Control Engineering

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Module - 12
Tutorial - 2
Lecture - 53

State Space Solution and Matrix Exponential

Hi everyone, today we'll be looking at how to solve for the states in State Space System. And in the process we'll also introduce what is called Matrix Exponential and we'll see how to solve for the Matrix Exponential.

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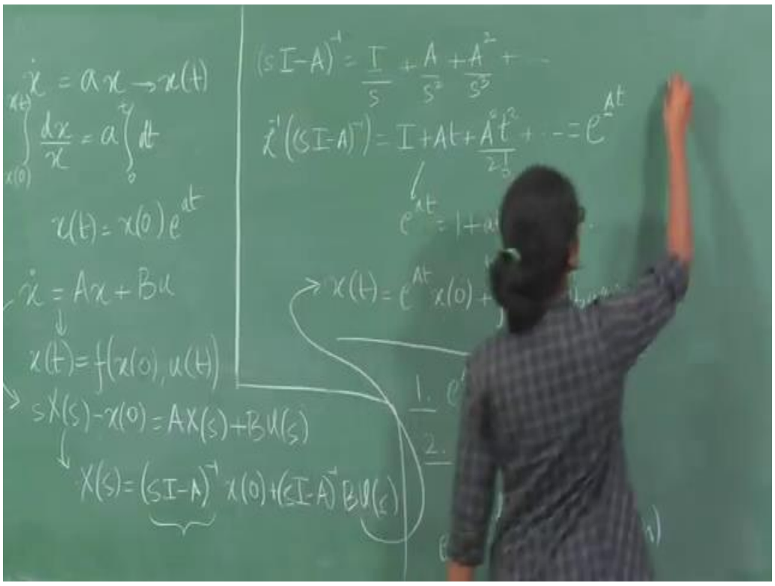

So, let's start with a simple scalar system $\dot{x} = ax$. Now, if we want to get a closed form expression for $x(t)$ then what we do is just integrate this. So, we have a $\int_{x(0)}^{x(t)} \frac{dx}{x} = a\int_0^t dt$. So, you solve this and you get $x(t) = x(0)e^{at}$.

So, this is simple because it's a scalar homogeneous system without any inputs. So, given a general LTI system $\dot{x} = Ax + Bu$ we want to arrive at a closed form expression for $x(t)$ as some function of the initial states $x(0)$ and the input $u(t)$ and of course, this system matrices.

Now, how do we go about doing this is to take the Laplace transform of the state space representation. So, we get $\boldsymbol{sX(s) - x(0) = AX(s) + BU(s)}$. And we all know we can solve for $\boldsymbol{X(s)}$ as $\boldsymbol{(sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)}$.

Now, to take this back to the time domain we take the inverse Laplace transform. In order to do that we need to know what the inverse Laplace transform of this function here is. So, we use this expression that. Now, you can check for yourselves that this is true because if you pre multiply by $\boldsymbol{(sI - A)}$ then you get identity. So, using this $\boldsymbol{L^{-1}\left((sI - A)^{-1}\right) = I + At + \frac{A^2t^2}{2!}+....}$

Now, this is similar to the scalar exponential expansion which is $\boldsymbol{e^{at} = 1 + at + \frac{a^2t^2}{2!}+.....}$ So, this is what we call the matrix exponential $\boldsymbol{e^{At}}$. Note that $\boldsymbol{A}$ is a matrix. So, using that and from here we say that $\boldsymbol{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}\, BU(\tau)d\tau}$.

Now, this $\boldsymbol{x}$ this the second part of this expression we get by using the property that multiplication in the s domain corresponds to convolution in the time domain. So, this is the closed form expression for your state $\boldsymbol{x(t)}$ as a function of your input, system matrices and the initial conditions. So, the question now is how do you compute this exponential, matrix exponential $\boldsymbol{e^{At}}$; because it's an infinite sum of matrices.

Now, the first method is quite obvious from the way we saw here that $\boldsymbol{e^{At}}$ is just the $\boldsymbol{L^{-1}\left((sI - A)^{-1}\right)}$. So, given a matrix $\boldsymbol{A}$ you can just compute $\boldsymbol{(sI - A)^{-1}}$ and take the inverse Laplace transforms of the elements of the matrix and you get the matrix exponential that is one method.

