Introduction to Nonlinear Dynamics

Prof. Gaurav Raina
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module -06
Lecture-26
2-Dimensional Flows, Limit Cycles, Lecture 3

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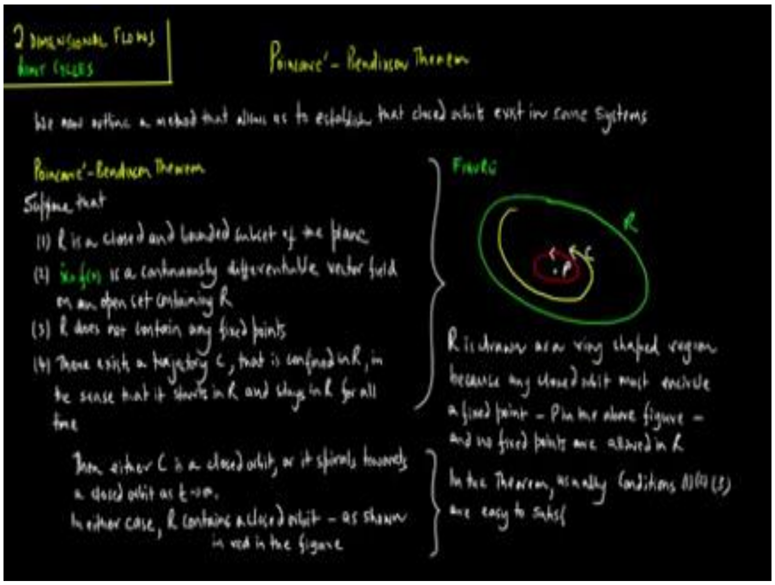

In this lecture, we focus on the Poincare Bendixson Theorem. We now outline a method that allows us to establish that closed orbits exits in some systems. So, we now state the theorem, suppose that number one, R is a closed and bounded subset of the plane. Number two, x dot = f of x is a continuously differentiable vector filed on an open set containing R. Number three, R does not contain any fixed points.

And number four, there exists a trajectories c that is confined in R, in the sense that it starts in R and stays in R for all time. Now let us plot a figure and visualise some of this, R is drawn as a ring-shaped region because any closed orbit must in circle a fixed point denoted as P in the above figure and no fixed points are allowed in R. Then either c is a closed orbit or its spirals towards a closed orbit, as t tends to infinity.

In either case R contains a closed orbit as shown in red, in the figure. In the theorem, usually conditions 1, 2, 3 are relatively easy to satisfy. But conditions 4 is actually difficult to satisfy.

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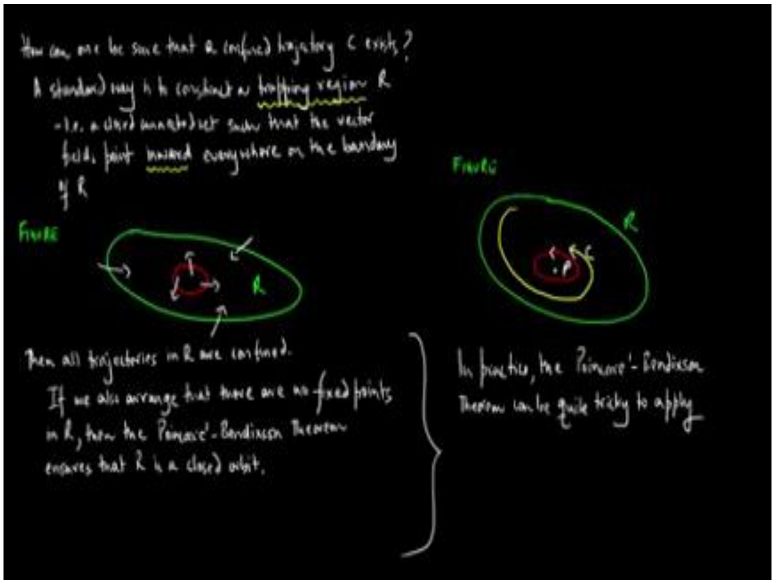

