Introduction to Nonlinear Dynamics
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Module -06
Lecture-29
2-Dimensional Flows, Bifurcations, Lecture 3

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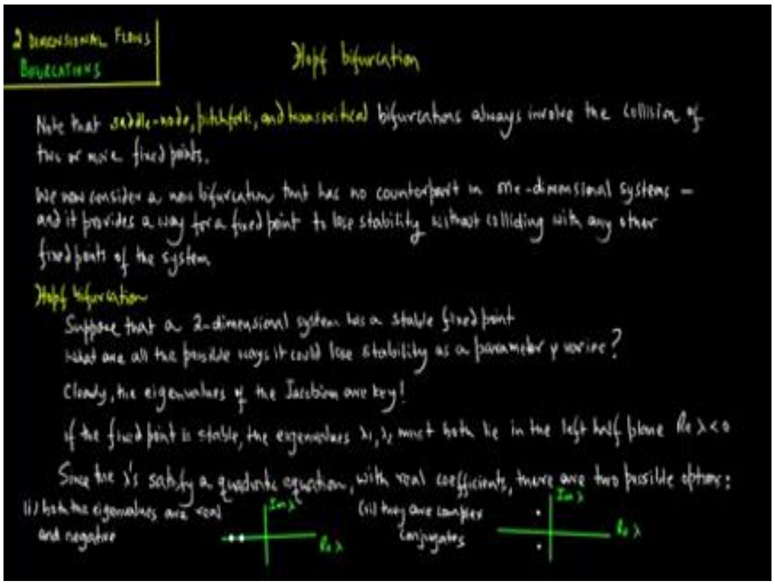

We now deal with the half bifurcation. Note that the Saddle node, the Pitchfork and the Transcritical Bifurcations always involve the collision of two or more fixed points, we now consider a new bifurcation that has no counter parts in one dimensional systems and it provides way for a fixed point to lose stability without colliding with any other fixed points of the system. So we now get on to the half bifurcation.

Suppose that a two-dimensional system has a stable fixed point, what are all the possible ways it could lose stability as a parameter meu varies. Now clearly the Eigen values of the Jacobian are key, if the fixed point is stable. The Eigen values lambda1 and lambda2 must both lie in the left half of plane, where real of lambda less than 0. Since the lambdas satisfy a quadratic equation with real coefficient.

There are two possible options one, both the Eigen values are real and negative. So, we plot the imaginary of lambda versus the real of lambda, you have two Eigen values which are real and negative. Number two, they are complex conjugates, so we plot again imaginary lambda and real of lambda and we have complex conjugate Eigen values.

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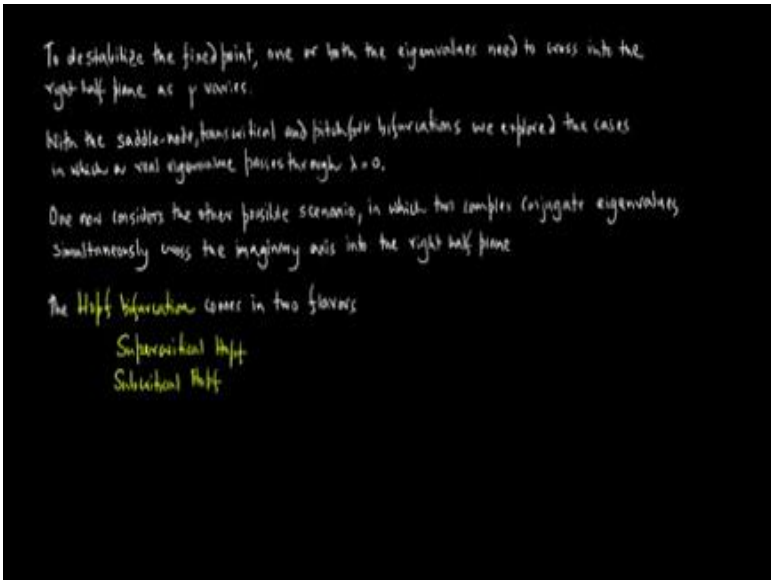

To destabilise the fixed point one or both of the Eigen values need to cross into the right half plane as meu varies. With the saddle nodes, transcritical and pitchfork bifurcations, we explored the cases in which the real Eigen value passes through lambda = 0. One now considers the other possible scenario in which two complex conjugate Eigen values simultaneously cross the imaginary axis into the right half plane. The half bifurcation comes in two flavours a supercritical half and a subcritical half.

