Introduction to Nonlinear Dynamics

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Module -03
Lecture-06
1-Dimension Flows, Flow on the line, Lecture 4

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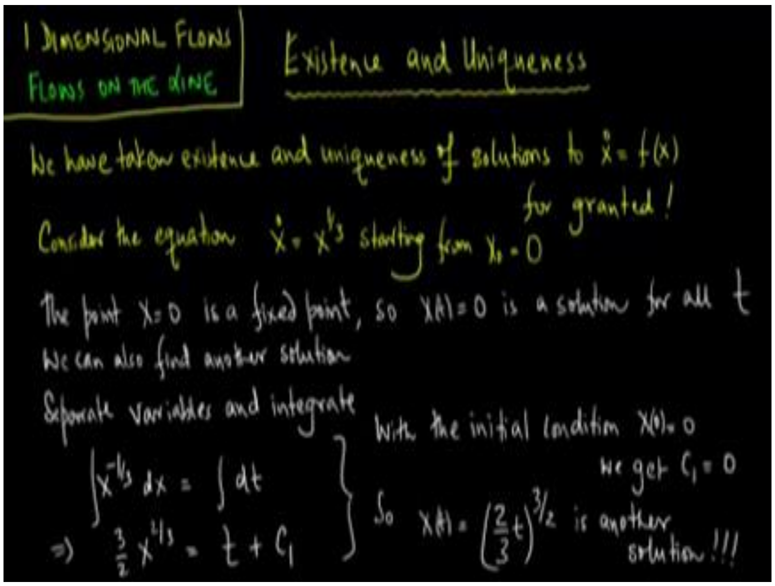

In this lecture we will first focus on existence and uniqueness of solutions. Now, so far we have actually taken the existence and the uniqueness of solutions to x dot = f(x) completely for granted. Consider the equation x dot = x(t)he one third starting from x(0) = 0. The point x = 0 is a fixed point. So x(t) = 0 is a solution for all t. We can also go ahead and find another solution.

We start by separating the variables and integrating to find, so evaluating the integral of x(t)he -1/3 and the integral of dt. We get 3/2x(t)he 2/3 = t + an arbitrary constant C1. So with the initial condition x(0) = 0, we get C1 = 0, so x(t) = 2/3 t to the power of 3/2 is another solution to this differential equation. So we have situation where we actually have two solutions to the differential equations. One is x(t) = 0 and the other is x(t) = 2/3 of t to the power 3/2.

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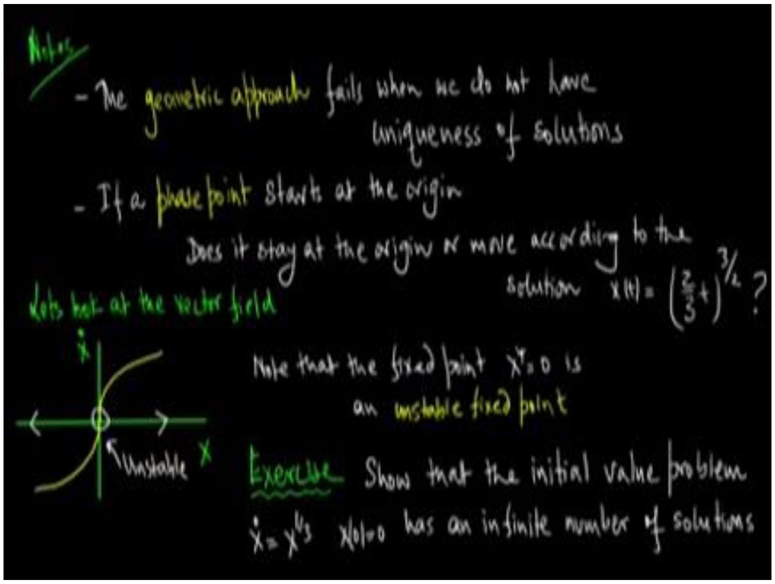

Here are some notes: The first point is that the geometric approach actually fails, when we do not have uniqueness of solutions to the differential equation. If a phase point starts at the origin, then does it stay at the origin for all time or does it move according to the solution, that we found that is x(t) = 2/3 of t to the power 3/2. Now let us take slightly closer look at the vector filed.

So, we now go ahead and plot x dot versus x that is the plot of x dot versus x. And we find that the fixed point would be repelling from both ends. So, note that the fixed point x star = 0 is in fact an unstable fixed point. Ok let us set out a much more challenging exercise for you. Can you go ahead and show that the initial value problem x dot = x(t)he 1/3, x(0) = 0 actually has an infinite number of solutions.