The second method is by using a diagonalization. So, here we assume that the matrix $\boldsymbol{A}$ has a distinct Eigen values in which case it can be written as $\boldsymbol{A = PDP^{-1}}$; where your $\boldsymbol{D}$ is a diagonal matrix with entries $\boldsymbol{(\lambda_1, \lambda_2, ..., \lambda_n)}$, where these are the Eigen values of the $\boldsymbol{A}$ matrix. So, once you why we do this diagonalization is because we can see that if we express $\boldsymbol{e^{At}}$ in terms of as something times $\boldsymbol{e^{Dt}}$ then your $\boldsymbol{e^{Dt}}$ is easy to compute because it's a diagonal matrix.

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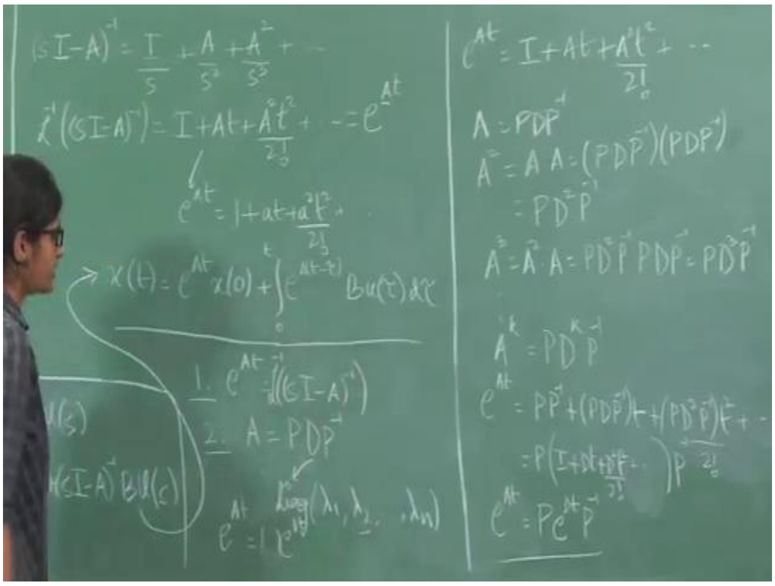

So, we see how we do that. You know that $\boldsymbol{e^{At}} = \boldsymbol{I} + \boldsymbol{At} + \frac{\boldsymbol{A^2t^2}}{\boldsymbol{2!}}+\ldots$ and now we know $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$. So, by the way $\boldsymbol{P}$ is any invertible matrix that accomplishes this diagonalization of the matrix $\boldsymbol{A}$. You can choose $\boldsymbol{P}$ to be the matrix of eigenvectors for a distinct eigen values yeah. So, $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$, $\boldsymbol{A^2} = \boldsymbol{A.A} = (\boldsymbol{PDP^{-1}})(\boldsymbol{PDP^{-1}}) = \boldsymbol{PD^2P^{-1}}$.

Similarly, $\boldsymbol{A^3} = \boldsymbol{A^2.A} = (\boldsymbol{PD^2P^{-1}})(\boldsymbol{PDP^{-1}}) = \boldsymbol{PD^3P^{-1}}$. So, in general we can say that $\boldsymbol{A^k} = \boldsymbol{PD^kP^{-1}}$. So, going back to the original matrix exponential $\boldsymbol{e^{At}}$ is $\boldsymbol{I}$ and we express the identity matrix as $\boldsymbol{P.P^{-1}}$ plus $\boldsymbol{At}$ and $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$ into $\boldsymbol{t}$ plus $\boldsymbol{A^2}$ which is $\frac{(\boldsymbol{PD^2P^{-1}})\,\boldsymbol{t^2}}{\boldsymbol{2!}}$ and so on (i.e., $\boldsymbol{e^{At}} = \boldsymbol{PP^{-1}} + (\boldsymbol{PDP^{-1}})\boldsymbol{t} + \frac{(\boldsymbol{PD^2P^{-1}})\,\boldsymbol{t^2}}{\boldsymbol{2!}}+\ldots$).