So how can only really be sure that a closed trajectory c actually exists, a standard way is to construct a trapping region R ie a closed connected set, such that the vector fields point inward everywhere on the boundary R. Now let us plot a figure to visualise this, then all the trajectories in R are confined, if we also arrange that there are no fixed points in R. Then the Poincare Bendixson Theorem ensures that R is a closed orbit. In practice, however the Poincare Bendixson Theorem can be quite tricky to apply.

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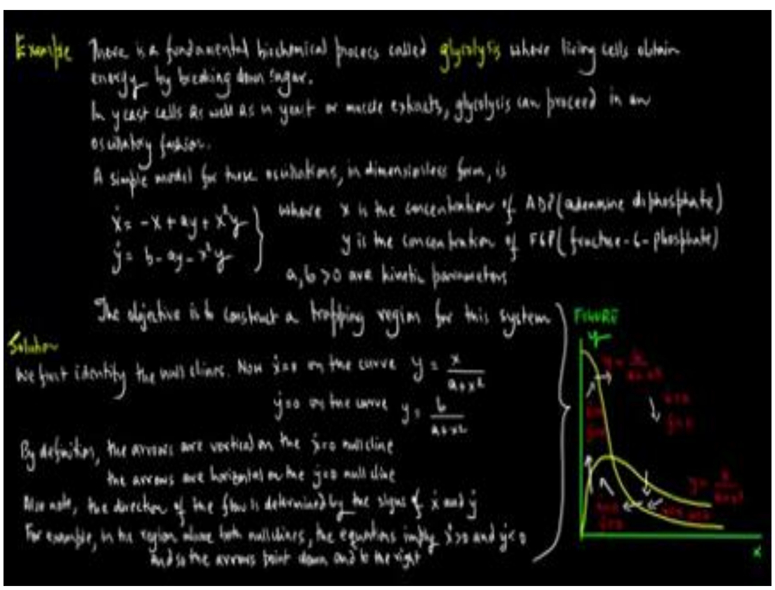

So we consider an example, there is a fundamental biochemical process called glycolysis. Where living cell obtain energy by breaking down sugar. In yeast cells as well as in yeast or muscles extracts glycolysis can proceed in an oscillatory fashion. A simple model for this oscillations in the dimensionless form is x dot = - x + ay +x square y, y dot = b -ay - x square y, where x is the concentration of ADP, which is Adenosine Di Phosphate and y is the concentration of F6P, which is Fructose 6 Phosphate, where a and b >0 are kinetic parameter.

The objective is to construct a trapping region for this particular system. So now we work towards its solution. We first identify the null clients, now x dot = 0 on the curve yields y = x / a + x square and y dot = 0, on the curve yields y = b / a + x square. So by definition the arrows are vertical on the x dot = 0 null client and arrows are horizontal on the y dot = 0 null client. Also note that the direction of the flow is determined by the signs of the x dot and y dot.

So, for example in the region above both the null clients, the equations imply x dot is greater than zero and y dot is less than zero. And so, the arrows point down and to the right. So, armed with this information, we draw a figure to visualise this.

