Introduction to Nonlinear Dynamics

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Module -03
Lecture-08
1-Dimension Flows, Flow on the line, Lecture 6

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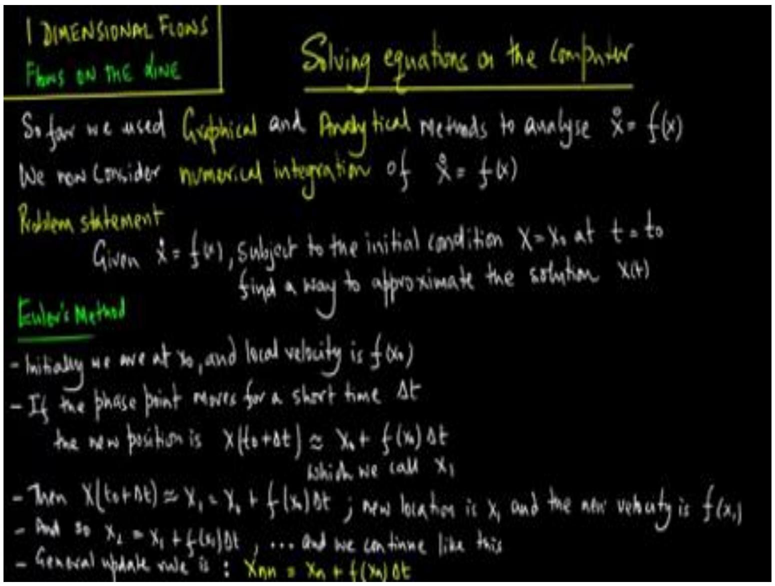

This lecture is on solving equations on the computer. So far we used graphical methods and analytical methods to analyse x dot=f(x). We now consider some numerical integration methods of x dot = f(x). Here is a general problem statement given x dot = f(x) subject to the initial conditions x = x0 at t = t0, find a way to approximate the solution x of t. Let us outline Euler's methods: Initially we are at x0 and the local velocity is f(x) not.

If the phase point moves for a short time delta t. The new position is x(t0) plus delta t is approximately is equal to x0 plus f(x0) times delta t, which we call x. Then x(t0) plus delta t is approximately x1which is equal to x0 plus f(x0) times delta t. So, the new location is actually x1. And so, the new velocity is f(x1) and so x2 = x1+ f(x1) times delta t and so we continue like this. So, the general update rule is then x of n + 1 =xn + f(xn) times delta t.

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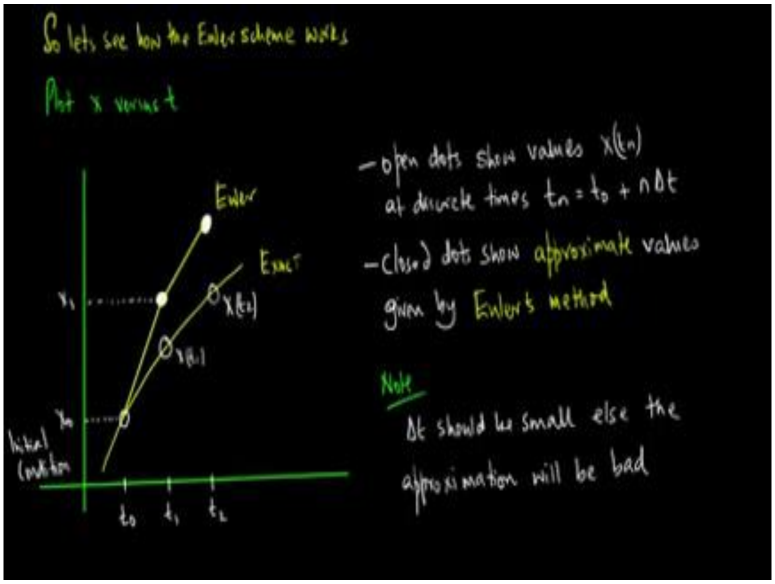

So let see how Euler's scheme actually works plot x of t versus t we identify t0, t1 and t2 that is the exact solution we identified x0 which is the initial conditions corresponding to t0 highlight x(t1) and x(t2). The open dot show values x(tn) at discreet times tn = t0 + n times delta t.

We then highlight values from Euler's scheme x1 and x2 that comes from Euler's scheme and we go ahead and connect the dots. The close dots show the approximate values given by Euler's method note that delta t should be small else the approximation will actually be bad. So here is a simple-minded representation of Euler's numerical schemes for approximating the solutions of a differential equations.