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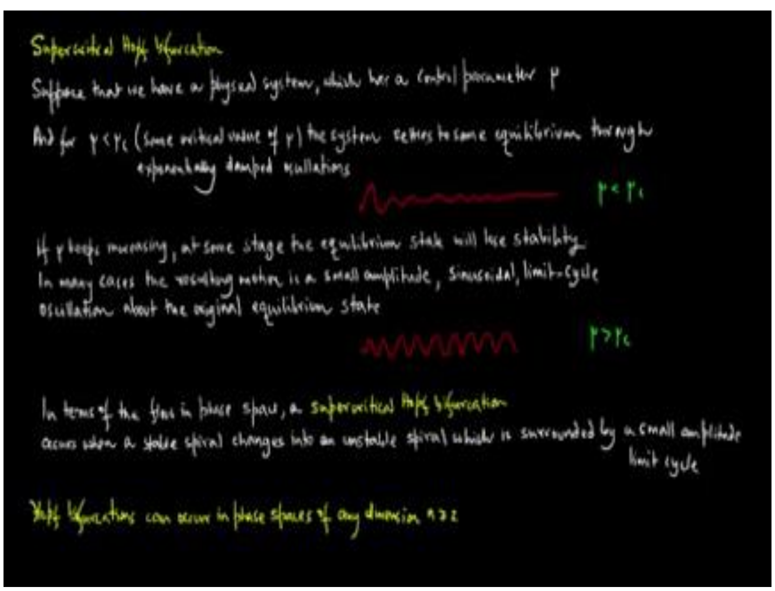

First, we discuss the supercritical half bifurcation. Suppose that we have a physical system which has a control parameter meu and for meu less than meu critical. For some critical value of meu, the system settles to some equilibrium through exponentially damped oscillations. So that is an example of damped oscillations when meu is less than meu critical. So if meu keeps increasing at some stage the equilibrium state will actually lose stability.

In many cases the resulting motion is a small amplitude sinusoidal limit cycle oscillation about the original equilibrium state. So that is the example of the limit cycle, where meu greater than meu critical. In terms of the flow in phase space a supercritical half bifurcation occurs when a stable spiral changes into an unstable spiral, which is surrounded by small amplitude limit cycle. Half bifurcations can occur in phase spaces of any dimension n greater than or equal to 2.

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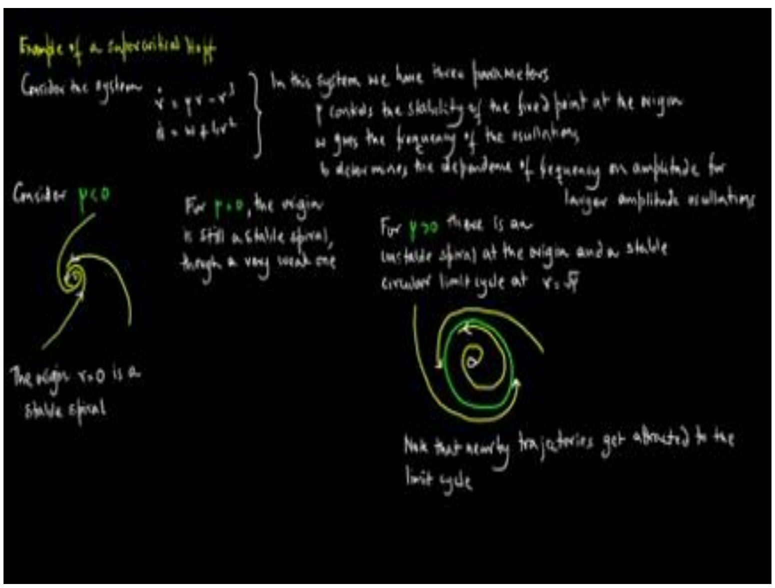

Now let us look at the example of a supercritical half. So, consider the system r dot = mue r - r cube, tita dot = omega + br squared. In this particular system we have three parameters, meu controls the stability of the fixed point at the origin. Omega gives the frequency of the oscillations and b determines the dependence of frequency on amplitude for larger amplitude oscillations.

So, consider meu less than 0, the origin r = 0 is a stable spiral. So, meu = 0, the origin is a still a stable spiral though a very weak one. So, for meu greater than 0, there is an unstable spiral at the origin and a stable circular limit cycle at the square root of meu. Note that nearby trajectories get attracted to the limit cycle.

