha_3 - a_3$, ok.

(Refer Slide Time: 42:08)



So, we will just write this down little formally now, ok. So, I have the characteristic polynomial of this form and if the system is not already in the controllable canonical form I will use the transformation, right the same thing $\boldsymbol{P^{-1}AP}$ or in this case I call it $\boldsymbol{T}$, it will be $\boldsymbol{T^{-1}AT}$, where I compute the transformation $\boldsymbol{T}$ in the following way. $\boldsymbol{T} = \boldsymbol{MW}$, where $\boldsymbol{M}$ and $\boldsymbol{W}$ are completely. $\boldsymbol{M}$ is just the controllable the controllability matrix, $\boldsymbol{W}$ comes from the entries of the $\boldsymbol{A}$ matrix in the following way.

I will not tell you how to do this, but these are very standard texts. So, once you have them in the controllable canonical form then $\boldsymbol{K}$'s are simply computed this way, right $\boldsymbol{\alpha}$'s and you can just compute. So, $\boldsymbol{\alpha}$'s are known to us, $\boldsymbol{a}$'s are unknown to us, so I can directly compute what is the $\boldsymbol{K}$ matrix. So, what we have seen in this edition is the definition of controllability, to check if the system is completely controllable or not and if the system is completely controllable how do I exactly do the control law.

So, the only control technique which we have learnt here is what is called as the pole placement. There are other formulas is typically called the Ackermann's formula. I will skip to those details, but what is important here is to know the concept. And once you know the concept things the even the Ackermann's formula is kind of quite easy to derive.

And you could also do the design directly by MATLAB. But before you do ask MATLAB to compute the gain matrix, the $\boldsymbol{K}$ matrix for you, it's very advantageous for you to know what is the; what is the procedure to compute the gain $\boldsymbol{K}$, right, ok.

So, in the next lecture we will see about what to do if the system is not controllable, is there any hope, and also to learn the concept of observability because so far we have been using things which says that the controller is of the form $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$. Who tells us what the $\boldsymbol{x}$ is? Is it really trivial? If $\boldsymbol{x}$ is not known to us completely say there are $\boldsymbol{n}$ states I can only measure two states out of those $\boldsymbol{n}$, is there any hope? So, these are the two things which we will quickly run through in the next lecture.

Thank you.