Introduction to Nonlinear Dynamics
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Module -06
Lecture-28
2-Dimensional Flows, Bifurcations, Lecture 2

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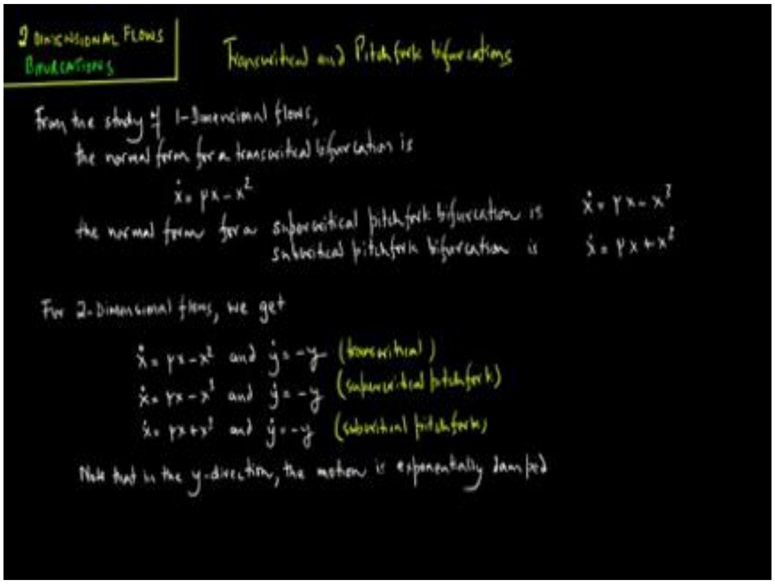

So, we are still dealing with two dimensional flows and our focus is still on bifurcations. So here we deal with Transcritical and Pitchfork Bifurcation. From the study of one dimensional flows, the normal form for a transcritical bifurcations is x dot = mue x - x squared. The normal form for a supercritical pitch fork bifurcation is x dot = meu x - x cube and a subcritical pitch fork bifurcation is x dot = meu x + x cube.

For two dimensional flows, we get x dot = meu x - x squared and y dot = -y and that is for transcritical bifurcation, x dot = mue x - x cube and y dot = -y and that is for a supercritical pitchfork bifurcation and x dot = meu x + x cube and y dot = -y and that is for a subcritical pitchfork bifurcation. Note that in the y direction the motion is always exponentially damped.

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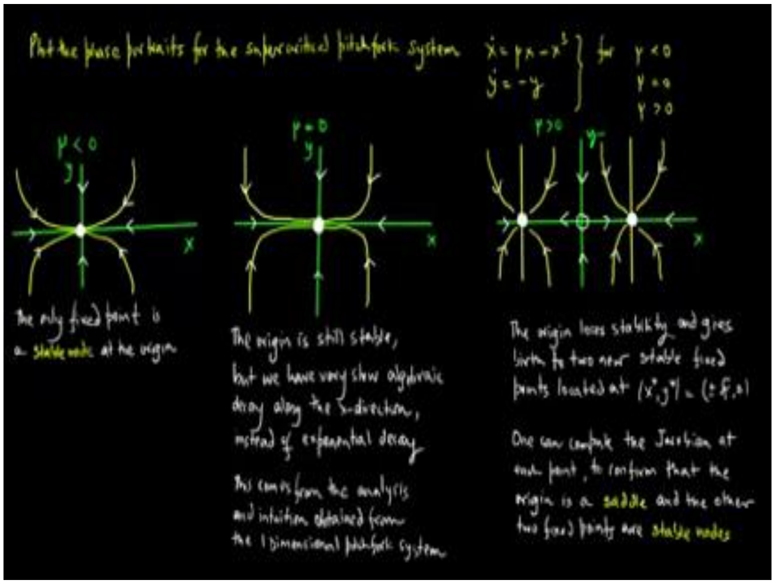

So, let us plot the phase portraits for the supercritical pitchfork system, x dot = meu x - x cube, y dot = -y for meu less than 0, meu = 0 and greater than 0. So, consider meu less than 0, here the only fixed point is a stable node at the origin. So, we have relatively straight forward phase portrait. So, plotting y versus x, we have stable node at the origin. With meu = 0, the origin is still stable, but we have very slow algebraic decay along the x direction instead of exponential decay, this comes from the analysis and the intuition obtained from the one-dimensional pitchfork system.

Now we look at meu greater than 0, here the origin loses stability and gives birth to two stable fixed points located at x* y star = plus minus square root of meu0. One can actually compute the Jacobian at each point to confirm that the origin is a saddle and the other two fixed points are in fact stable nodes. Now let us plot the phase portrait for the system, we highlighted the saddle at the origin and those are the two stable nodes. So that fills out the full phase portrait for the system.

