Introduction to Nonlinear Dynamics

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Module -06
Lecture-27
2-Dimensional Flows, Bifurcations, Lecture 1

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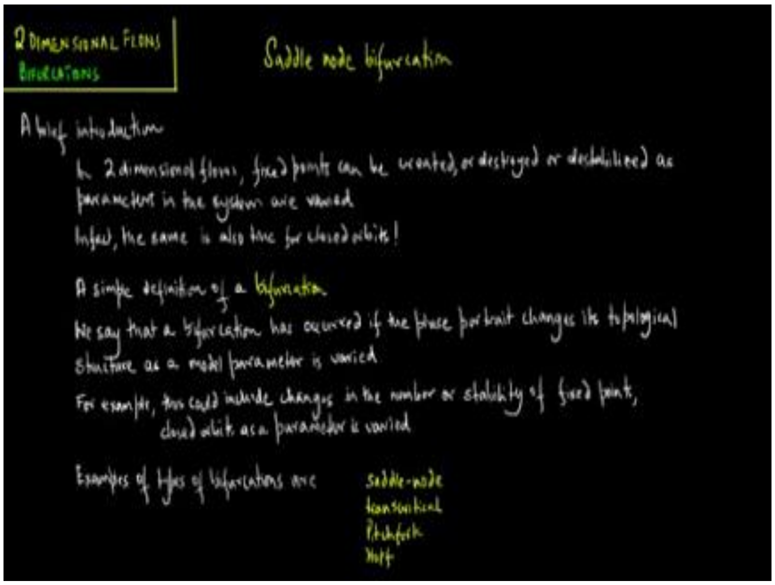

We are still dealing with two dimensional flows and now our focus will be on bifurcations. In this lecture focus will on saddle node bifurcations. We start with a very brief introduction to bifurcations. In two dimensional flows fixed points can be created or destroyed or destabilised as parameters in the system are varied. In fact, the same is also true for closed orbits. So, we offer a very simple definition of a bifurcation.

We say that a bifurcation has occurred if the phase portrait changes its topological structure as a model parameter is varied, for example this could include changes in the number or stability of fixed points or closed orbits as a parameter is varied. So here are some examples of types of bifurcations, we have a saddle node, we can have transcritical, we can have a pitchfork and you can have a half.

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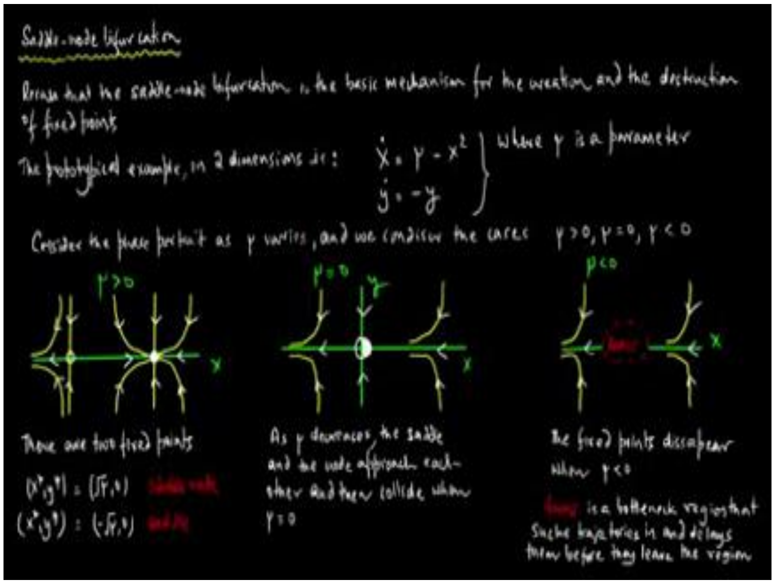

So, look at a saddle node bifurcation in some more detail. Recall that the saddle node bifurcation is the basic mechanism for the creation and the destruction of fixed points the prototypical example in two dimensions is x dot = meu - x square y dot = -y, where meu is a parameter. So now let us consider the phase portrait as meu varies. And we consider the cases, meu greater than zero, equal to zero and less than zero. We first start by looking at the case meu is greater than zero.

Now, here there are two fixed points x* y* = square root of meu0 and x* y* = -square root of meu0. As we note that, we have a stable node and a saddle. Let us look at meu = 0. As meu decreases the saddle and the node approach each other. And then collide when meu = 0. And with meu less than zero, the fixed points actually disappear when meu is less than zero. So, the fixed points disappear and we have the ghost region which appear in the phase portrait.