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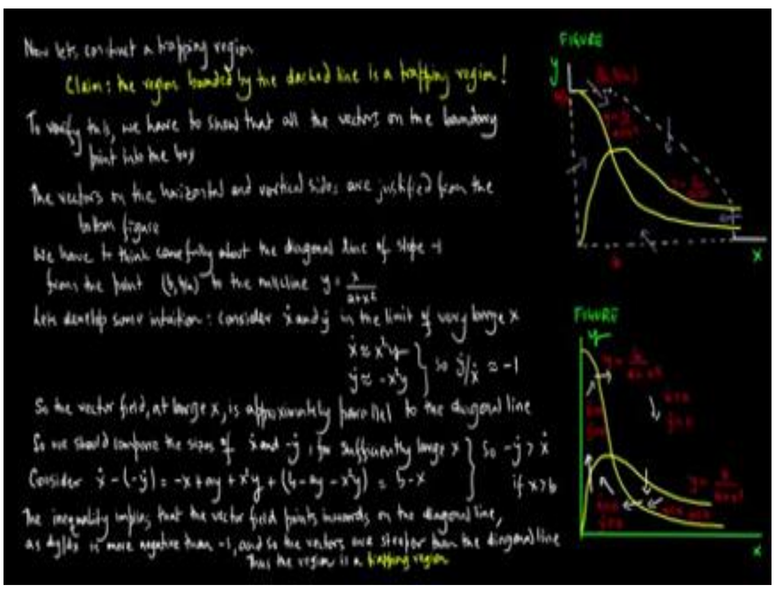

So now let us go ahead and construct a trapping region. So, the figure that we draw now will be an attempt to construct a trapping region. So, the claim is that the region bounded by the dashed line is in fact a trapping region. To verify this, we have to show that all the vectors on the boundary in fact point into the box. The vectors on the horizontal and the vertical sides are justified from the bottom figure, but we have to think carefully about the diagonal line of slope -1 from the point b b on a to the null client y = x / a + x square.

Let us develop some intuition now, so consider x dot and y dot in the limit of very large x. So, x dot is approximately x square y and y dot is approximately -x square y, so y dot / x dot is approximately -1. So, the vector field at larger x is approximately parallel to the diagonal line. So, we should compare the sizes of x dot and -y dot for sufficiently large x. So, consider x dot -, -y dot which is = -x + ay +x square y + b -ay -x square y which is = b -x.

So -y dot is greater than x dot, if x>b. The inequality implies that the vector field points inwards on the diagonal line as dy dx is more negative than -1. And so, the vectors are steeper than the diagonal line and thus the region is in fact a trapping region.
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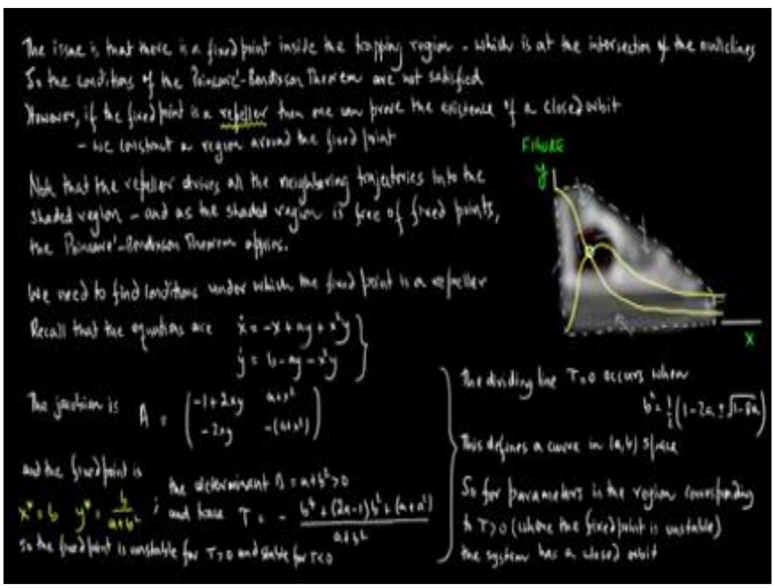

The issue is that there is a fixed point inside the trapping region, which is at the intersection of the null clients. So, the conditions of the Poincare Bendixson Theorem are not satisfied. However, if the fixed point is a repeller then one can prove the existence of a closed orbit and we can construct a region around the fixed point. Now this can be visualised in the following figure. Note that the repeller drives all the neighbouring trajectories into the shaded region.