So, we just do some jugglery here. So, take $\boldsymbol{P}$ and $\boldsymbol{P^{-1}}$ on either side outside. So, you have $\boldsymbol{e^{At}} = \boldsymbol{P}\left(\boldsymbol{I} + \boldsymbol{Dt} + \frac{\boldsymbol{D^2t^2}}{\boldsymbol{2!}} + \cdots\right)\boldsymbol{P^{-1}}$. So, we basically have that $\boldsymbol{e^{At}} = \boldsymbol{P}\,\boldsymbol{e^{Dt}}\boldsymbol{P^{-1}}$. So, since these are diagonal matrix we can use this way to compute the matrix exponential $\boldsymbol{e^{At}}$. We’ll now look at the third method.

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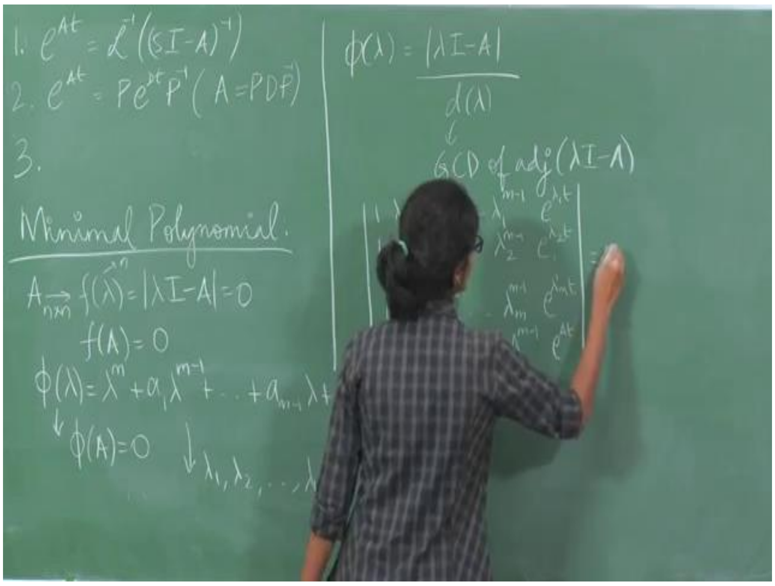

So, the third method for computing the matrix exponential is using the minimal polynomial. Now, given a matrix $\boldsymbol{A}$ from Cayley Hamilton theorem we know that the matrix satisfies it's characteristic equation. So, let's say $\boldsymbol{f(\lambda) = |\lambda I - A| = 0}$ is the characteristic equation of the matrix $\boldsymbol{A}$. So, we know that $\boldsymbol{f(A) = 0}$. Now, for an $n \, X \, n$ matrix $\boldsymbol{A}$ this polynomial is of order n, but there could be a polynomial of lower order for which the matrix satisfies.

So, let's say there is another polynomial $\boldsymbol{\varphi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m}$. So, this $\boldsymbol{m \leq n}$ and this polynomial is such that $\boldsymbol{\varphi(A) = 0}$. So, it's called a minimal polynomial; if the for the smallest $\boldsymbol{m}$ for which the matrix satisfies this equation $\boldsymbol{\varphi(A) = 0}$

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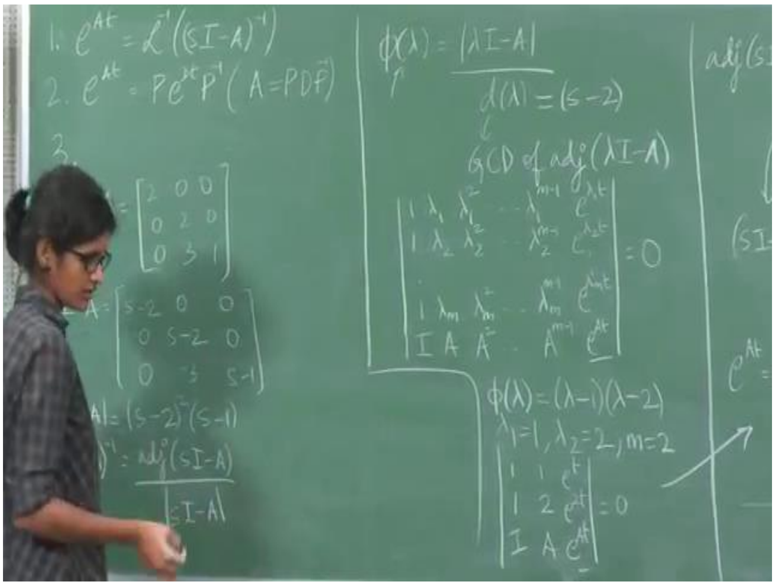