So, the ghost is a bottle neck region that sucks trajectories in and delays them before they actually leave the region.

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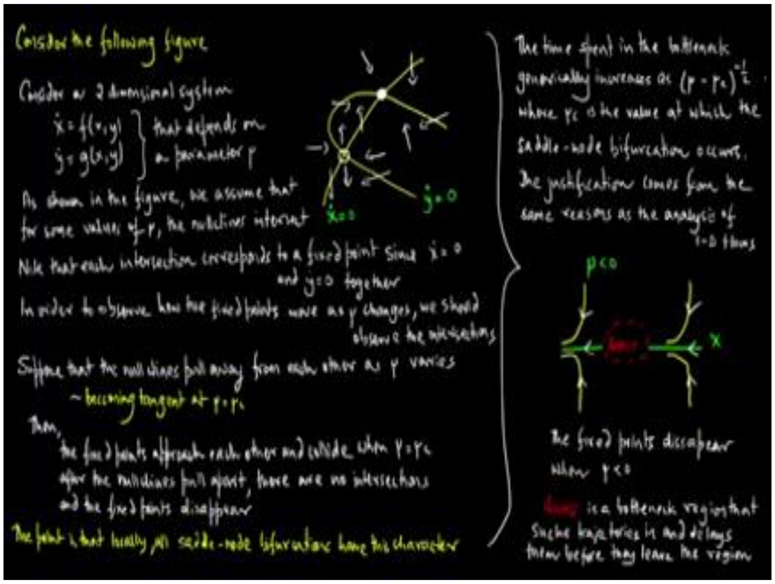

The time spent in the bottle neck generically increases as meu - meu critical to the power -1/2, where meu critical is the value at which the saddle node bifurcation actually occurs. The justification comes from the same reasons as the analysis of the one-dimensional flows. Now consider the following figure, let us consider a two-dimensional system x dot = f of xy and y dot = gs of xy that depends on a parameter meu as shown in the figure, we assume that for some values of meu, the null client intersect.

Note that each intersection corresponds to a fixed point, since x dot = 0 and y dot = 0 together, in order to observe, how the fixed points move as meu changes. We should be observing the intersections. Now suppose that the null clients pull away from each other as meu varies. Essentially becoming tangent at meu = meu critical. Then the fixed points approach each other and collide when meu = meu critical and after the null clients pull apart there are no intersections and the fixed points actually disappears.

The essential point is that locally, all saddle node bifurcations have this character.

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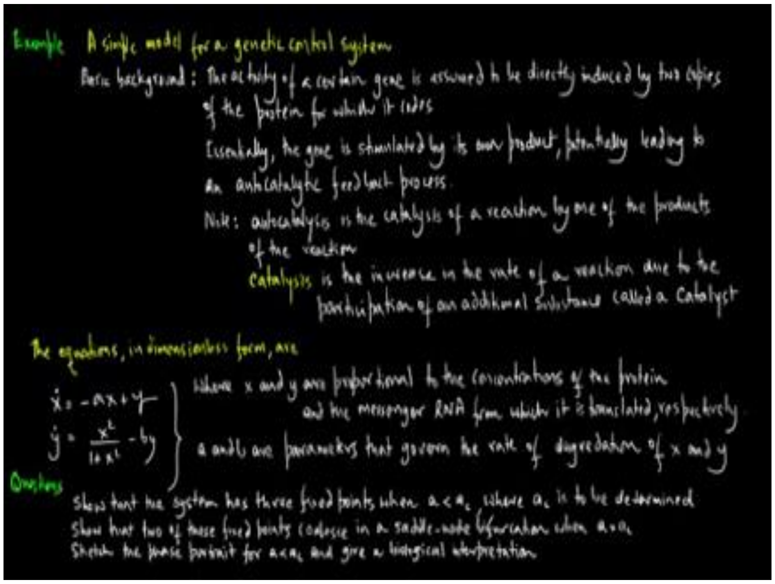

So let us consider an example, so we consider a simple model for a genetic control system. We start with some basic background. The activity of a certain gene is assumed to be directly induced by two copies of the protein for which it codes. So, essentially the gene is stimulated by its own product potentially leading to an auto catalytic feedback process. Note autocatalysis is the catalysis of a reaction by one of the products of the reaction. And catalysis is the increase in the rate of a reaction due to the participation of an additional substance called a catalyst.