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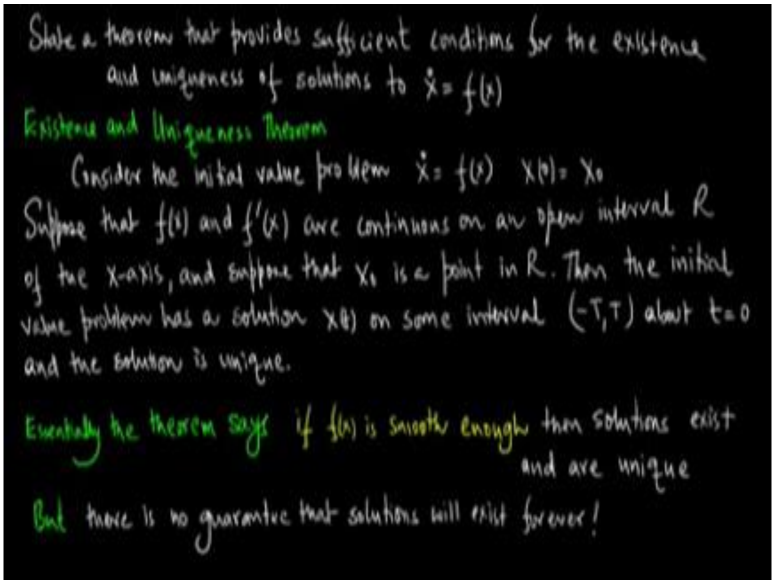

Now let us go ahead and state the theorem that provides sufficient conditions for the existence and the uniqueness of solutions to x dot=f(x). So let us call this the existence and the uniqueness theorem, consider the initial value problem x dot=f(x), where x(0) = x not and suppose that f(x) and f prime of x are continuous on an open interval R of the x axis and suppose that x not is a point in R.

Then the initial value problem has a solution x(t) on some interval -T to T about t =0 and the solution is unique. So, let us take a minute to just absorb this theorem. What does it really tell us and how can we translate it into more plain English. So essentially what the theorem says is that is if f(x) is smooth enough, then the solutions will exist and they will be unique.

But there is actually no guarantee that the solutions will exists for ever. Now this is a really important point to remember, so we talking about the existence and the uniqueness of the solution. But remember that at this point of time, we may not able to guarantee that the solution will actually be exist for ever, it may only exist for a short period of time.

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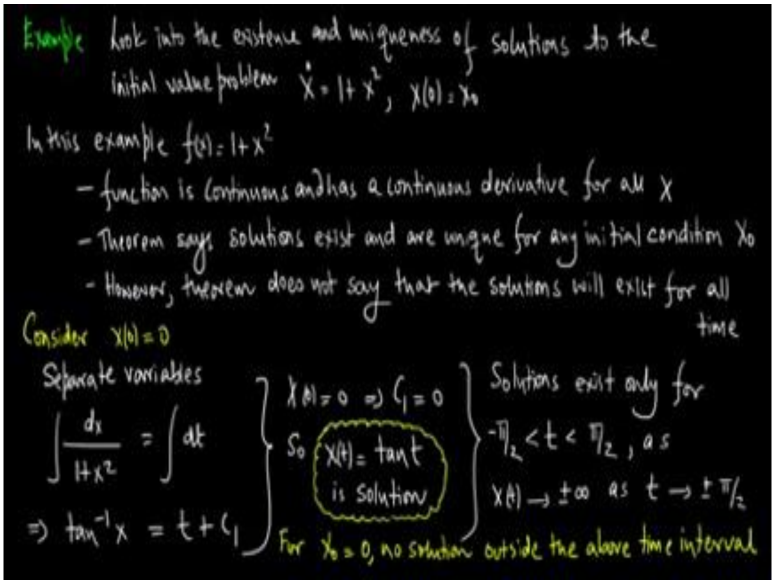

Now let us consider an example. Let us look into the existence and the uniqueness of solutions to the initial value problem x dot = 1 + x square where x(0) = x not. In this example f(x) is 1 + x square, so the function is continuous and has a continuous derivative for all x. The theorem says that solutions would exist and be unique for any initial condition x not. However, the theorem does not say that the solutions will exist for all time. So let us consider x(0) = 0, we separate the variables, so we get dx / 1+ x square.

We integrate that and we integrate dt, this gives us tan inverse of x = t + an arbitrary constant. With x(0) = 0. We get C1 = 0, so x(t) = tan t turns out to be a solution for the initial value problem. But the solutions will only be exist for t when t is actually sandwich between pi on two and - pi on two, as x(t) will tend + or - infinity as t tends to + or - pi on two, for x(0) = 0, there is in fact no solution outside the above the time interval. So essentially we have an example here, where the solutions exist but they only exist for a certain time interval.