Now, for all computational purposes we can just use this $\boldsymbol{\varphi(\lambda)}$ equals $|\boldsymbol{\lambda I - A}|$ which is nothing but the characteristic equation divided by some $\boldsymbol{d(\lambda)}$ (i.e., $\boldsymbol{\varphi(\lambda)} = \frac{|\boldsymbol{\lambda I - A}|}{\boldsymbol{d(\lambda)}}$). Now, this $\boldsymbol{d(\lambda)}$ is the greatest common divisor of $\boldsymbol{adj(\lambda I - A)}$.

Now, why do we use this to find exponential of $\boldsymbol{At}$ is we use this formula called the Sylvester interpolation formula which is which basically says that, $\boldsymbol{\varphi(\lambda)}$ which is the minimal polynomial of a matrix $\boldsymbol{A}$ assume that it is in this form. And let's say its roots are $(\boldsymbol{\lambda_1, \lambda_2, \ldots \lambda_m})$. So, in that case your matrix exponential can be expressed using this formula. So, basically this whole thing is a matrix. You find the determinant equate it to $\boldsymbol{0}$ and one of the entries is $\boldsymbol{e^{At}}$. So, we get $\boldsymbol{e^{At}}$ in terms of all the lower powers of $\boldsymbol{A}$ until $(\boldsymbol{m-1})$.

So, we'll see how these 3 methods work with an example. So, we'll start with the first method which is just using inverse Laplace transforms. So, $\boldsymbol{sI - A}$ and minus $\boldsymbol{A}$ to find the inverse, $(\boldsymbol{sI - A})^{-1}$ you first calculate the determinant which turns out to be just the product of diagonal terms $|\boldsymbol{sI - A}| = (\boldsymbol{s-2})^2(\boldsymbol{s-1})$. So, and then your $(\boldsymbol{sI - A})^{-1} = \frac{\boldsymbol{adj(sI-A)}}{|\boldsymbol{sI-A}|}$ ok.

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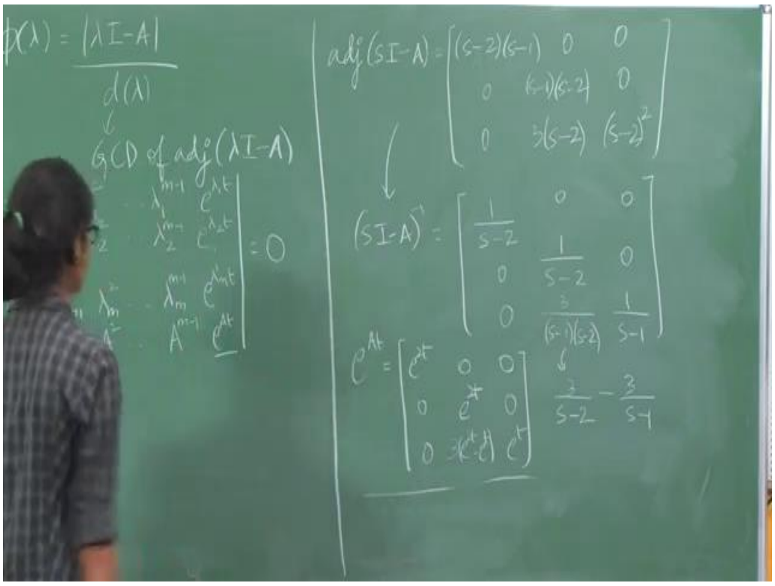

So, $\boldsymbol{adj}(\boldsymbol{sI}-\boldsymbol{A})$ and then you get $(\boldsymbol{sI}-\boldsymbol{A})^{-1}=\frac{\boldsymbol{adj}(\boldsymbol{sI}-\boldsymbol{A})}{|\boldsymbol{sI}-\boldsymbol{A}|}$ which is. And $\boldsymbol{e^{At}}$ is the inverse Laplace transform of this. So, $\boldsymbol{L^{-1}}\left(\frac{\boldsymbol{1}}{\boldsymbol{s-2}}\right)=\boldsymbol{e^{2t}}$. Similarly, you take an inverse Laplace transform of every element in the matrix. We know how to compute the inverse Laplace transform of this by expanding it in terms of partial fractions. So, that will just be $\boldsymbol{3}(\boldsymbol{e^{2t}}-\boldsymbol{e^{t}})$. So, that's the matrix exponential computed using inverse Laplace transforms.