So the equations in the dimension less form are x dot = -ax + y and y dot = x square / 1+ x square - by where x and y are proportional to the concentrations of the protein and the messenger RNA from which it is translated respectively, a and bare parameters that govern the rate of degradation of x and y. So here a list of questions, show that the system has three fixed points. When a is less than a critical where a critical is to be determined.

Show that two of these fixed points coalesce in a saddle node bifurcation when a = a critical and sketch the phase portrait for a less than a critical and give a biological interpretation.

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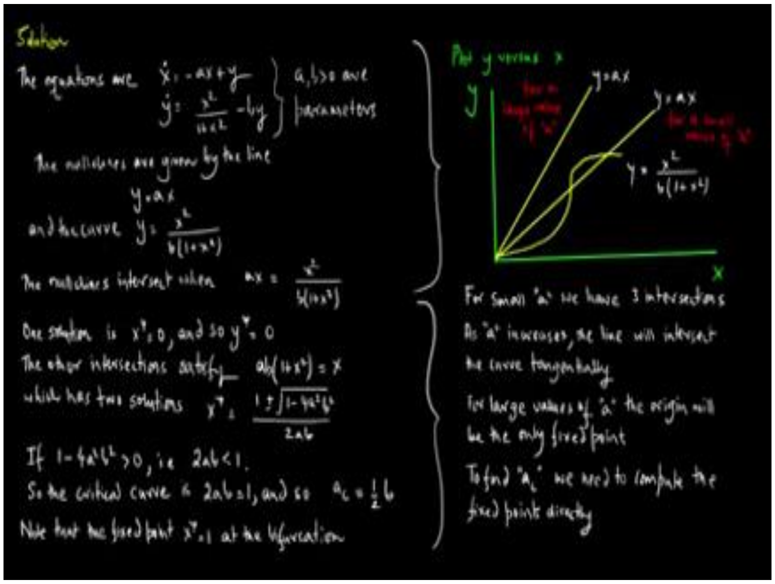

Now let us look at the solution in some detail. The underline equations are x dot = - ax + y and y dot = x square /1 + x square - by, where a and b greater than zero are system parameters. The null clients are given by the line y = ax and the curve y = x square / beta is 1+ x square. Let us plot y versus x and that is y = x square / beta into 1+x square and that is the line y = ax. Note that this is for a small value of a.

We plot another line y=ax and this is for a large value of a. For small a, we have three intersections as a, increases the line will intersect the curve tangentially. For larger values of a, the origin will be the only fixed point and to find a critical. We need to compute the fixed point directly. Now note that the null clients will intersects when ax = x square / beta is 1+ x square.

So, one solution is x* = 0 and so y* = 0 and the other intersections satisfy ab times 1+ x square = x, which actually has two solutions x* = 1 plus minus 1-4a square b square, square root / 2ab.
If 1-4a square b square is greater than zero, i e 2ab is less than one. So, the critical curve is 2ab = 1 and so a critical = 1/2 times b. Note that the fixed points x* = 1 at the bifurcation.
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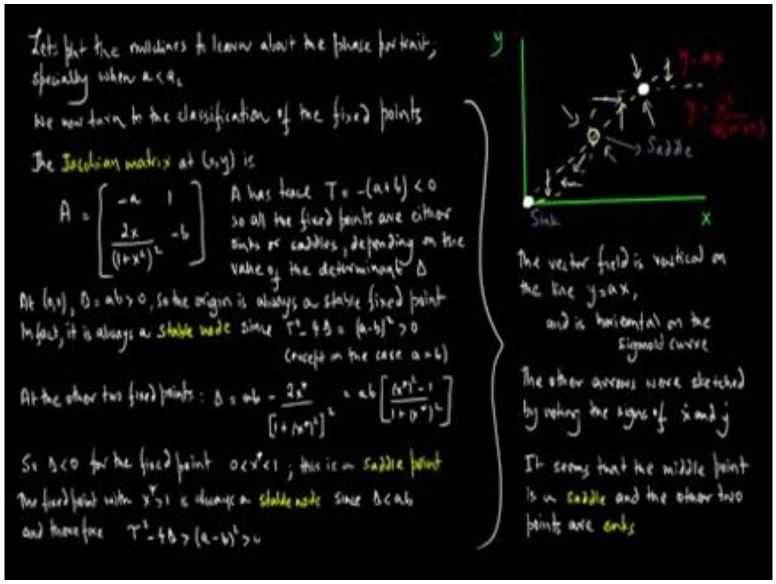

