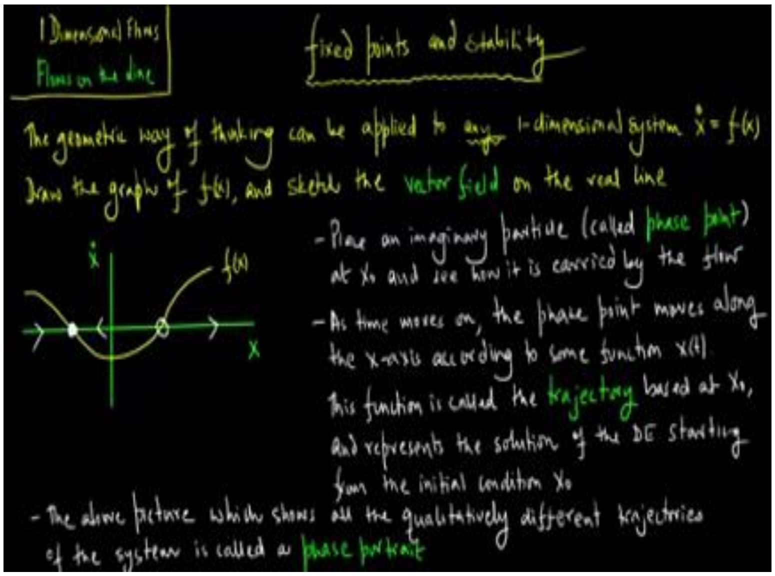
Introduction to Nonlinear Dynamics
Prof. Gaurav Raina
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module -02
Lecture - 04
1-Dimension Flows, Flow on the line, Lecture 3

(Refer Side Time: 00:01)

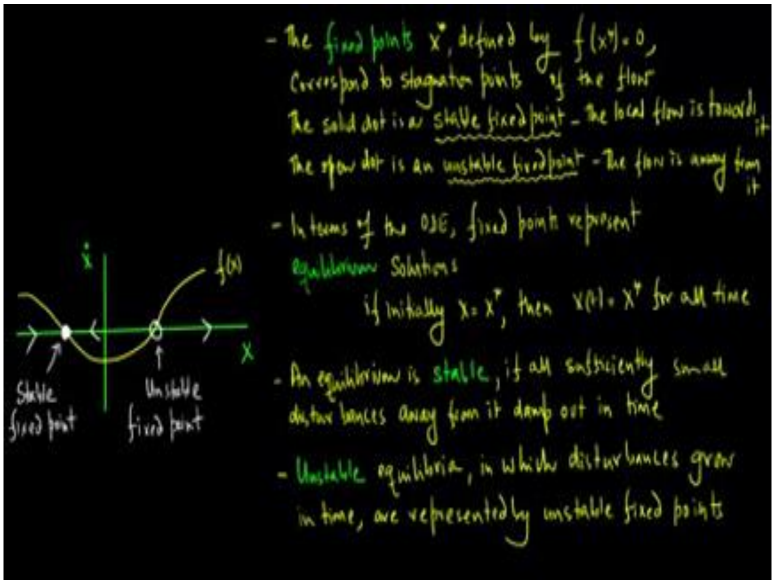


Now welcome back to one dimensional flows, where we are still dealing with flows on the line. The focus of this lecture will be on fixed points and stability. Now the geometric way of thinking can readably be applied to any one dimensional system x dot is equal to f of x. Now essentially you just need to draw the graph of f of x and sketch the vector field on the real line.

So, just as an example, x dot verses x and you plot any arbitrary function f of x may be, highlight the fixed points of the system and draw the direction of the flows. Now what we do is you place an imaginary particle which is called a phase point at x0 and see how it is actually carried by the flow. Now, as time moves on the phase point moves along the x axis according to some function x(t).

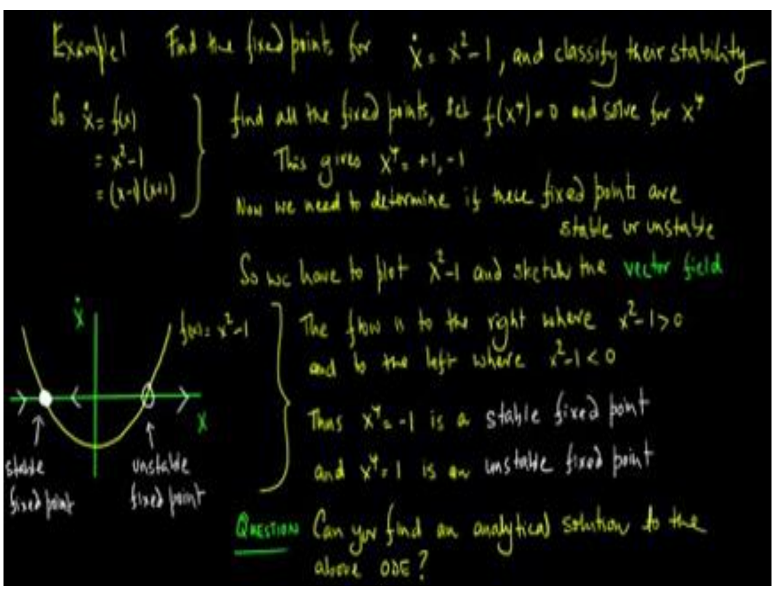
Now this function is called the trajectory based at x0 and represents the solution of the ordinary differential equations starting from the initial condition x0. The above picture which shows all the qualitatively different trajectories of the system is called a phase portrait.