We will solve it using the minimal polynomial way as well. So, to do that like I had mentioned you find the minimal polynomial first which is $\frac{|\boldsymbol{\lambda I}-\boldsymbol{A}|}{\boldsymbol{d}(\boldsymbol{\lambda})}$. Now, so we know that $\boldsymbol{adj}(\boldsymbol{sI}-\boldsymbol{A})$ is this. And clearly $(\boldsymbol{s}-\boldsymbol{2})$ is a common factor of this adjoint matrix. So, your $\boldsymbol{d}(\boldsymbol{\lambda})=\boldsymbol{s}-\boldsymbol{2}$ and we know $|\boldsymbol{\lambda I}-\boldsymbol{A}|=(\boldsymbol{s}-\boldsymbol{2})^{2}(\boldsymbol{s}-\boldsymbol{1})$.

So, we have $\boldsymbol{\varphi}(\boldsymbol{\lambda})=(\boldsymbol{\lambda}-\boldsymbol{1})(\boldsymbol{\lambda}-\boldsymbol{2})$. So, we know our $\boldsymbol{\lambda}$'s here $\boldsymbol{\lambda_1}=\boldsymbol{1}$, $\boldsymbol{\lambda_2}=\boldsymbol{2}$ and we use this formula to calculate $\boldsymbol{e^{At}}$. So, you can just substitute here. So, here you are $\boldsymbol{m}=\boldsymbol{2}$ because $\boldsymbol{\varphi}(\boldsymbol{\lambda})$ is of order $\boldsymbol{2}$. So, you have **1 1**.

So, you can solve this determinant equation the way you usually solve for determinants and get $\boldsymbol{e^{At}}$ in terms of $\boldsymbol{A}$ and these exponentials. You can cross check that you get the same answer as this. Now, going to method 2 which is diagonalization ok.

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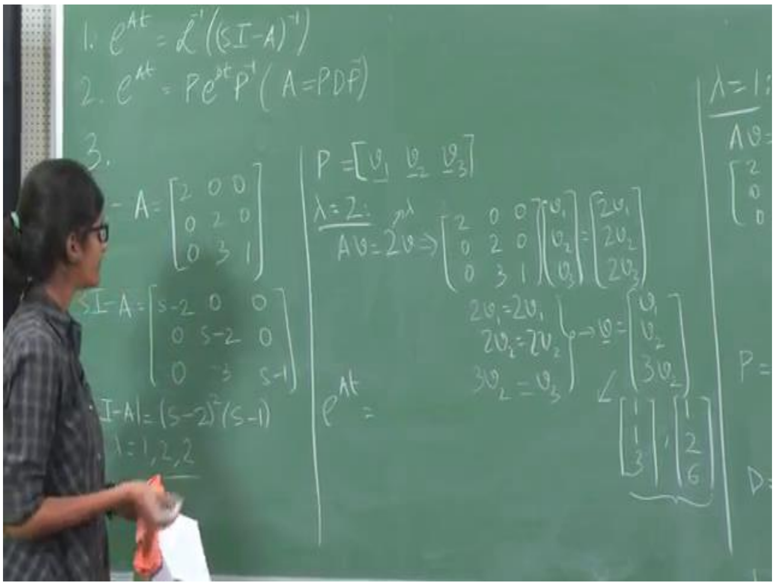

So, method 2 including diagonalization the matrix. So, firstly we find out the eigen values of the matrix $\boldsymbol{A}$. We have $|\boldsymbol{sI} - \boldsymbol{A}|$ here. The eigen values are **1**, **2** and **2**. There is a repeated eigen value which is **2**. Now, the matrix $\boldsymbol{P}$ which is used for diagonalization is a matrix of eigen vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$. So, we solve for these Eigen vectors for each Eigen value say for $\boldsymbol{\lambda} = \mathbf{2}$: $\boldsymbol{Av} = \mathbf{2}\boldsymbol{v}$ this is just $\boldsymbol{\lambda}$. $\boldsymbol{Av} = \boldsymbol{\lambda v}$ will give you. So, when you solve this you get $2\boldsymbol{v_1} = 2\boldsymbol{v_1}$.