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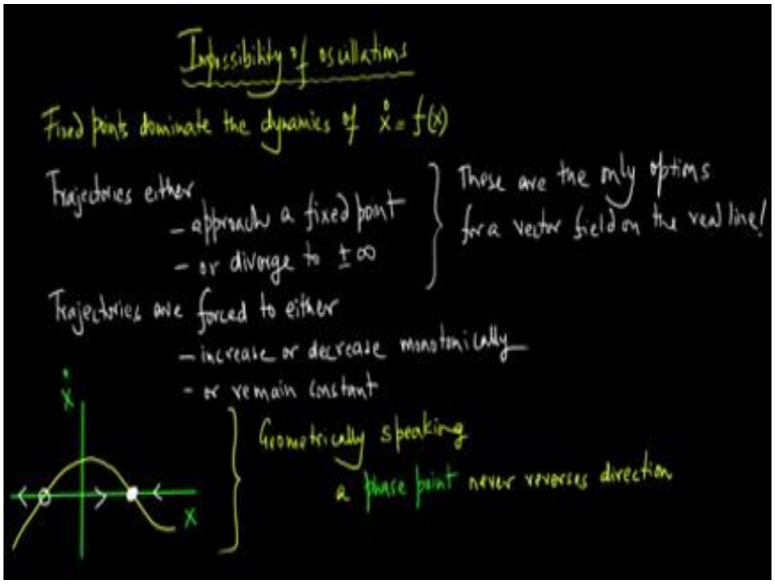

Our next sub heading is the impossibility of oscillations. By now we are quite familiar that fixed points dominate the dynamics of x dot=f(x). So trajectories either approach a fixed point or they diverge to + or - infinity. In fact these are the only options for a vector field on the real line. Trajectories are forced to either increase or decrease monotonically or actually just remain constant.

So geometrically speaking if you plot x dot versus x, that is an arbitrary function f(x) and we can identify the stable and the unstable fixed points. So geometrically speaking a phase point actually never reverses direction and this is one key point to note.

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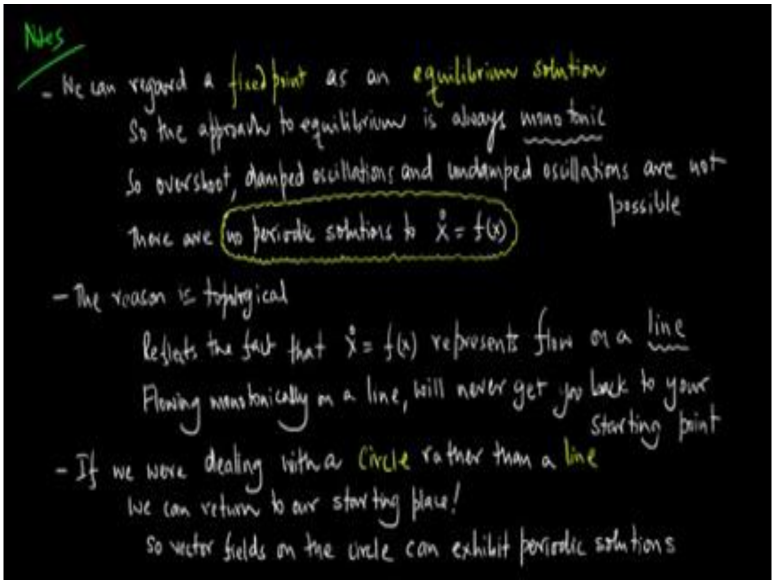

Here are some notes, we can regard a fixed point as an equilibrium solution. So, the approach to equilibrium is always monotonic. So, overshoot damped oscillations and undamped oscillations are not possible. So, there are actually no periodic solutions to x dot = f(x) and this is a really key point, so we highlight it. The reason in fact is topological, it reflects the fact that x dot = f(x) represents a flow on a line.

So flowing monotonically on a line will actually never get you back to your starting point. If we were in fact dealing with a circle rather than with a line then we can return to our starting place. So in that sense vector fields on the circle can actually exhibits periodic solutions.

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Now there were two main points that you wanted to make in this lecture. Point number one was about the existence and the uniqueness of the solutions. Hence, often a tendency to actually take existence and uniqueness little bit for granted, so to that end we produced two examples. In one examples, we showed that you could have multiple solutions and in the second example we showed that while the solution would exist it would not necessarily exist for all time ok.

So, I think this is more a question of being aware and being careful that actually have a differential equation. Please do spend a till bit of time thinking both about existence and uniqueness. The second point that we made was about the impossibility of oscillations. Now, essentially when you are dealing with one dimensional flows. If you have flow on the line then you are not really going to come back same place again.

So, the trajectories would either go off to plus minus infinity or would go towards a fixed point. Of course, if you dealing with flows on a circle you could have oscillations, but the main point to remember here is that you would not get oscillations in a one dimensional equation of the form x dot = f(x), as long as it was flowing on the real line.