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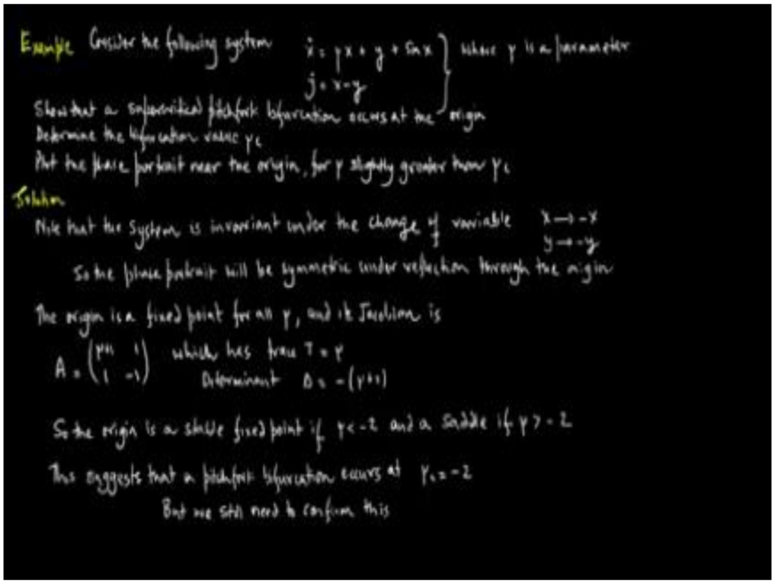

So, we look at an example now, consider the following system, x dot = meu x + y + sine x and y dot = x - y, where meu is a model parameter. Show that a supercritical pitchfork bifurcation occurs at the origin. Determine the bifurcation value meu critical and plot the phase portrait near the origin for meu slightly greater than meu critical. So here is the solution, note that the system is invariant under the change of the variable x to -x, y to -y.

So, the phase portrait will be symmetric under the reflection through the origin. The origin is in fact the fixed point for all meu and its Jacobian is A = meu +1 1 1 and -1, which as trace tou = meu and determinant delta = -meu +2. So, the origin is a stable fixed point if meu is less than -2 and a saddle, if meu is greater than -2. So, this suggests that a pitchfork bifurcation occurs at meu critical = -2, but we still actually need to confirm this.

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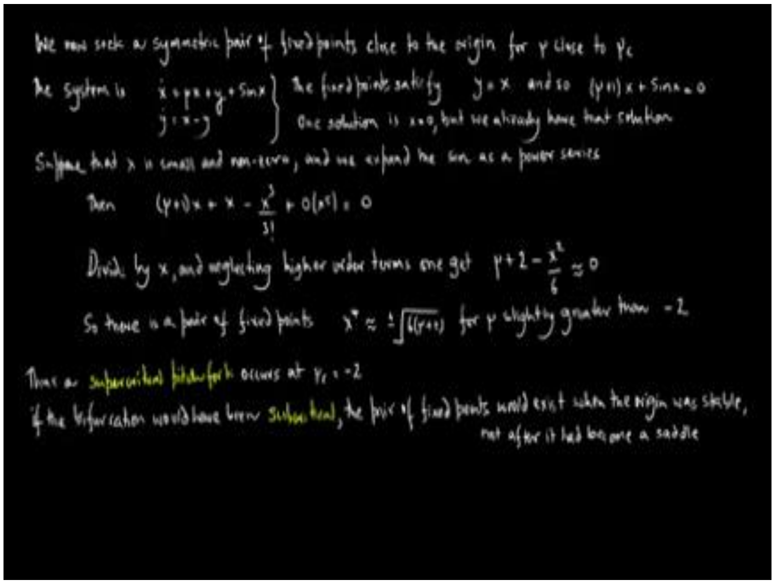

We now seek a symmetric pair of fixed points close to the origin for meu close to meu critical. Now recall that the system is x dot = meu x +y + sine x and y dot = x -y. The fixed points satisfy y = x and so meu +1x + sine x = 0. One solution is x = 0, but we already have that solution. So, suppose that x is small and non zero and we expand the sine term as a power series then meu +1 x + x - x cube by 3 factorial + order x to the 5 terms = 0.

So, dividing by x and then neglecting higher order terms one gets meu +2 -x squared on 6 is approximately is 0.   So, there is a pair of fixed points x* plus minus 6 times meu +2 square root for meu slightly greater than -2. Thus, a supercritical pitchfork bifurcation occurs at meu critical is -2. If the bifurcation would have been subcritical, then the pair of fixed points would exist when the origin was stable, not after it had become a saddle.