(Refer Side Time: 02:04)



The fixed points x* defined by f of x* is equal to zero correspond to stagnation points of the flow. The solid dot is a stable fixed point and the local flow is towards the stable fixed point. The open dot is an unstable fixed point and the local flow is away from it. In terms of the ordinary differential equation fixed points represents equilibrium solutions, if initially x is equal to x* then x(t) is equal to x* for all t and equilibrium is stable if all sufficiently small disturbances away from it actually dumped out in time. Unstable equilibrium in which disturbances grow in time are represented by unstable fixed points.

(Refer Side Time: 03:32)

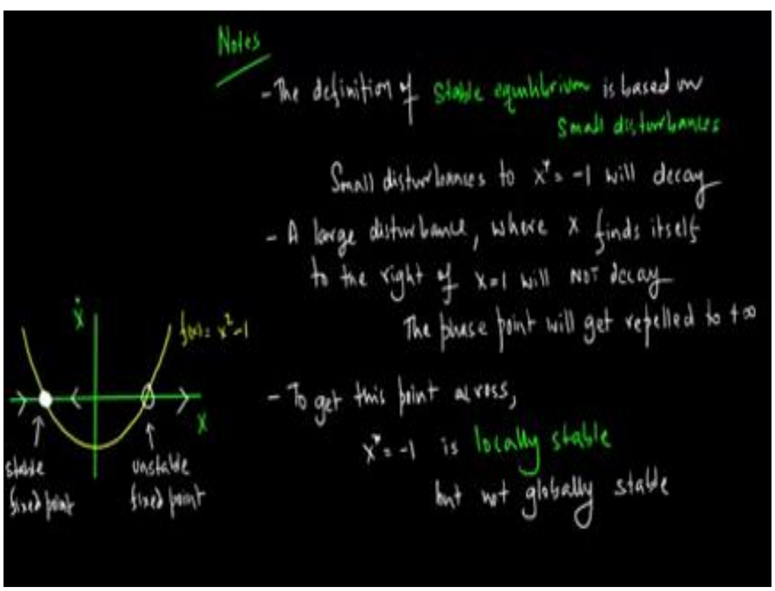


Now let us consider some examples, we talked about some theory, let us look at some examples now. So find the fixed points for x dot is equal to x square -1 and classify the stability of those fixed points. So x dot is equal to f of x, which is equal to x square -1, which is equal to x -1 times x + 1. So to find all the fixed points, we set f of x* is equal to zero and solve for x*.

This gives x* is equal to + 1 and -1 the algebra is straight forward. Now we need to determine if these fixed points are actually stable or unstable. So we have to plot x square -1 and sketch the resulting vector field. So we go ahead and plot x dot versus x and make a plot of f of x is equal to x square -1, highlight the direction of the flows and highlight the fixed points. The flow is to the right where x square -1 is greater than zero and to the left where x square -1 is less than zero.

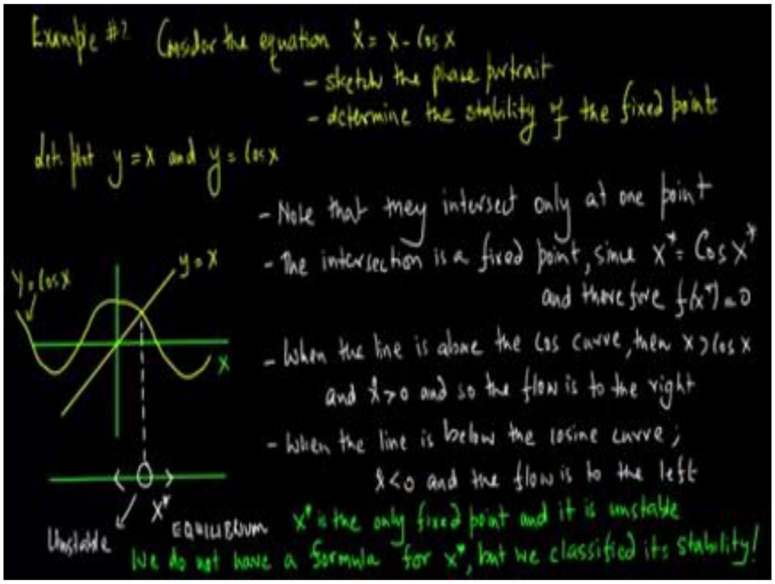
Thus x* is equal to -1 is a stable fixed point and x* is equal to one is an unstable fixed point. Now we go ahead and highlight the stable fixed point and the unstable fixed point in the diagram. Now here is a question for you can you actually find an explicit analytical solution to the above ordinary differential equation? Now note that we have we are able classify stability of fixed points without actually knowing the analytical solution. But do go ahead find, try and see an explicit analytic solution to the above ODEs.

