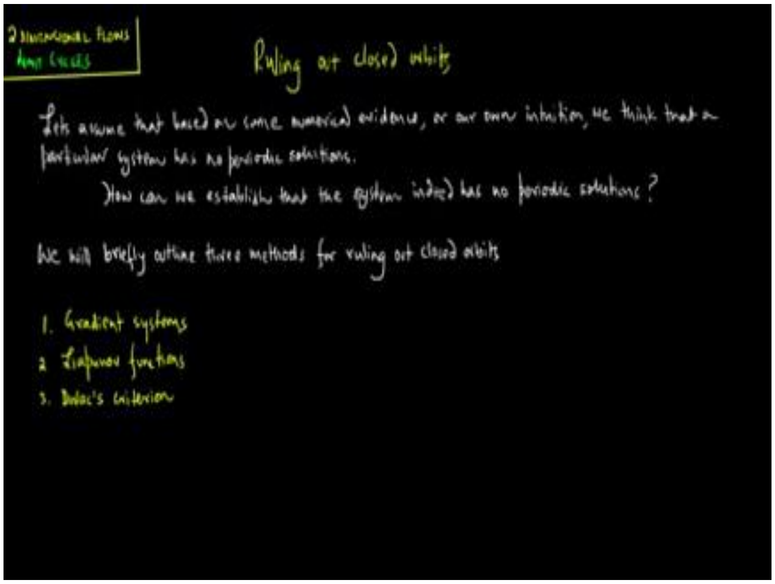
A brief Introduction to Modelling
Prof. Gaurav Raina
Department of Electrical Engineering
Indian Institute of Technology, Madras

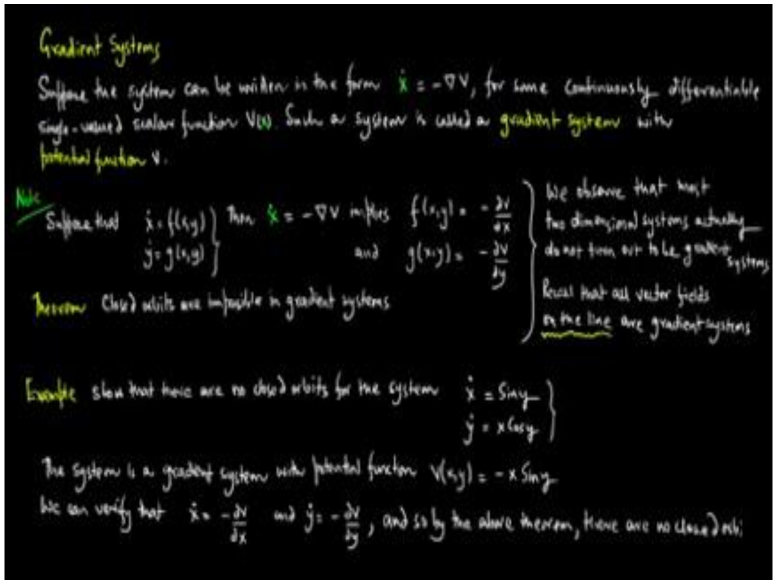
Module -06
Lecture-25
2-Dimensional Flows, Limit Cycles, Lecture 1

(Refer Slide Time: 00:01)



We focus is on ruling out closed orbits. So, let us assume that based on some numerical evidence or our own intuition, we think that a particular system has no periodic solutions. So how can we establish that the system indeed has no periodic solutions? We will briefly outline three methods for ruling out closed orbits. Number one Gradient systems number two is Lyapunov functions and number three is Dulac's criteria.