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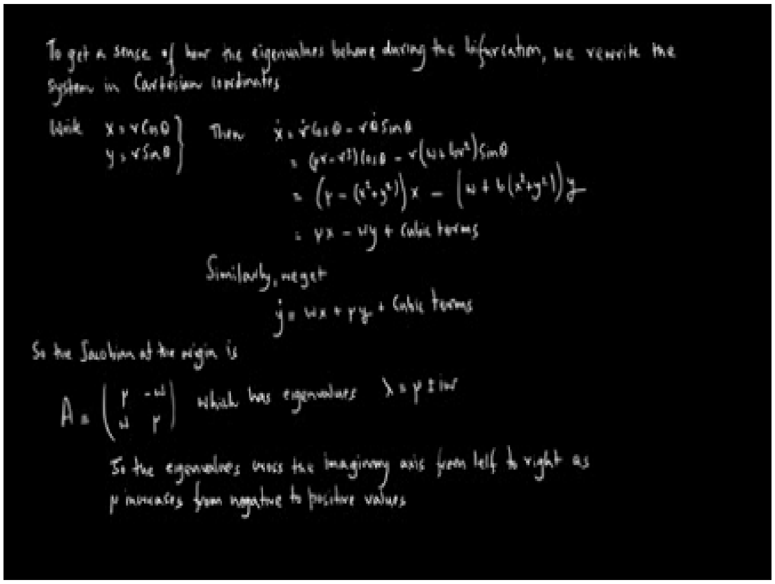

To get a sense of how the Eigen values behave during the bifurcation. We rewrite the system in Cartesian coordinates. So, we write x = r cos tita and y = r sine tita, then x dot = r dot cos tita - r tita dot sine tita, which = mue r - r cube times cos tita - r times omega + br squared times sine tita, which is = meu -x squared +s y squared and to the whole term times x - omega + b times x squared + y squared and the whole term times y, which is = meu x -omega y + cubic terms.

And similarly, we get y dot = omega x + meu y + cubic terms. So, the Jacobian at the origin is A = meu -omega omega meu, which has Eigen values lambda = meu plus minus i omega. So, the Eigen values actually cross the imaginary axis from left to right as meu increase from negative to positive values.

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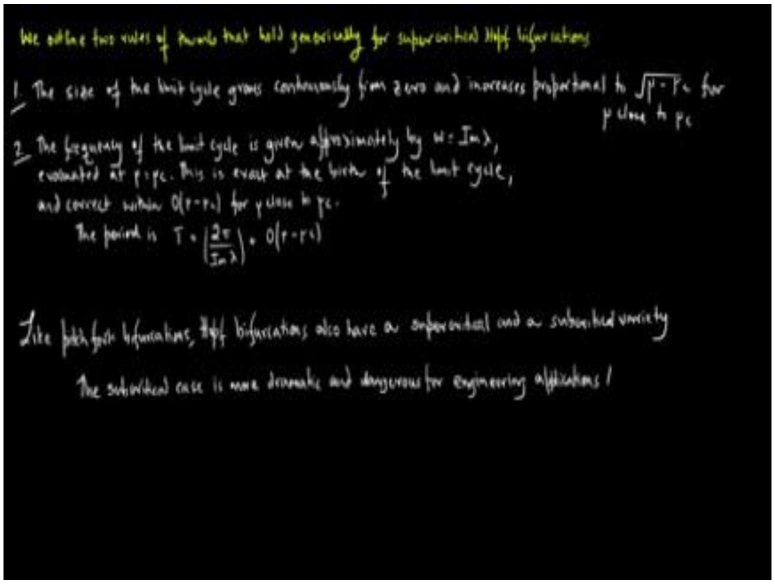

We now outline two rules of thumb that hold generically for supercritical half bifurcations. The first is that the size of the limit cycle grows continuously from zero and increases proportional to meu - meu critical square root, for meu close to meu critical. The second is that the frequency of the limit cycle is given approximately by omega = imaginary of lambda evaluated at meu = mue critical.

This is exact at the birth of the limit cycle and correct within order meu minus meu critical, for meu close to meu critical. The period is capital T = 2pi / imaginary lambda + order meu - meu critical terms. Now like pitchfork bifurcations, half bifurcations also have a supercritical and a subcritical variety. The subcritical case is much more dramatic and dangerous for engineering applications.

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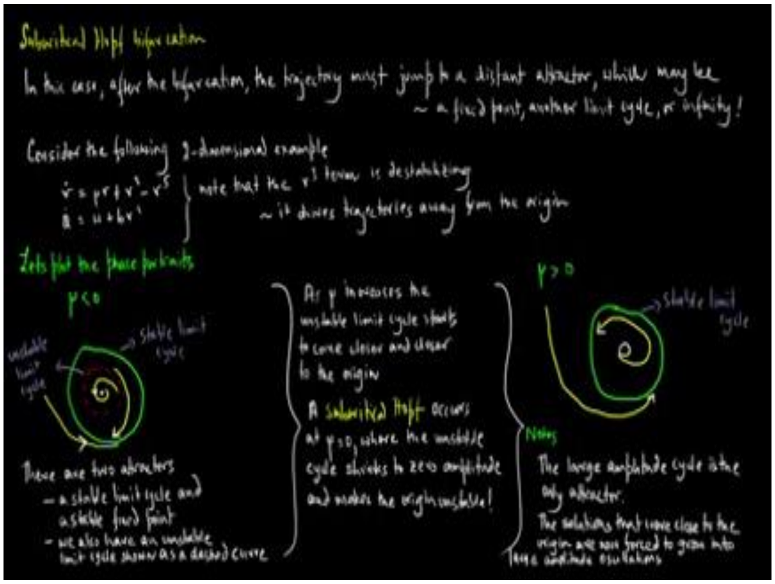