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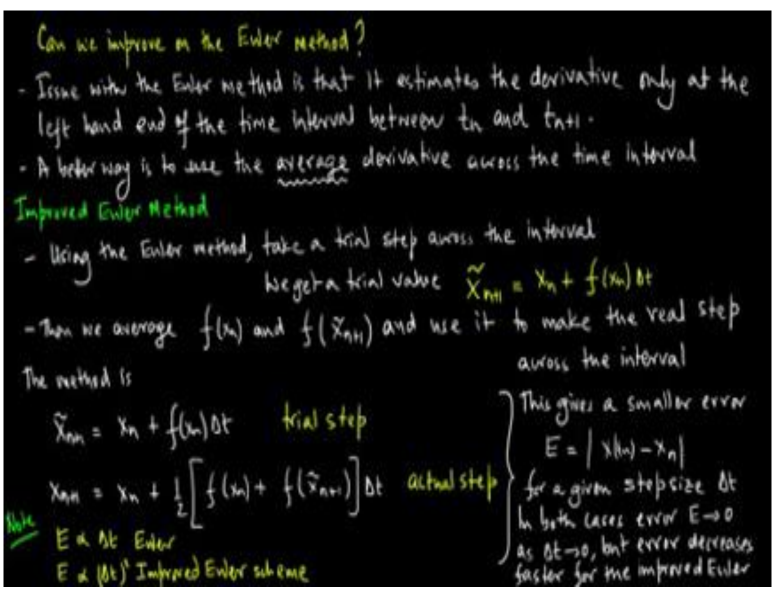

So it is natural to ask, we can actually improve on the Euler method. The issue with the Euler method is that it estimates the derivatives only at the left-hand end of the time interval between tn and tn +1. A better way is to use the average derivative across the time interval. So here is the improved Euler method: using the Euler method take a trial step across the interval and we will get a trial value x tilde n +1 = xn + f(x)n times delta t.

Then the average f(x)n and f(x) tilde n+1 and use it to make the real step across the interval. The method is as follows x tilde n+1 = xn + f(x)n times delta t, which is the trial step when xn+1 = xn + 1/2 of f(x)n + f(x) tilde n+1 times delta t and this is the actual step. This gives the smaller error e which is x(tn) - xn for a given step size delta t. Now in both the cases the error e tends to 0 as delta t tends to 0, but the error decreases faster for the improved Euler scheme.

Note that the error is proportional to delta t in the Euler method and it is proportional to delta t squared in the improved Euler scheme.

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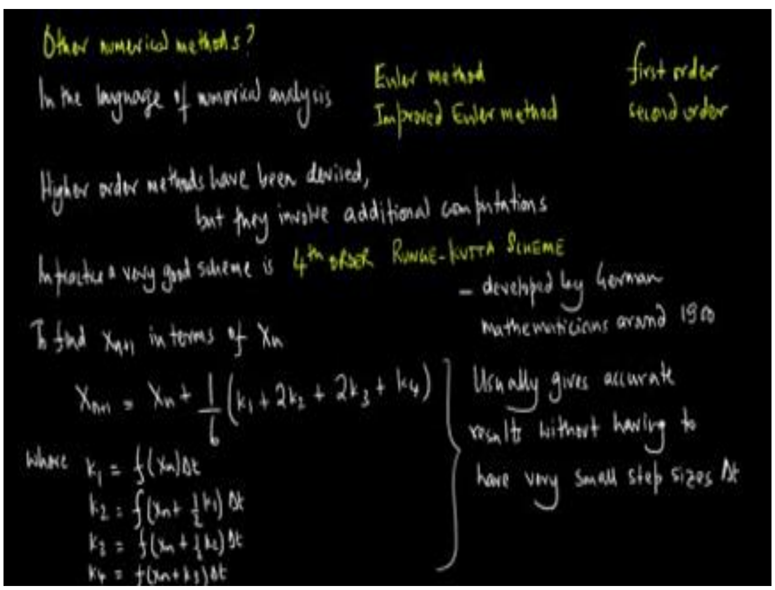

Now other numerical methods that one could use, now before mentioning them, we mention that in the language of numerical analysis the Euler method is a first order method, and the improved Euler method is a second order method. Higher order methods have been devised in the literature, but they actually involve additional computations. In practice a very good scheme is the fourth order Runge Kutta scheme. This was actually developed by German mathematicians working in approximately 1900.

So the objective is to find the xn+1 in terms of xn, so xn+1 = xn+ one upon six times k1 + 2 times k2 + 2 times k3 + k4, where k1 is f(xn) time delta t, k2 is f(x)n +1/2 k1 times delta t, k3= f(xn) + 1/2 k2 times delta t and k4 = f(xn) + k3 times delta t. Now usually this gives us very accurate results without having to rely on very small step sizes delta t.

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This lecture was centred around using the computer to solve our differential equations using numerical methods. We seen variety of analytical method to develop intuition about nonlinear equations of the form x dot = f(x). But it can sometimes rather difficult to develop intuition purely analytically because non-linearity may be just very, very strange. So, it is perfectly fare and it is perfectly sensible to actually use the computer to actually simulate the differential equation to actually develop some insights about how the equation would actually behave.

So to that end, we highlighted couple of numerical schemes. We started off with very simple Euler methods, we introduced you to the improved Euler method and we also mentioned that in practice a good compromise between accuracy and efficiency is actually obtained by the fourth order Runge Kutta method. Now interestingly these two mathematicians, German mathematicians actually devised the scheme in 1901 that was way before computers, were actually devised.

In fact, computers played a extremely crucial role in the popularisation of nonlinear dynamics and that was done by very famous paper by Murray 1963, where he had numerical computations of a model, that he has devised for atmospheric dynamics. To show that it had all kind of very strange behaviour. So, the lesson from this particular lecture is that numerical schemes and computer cannot be a substitute for an analysis. But in fact, there are excellent complements to the analysis.