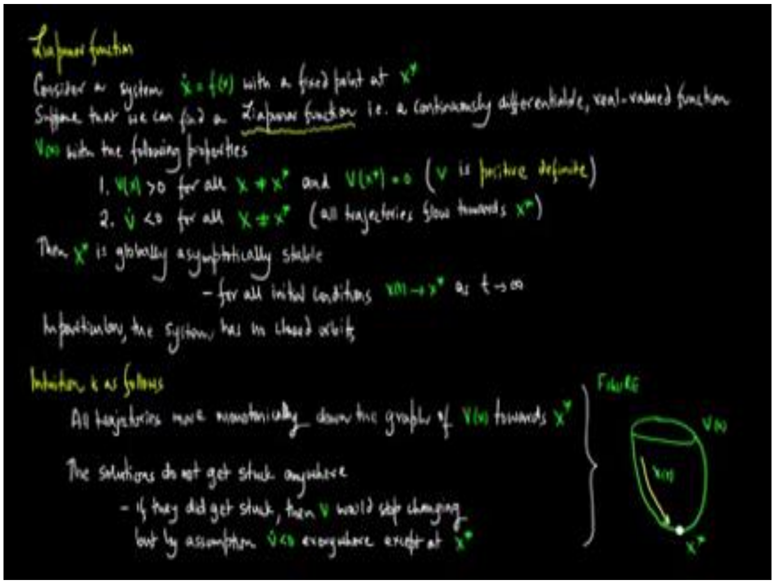
(Refer Slide Time: 00:59)



So, let us look at Gradient systems, suppose the system can be written in the form x dot = -del v, for some continuously differentiable single valued scalar function v of x. Such a system is called a Gradient system with potential function v. Note suppose that x dot = f(x)y, y dot = gs of xy then x dot = -del v implies f(x)y -dv dx and gs of xy is -dv dy. So, there is a theorem which states that closed orbits are impossible in gradient systems.

We observe that, in fact most two dimensional systems actually do not turn out to be gradient systems. Recall that all vector fields on the line are gradient systems. Let us consider an example, show that there are no closed orbits for the system x dot = sine y and y dot = x cos y. The system is a gradient system with potential function v of xy = -x sine y and we can readily verify that x dot = -dv dx and y dot = -dv dy and so by the above theorem there are no closed orbits.

(Refer Slide Time: 03:18)

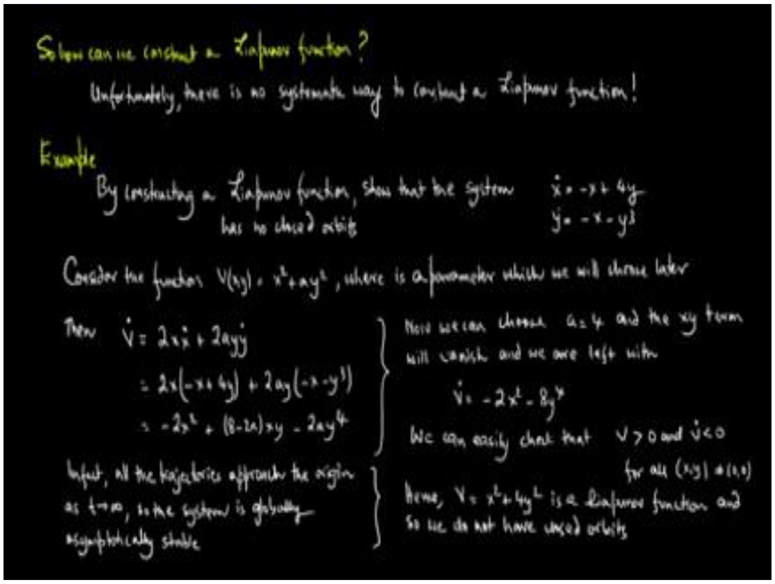


Have we discussed about Lyapunov function, consider a system x dot = f(x) with a fixed point at x*. Suppose that we can find a Lyapunov function that is a continuously differentiable real valued function v(x)with the following properties, 1. The effects is greater than zero for all x0 = x* and v(x*) =0, so v is positive definite. 2. v dot is less than zero for all x0 = x*, so all trajectories flow towards x*.

Then x* is globally asymptotically stable, in sense that for all initial condition x of t tends to x* as t tends to infinity. In particular, the system has no closed orbits, there are conclusion is as follows; all trajectories move monotonically down the graph of v(x)towards x*. So, let us visualise this through a figure, so that is the equilibrium point x*, that is the graph of v(x)and the trajectories move monotonically towards the equilibrium x*, the solutions actually do not get stuck anywhere.