(Refer Side Time: 05:53)



Now here are some notes the definition of stable equilibrium is actually based on the notion of small disturbances. So small disturbances to x* is equal to -1, will actually decay a large disturbance, where x finds itself to the right of x = 1 will actually not decay. In fact the phase point will get repelled towards plus infinity to get this point across x* is equal to -1 is locally stable, but not globally stable. In the sense, that we will not have convergence from any arbitrary initial conditions.

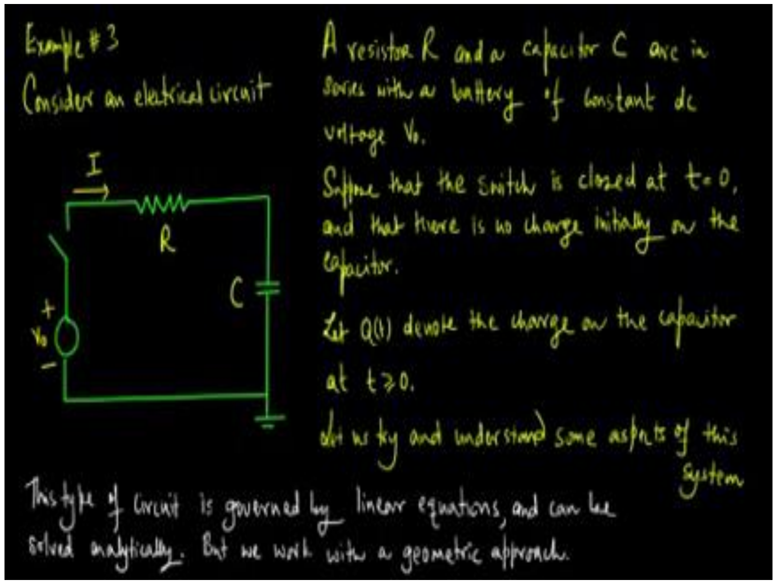
(Refer Side Time: 07:00)



Now let us consider another example, consider the equation x dot is equal to x - cos x, sketch the phase portrait and determine the stability of the fixed points. Now let us plot y = x and y = cos x and that is the curve for y = x and that is simple minded curve for y = cos x. Now note that they intersect only at one point. The intersection is a fixed point since x* is equal to cos x* and therefore f of x* is equal to zero.

When the line is above the cosecant curve, when x is greater than cos x and x dot is greater than zero and so the flow is to the right when the line is below the cosine curve and x dot is less than zero and the flow is to the left. Now, we go ahead and actually identify the fixed point x*, now x* is the only fixed point and it is unstable. We do not have a formula for x*, but we were still able to classify its stability.

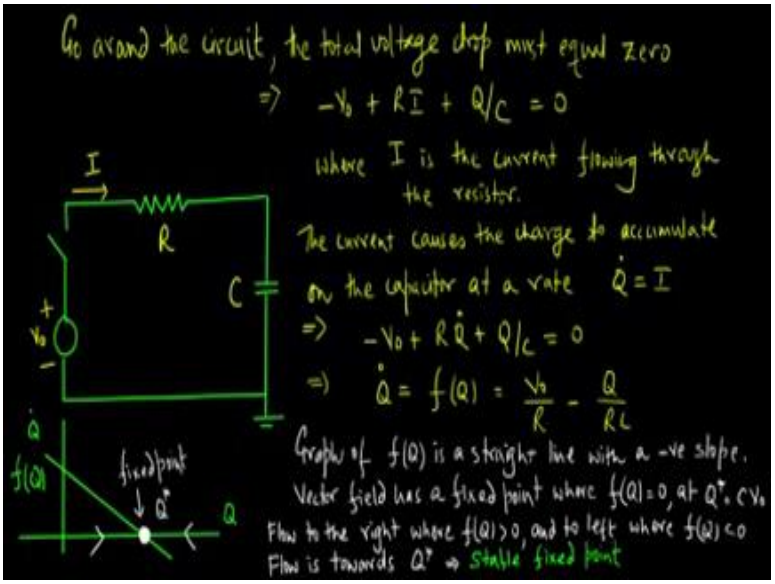
(Refer Side Time: 08:59)



Now let us look at our final example and consider an electrical circuit, so let us go ahead and actually make a diagram of our electrical circuit. We have resistor R and capacitor C which are in series with a battery of constant dc voltage V not. Now suppose that switch is closed at t is equal to zero and that there is no charge initially on the capacitor. Let Q of t denote the charge on the capacitor at the t greater than or equal to zero.