So, here we see that our Eigen vector $\boldsymbol{v}$ is of this form $\begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ 3\boldsymbol{v_2} \end{bmatrix}$. So, there are no constraints on the quantity on the first and the second coordinates. So, you see that the null space for this eigen value which is $\boldsymbol{\lambda} = \mathbf{2}$ has dimension **2**. So, basically what I am saying is that for eigen value **2** you will have **2** eigen vectors, because **2** quantities are not constrained. So, you can just put in any values for $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. I will just choose. So, these are the two eigenvectors corresponding to the repeated eigen value **2**.

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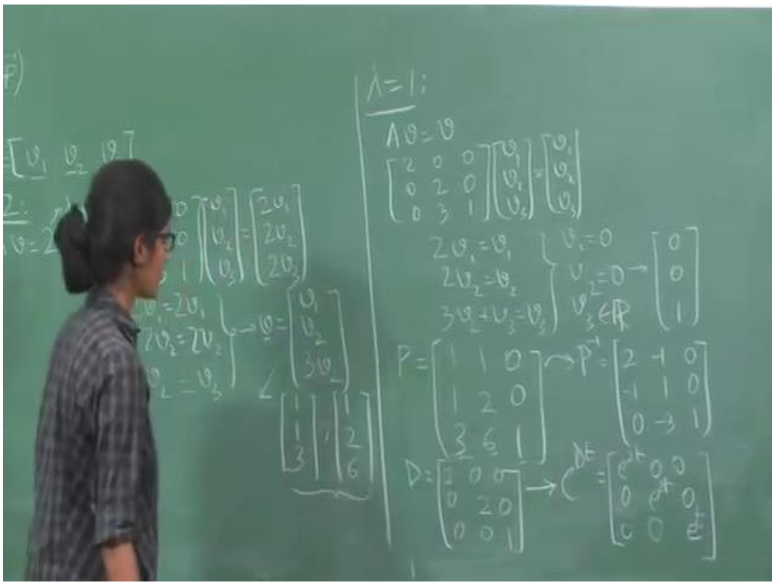

Now, we compute for $\boldsymbol{\lambda = 1}$. So, solving this equation you get that $\boldsymbol{v_1 = 0}$, $\boldsymbol{v_2 = 0}$ and $\boldsymbol{v_3} \in \boldsymbol{R}$. So, we choose our Eigen vector to be $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$. So, your matrix $\boldsymbol{P}$ finally, looks like $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{6} & \mathbf{1} \end{bmatrix}$.

So, from here you can compute $\boldsymbol{P^{-1}}$ as the regular adjoint by determinant. I will give you the numbers ok. So, once you have $\boldsymbol{P}$ and $\boldsymbol{P^{-1}}$ inverse you can your $\boldsymbol{D}$ is a nothing but a diagonal matrix with these Eigen values. So, note that the eigen values need to be in the same order as your eigen vectors. So, these two eigenvectors correspond to $\boldsymbol{\lambda = 2}$

So, $\mathbf{2}$ and $\mathbf{2}$ and this is for $\boldsymbol{\lambda = 1}$ and the remaining entries are all $\mathbf{0}$. So, once you have again here you can cross check that $\boldsymbol{PDP^{-1}}$ will give you $\boldsymbol{A}$ and from here your $\boldsymbol{e^{Dt}} = \begin{bmatrix} \boldsymbol{e^{2t}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e^{2t}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e^{t}} \end{bmatrix}$. So, that’s just the individual exponentials along the diagonal.

So, once you have $\boldsymbol{e^{Dt}}$ you compute $\boldsymbol{e^{At} = Pe^{Dt}P^{-1}}$. And we had obtained our $\boldsymbol{e^{At}}$ earlier on using the other two methods before. So, you can again check that it matches with the value that you get here. So, yeah so today we learnt about why we need matrix exponentials and how do you compute them using an example.

Thank you.