Now let us plot the null client to learn about the phase portrait, specially when a is less than a critical. So now let us plot y versus x, that is the line y = ax. And now we plot the sigmoid shape curve which is y = x square / b times 1+ x square. The vector field is vertical on the line y = ax and is horizontal on the sigmoid curve. The other arrows were sketched by noting the signs of the x dot and y dot. It seems that the middle point is a saddle and the other two points are sinks.

So, we highlight the sink and that is the saddle. We now turn to the classification of the fixed points. The Jacobian matrix at x y, is A = - a 1 2x / 1+ x square whole square -b, a has trace, - a + b which is less than 0. So, all the fixed points are either sinks or saddles depending on the value of the determinant delta. At 0 0, delta is ab which is greater than zero. So, the origin is always a stable fixed points, in fact it is always a stable node.

Since tau square -4delta = a-b whole square greater than 0, except in the case a = b. At the other two fixed points delta = ab -2x* / 1+ x* square whole square = ab times x* square -1/ 1 + x* square. So, delta is less than 0, for the fixed point x* < 1 and greater than 0. So, this is the saddle point, the fixed point with x* > 1 is always a stable node. Since delta is less than ab and therefore tau square -4delta is greater than a -b whole square which is greater than 0.

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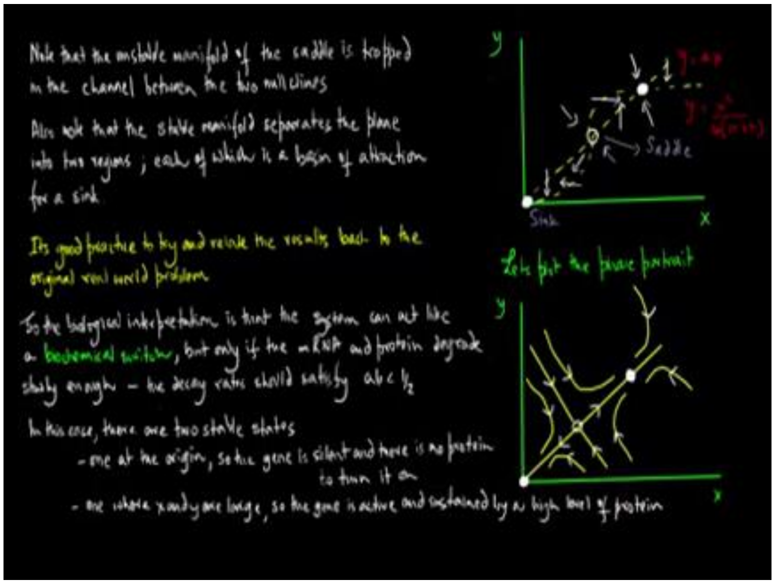

Let us plot the phase portrait, so plot y versus x that is the sink that is the saddle that is the sink. Note that the unstable manifold of the saddle is trapped in the channel between the two null clients. Also note that the stable manifold separates the plane into two regions, each of which is a bastion of attraction for a sink. Now it is always a good practice to try and relate the results back to the original real world problem.

So, the biological interpretation is that system can act like a biochemical switch, but only if the mRNA and protein degrade slowly enough ie the decay rates should satisfy ab less than a 1/2. In this case, there are two stable states, one at the origin. So, the gene is silent and there is no protein to actually turn it on and one where x and y are large, so the gene is active and sustained by a high level of protein.

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In this lecture, we discussed the Transcritical and the Pitchfork Bifurcations in the two-dimensional flows. Now these are bifurcations that have counterparts in one dimension flows as well and know that the pitchfork comes in two variants a Supercritical and a Subcritical. Now if you want to construct the normal forms for these bifurcations in two dimensions, all you do is look at the moment forms for these bifurcations in one dimension and add the equation y dot = -y.

And we will have the normal form in two dimensional flows for these bifurcations. So, to that end there is not much of a conceptual leap of faith in going from one dimensional flows to two dimensional flows except to mention that it eminently could be possible, that the algebra actually gets much more involved in typical two dimensional examples.