We now consider a subcritical half bifurcation. In this case after the bifurcation the trajectory must jump to a distance attracter which may be a fixed point, another limit cycle or of to infinity. So consider the following two dimensional examples, r dot = meu r + r cube - r5 and tita dot = omega +br squared. Note that the r cube term is destabilising ie it drives trajectories away from the origin.

Now let us plot the phase portrait with meu less than 0. We have a stable limit cycle and we have a unstable limit cycle. In this system, there are two attractors a stable limit cycle and a stable fixed point. We also have an unstable limit cycle shown as a dashed curve. As meu increases the unstable limit cycle starts to come closer and closer to the origin. A subcritical half occurs at meu = 0, where the unstable cycle shrinks to zero amplitude and makes the origin unstable.

Now we consider meu greater than 0, we are left with the stable limit cycle. Here is some notes the large amplitude limit cycle is the only attractor in the system. The solutions that were close to the origin are now force to grow into large amplitude oscillations.

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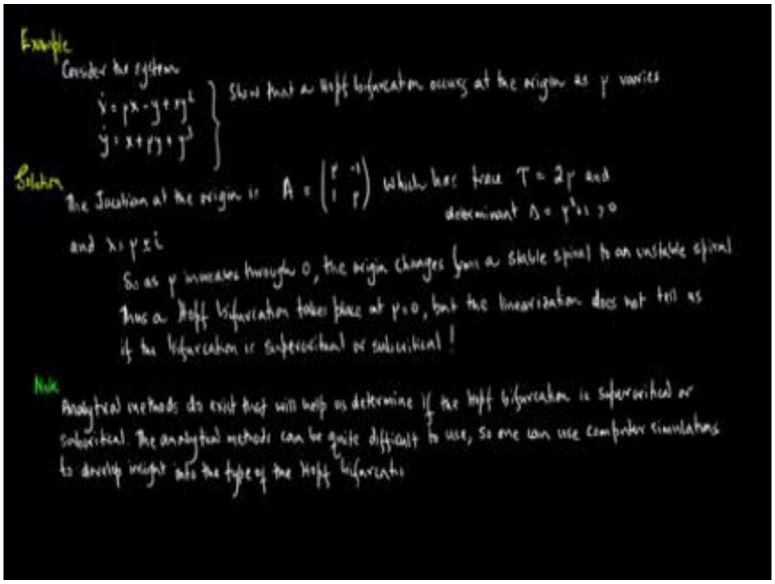

So, let us look at an example, consider the system x dot = meu x -y + xy squared and y dot = x +meu y + y cube, show that a half bifurcation occurs at the origin as meu varies. So, let us look at the solution, the Jacobian at the origin is A = meu -1 1 meu, which has trace 2meu and determinant meu squared +1 is greater than 0 and lambda = meu plus minus i.

So, as meu increases through zero the origin changes from a stable spiral to an unstable spiral. Thus, a half bifurcation takes places at meu = 0, but the linearization actually does not tell us if the bifurcation is supercritical or subcritical. Now we note that the analytical methods do exists that will help us to determine, if the half bifurcation is supercritical or subcritical. But the analytical methods can be quite difficult to use. And so, one can first use computers simulations to develop insight into the type of the half of bifurcation.

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Now the topic of discussion in this lecture was the half bifurcation. The half bifurcation is a very important one in both science and in engineering applications. Essentially it is bifurcation that leads to the birth of a limit cycle as a parameter in the system varies. It is worth mentioning that this bifurcation only occurs in dimensions two or higher. So, it does not have a counter part in one dimensional flows. This bifurcation comes with two variants a supercritical half and the subcritical half.

In a supercritical half, we get the birth of a small amplitude stable limit cycle as a parameter in the system varies. In the subcritical half is more dramatic bifurcation, where after the bifurcation trajectory may actually go off to a completely different fixed point, may go off to a distant limit cycle or may actually go off to the infinity. So, between these two variants the subcritical half is the one that should be avoided specially in engineering applications.