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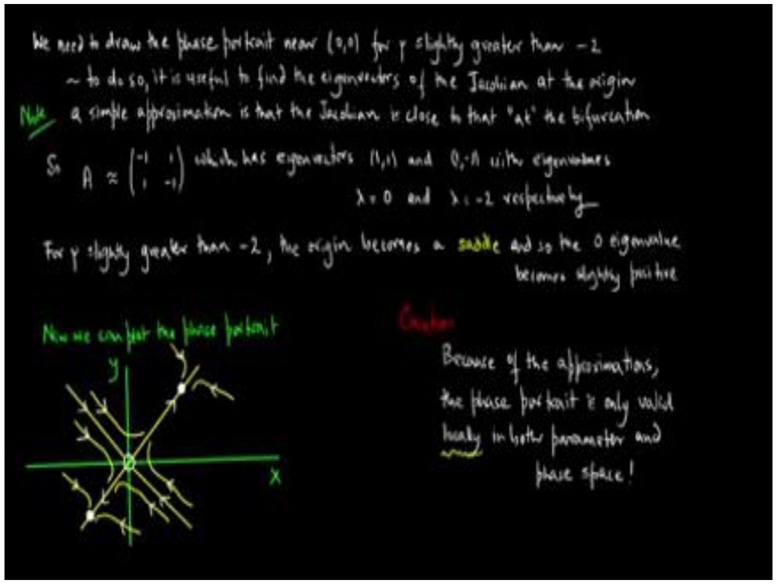

We need to draw the phase portrait near 0 0, for meu slightly greater than -2. To do so, it is actually useful to find the Eigen vectors of the Jacobian at the origin. So, we make a note that a simple approximation is that the Jacobian is close to that at the bifurcation. So, a is approximately -1 1 1 -1, which has Eigen vectors 1 1 and 1 and -1 with Eigen values lambda = 0 and lambda = -2 respectively.

So, meu slightly greater than -2 the origin becomes a saddle. And so, the zero Eigen value becomes slightly positive. So now we can go ahead and plot the phase portrait for the system. So, we plot y versus x, the origin is a saddle and that now completes the phase portrait. We have word of caution because of the approximations the phase portrait is only valid locally in both parameter and phase space.

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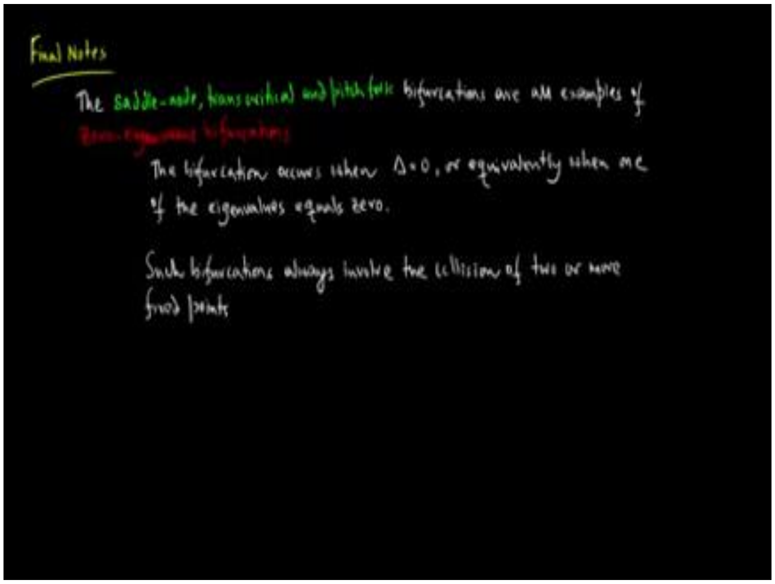

We end with some final notes, the saddle node, the transcritical and the pitchfork bifurcation are all examples of zero Eigen value bifurcations. The bifurcation occurs when delta = 0 or equivalently when one of the Eigen values equals 0. Such bifurcations always involves the collision of two or more fixed points.

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In this lecture, we discussed Transcritical and the Pitchfork Bifurcations in two dimensional flows. Now these are bifurcations that have counter parts in one dimensional flows as well and we note that the pitchfork comes in two variant a supercritical and a subcritical. Now if you want to construct the normal forms for these bifurcations in two dimensions. All you do is that look at the normal forms for this bifurcations in one dimension and add the equation y dot = -y and we will have the normal form in two dimensional flows for these bifurcations.

So, to that end, there is not much of a conceptual leap of faith in going from one dimensional flows to two dimension flows except to mention that it eminently could be possible that the algebra actually gets much more involved in typical two dimensional examples.