Now let us try and understand some aspects of this particular system. Now, this type of circuit is actually governed by linear differential equations and can actually be solved analytically. However, we work with a geometric approach to try and get some intuition about the about the above circuit.

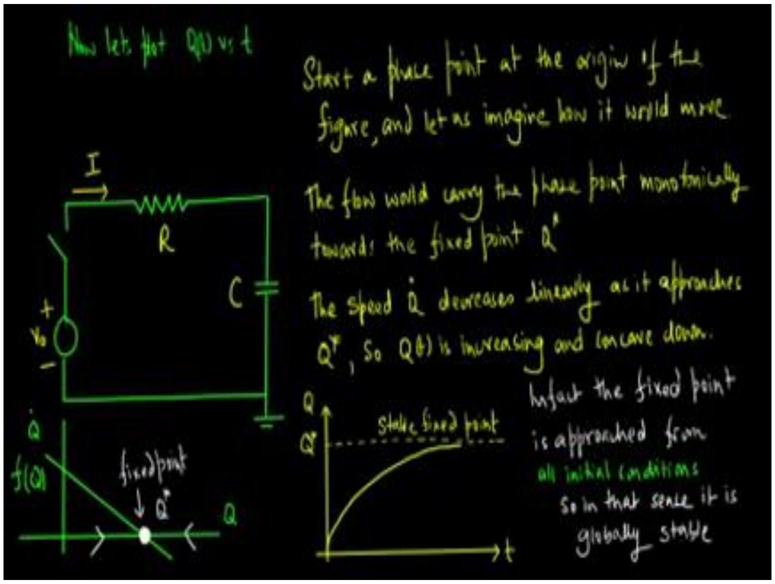
(Refer Side Time: 10:31)



So now, if you go around the circuit, the total voltage drop must equal zero. Which gives us -V not +RI + Q/C = 0. Where, I is the current flowing through the resistor. The current causes the charge to accumulate on the capacitor at a rate Q dot is equal to I, this gives us -V not + RQ dot + Q/C = 0. Rewriting this we get Q dot is equal to f is the function Q = Vnot / R - Q / RC. Now the graph of the f of Q is just a straight line with negative slope.

Now that is the plot of Q dot versus Q and we plot f of Q on this plot the vector field as a fixed point where f of Q = 0 at Q star = CVnot. So highlight the fixed point the flow is to the right where f of Q is greater than zero and the flow is to the left where f of Q is less than zero. So the flow is towards Q star and hence we have stable fixed point.

(Refer Side Time: 12:34)



Now let us plot the Q of t versus t, start a phase point at the origin of the figure and let us imagine how it would move the flow would carry the phase point monotonically towards the fixed point Q star. The speed Q dot decreases linearly as it approaches Q star. So Q of t is increasing and concave down. Now let us go ahead and plot Q of t versus t and highlight the equilibrium value Q star and denote the stable fixed point and that is the curve for Q of t versus t. Now in fact the fixed point is approached from all initial conditions and so in that sense it is actually globally stable.

(Refer Side Time: 13:54)

Ok so let us have some final comments and remarks about this lecture. Now this lecture was centred around fixed points and stability and we started dealing with stability in little bit more detail because stability is key concept in study of control and nonlinear dynamics. Now we have these special points referred to as fixed points and we were able to classify whether fixed points was stable or they were unstable.

And we were able to do this without any formal algebraic technique. But there was one point of subtlety that arouse and that was of local versus global stability. Now let just take an example, for example we got this pen of mine, and let assume that the tip of the pen is a stable fixed point. So if I found myself to be a silent neighbourhood of this stable fixed point. Then there is decent chance that I would actually get attracted towards that stable fixed point.

On the other hand, if there was a large disturbance and I found myself rather far away from this particular stable fixed point, then it is not very clear that the trajectories would actually converge towards to the stable fixed point. In fact one of our examples highlighted that, you could actually diverge away from that stable fixed point, they could actually have large disturbance. So there is one subtle point which came out in the lecture and that was of local versus global stability and that should be one of the key take away from this lecture.