And as the shaded region is free of fixed points the Poincare Bendixson Theorem actually applies. Now we need to find conditions under which a fixed point is actually a repeller. Recall that the equations are x dot = - x + ay + x square y and y dot = b - ay -x square y. The Jacobian of this system is a = -1 + 2xy a + x square -2xy -a + x square and the fixed point is x* = b and y* = b / a + b square.

The determinant of the system delta = a + b square which is greater than zero and the trace tou = - b to the 4 + 2a -1 times b square + a + a square / a + b square. And so, the fixed point is in fact unstable for tou greater than zero and will be stable for tou less than zero. The dividing line tou =0, occurs when b square = 1/2 times 1-2a + or - 1 - 8a square root and this defines a curve in the a b space.

So, for parameters in the region corresponding to tou greater than zero that is where the fixed point is unstable the system has a closed orbits.

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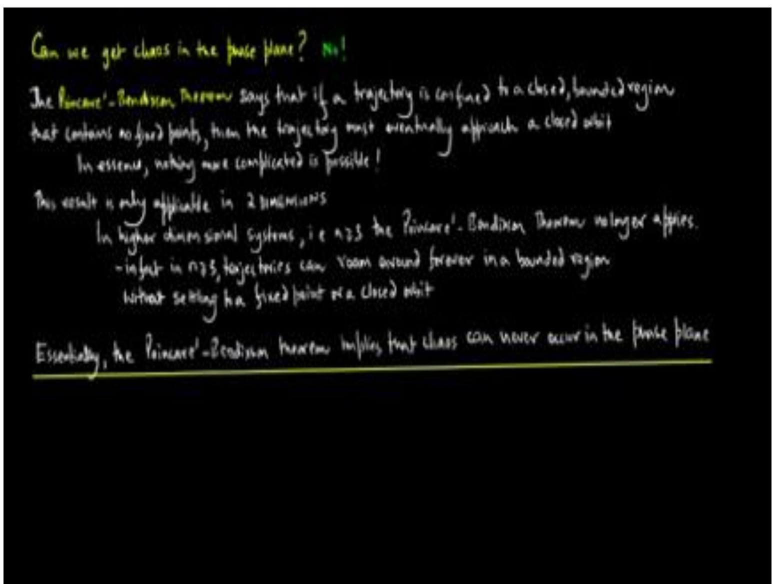

So can actually get chaos in the phase plane and the short answer is no. The Poincare Bendixson Theorem says that if a trajectory is confined to a closed bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. In essence nothing more complicated is possible. The result is only applicable in two dimensions, in higher dimensional system ien greater than or equal to 3.

The Poincare Bendixson Theorem in fact no longer applies. In fact, in n greater than or equal to 3 the trajectories can roam around forever in the bounded region without actually settling to a fixed point or a closed orbit. So, essentially the Poincare Bendixson Theorem implies that chaos can never occur in the phase plane.

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Now closed orbits are very important objects scientifically and they occur in numerous models in science and engineering, specially in places where we actually, have models which exhibits self sustained oscillations. So, it is important to have methods which allow us to talk about the existence of such closed orbits.   And the Poincare Bendixson Theorem is an important result in nonlinear dynamics in this directions essentially what the Poincare Bendixson Theorem tells us the following:

Let assume we have a closed bounded region, inside that region, we do not have a fixed point, but we have a trajectory which starts inside this region, then this trajectory has to eventually approach a closed orbit. This result is only applicable in two dimensions. So if you are looking at high dimensional systems for example dimension 3 or higher, then you can be a trajectory, which start within the closed bounded region.

But you do not have to get a fixed point or approach a closed orbit. But you actually keep moving on randomly or chaotically forever. So essentially what this says is that this form of random or chaotic behaviours cannot actually happen in a two-dimensions. But can happen in dimensions 3 or higher and in particular key result and one key take away from the Poincare Bendixson Theorem is that such chaotic phenomena will not happen in two dimensions.