If they did get stuck then v would actually stop changing, but by assumption v dot is less than zero, everywhere except at x*.

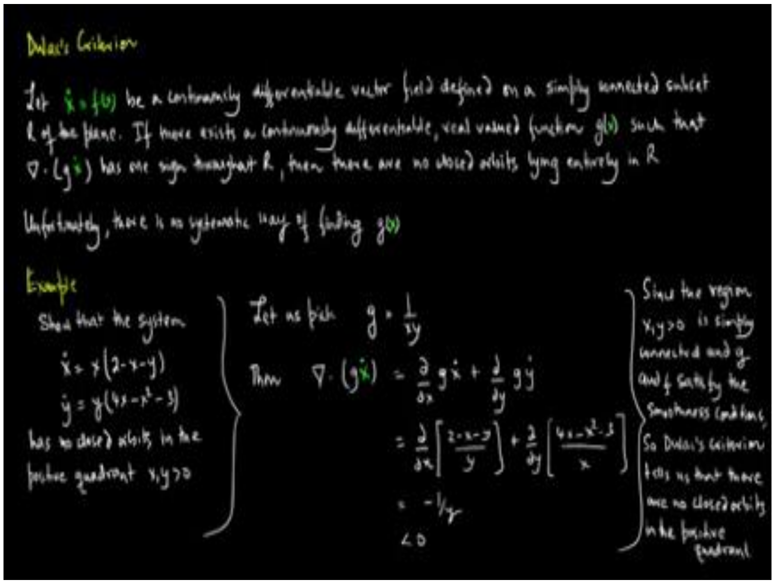
(Refer Slide Time: 05:45)



So how can we actually construct a Lyapunov function, unfortunately there is no systematic way to construct a Lyapunov function. So, let us consider an example, by constructing a Lyapunov function, show that the system x dot = -x + 4y and y dot = -x -y to the cube has no closed orbits. Consider the function v of xy = x square + ay square where a is a parameter which we will choose later. Then v dot = 2x x dot + 2ay y dot which is = 2x times -x +4y + 2ay times -x -y cube which is = -2x square +8 -2a times xy -2ay to the 4.

So now we can choose a = 4 and the xy term they will vanish and we are left with v dot = -2x square -8y to the 4, so we can easily check that v >0 and v dot <0 for all xy not equal to 0 0. Hence v = x square +4y square is indeed a Lyapunov function and so we do not have closed orbits. In fact, all the trajectories approach the origin as t tends to infinity and so the system is in fact globally asymptotically stable.

(Refer Stable Time: 08:10)



So now we discuss Dulac’s criterion, let x dot = f(x) be a continuously differentiable vector field defined on a simply connected subsect R of the plane. If there exists the continuously differentiable real value function g(x) such that delta times g x dot has one sign throughout R, then there are no closed orbits lying entirely in R. Unfortunately, there is no systematic way of finding g(x).

Let us consider an example, show that the system x dot = x times 2- x - y and y dot = y times 4x - x square -3 has no closed orbits in the positive quadrant xy greater than zero. So, let us go ahead and pick g = 1 / xy, then del dot g x dot = ddx g times x dot + ddy g times y dot = ddx times 2- x - y / y + ddy of 4x -x square -3 / x = -1 upon y which is less than zero. Since the region xy>0 is simply connected and g and f satisfy the smoothness conditions, so Dulac’s criteria tell us that there are no closed orbits in the positive quadrant.

(Refer Slide Time: 10:59)

Now we will have a model of the real world is quite use full to know if we can actually rule out closed orbits. So, in this lecture, we outlined three methods that can be useful for ruling out those topics, number one was Gradient systems so you show your system is a Gradient system, the second is based on the method of Lyapunov functions and third was based on Dulac's criteria. Now all these three are very, very powerful theoretical ideas.

And when work they can be very powerful except that the only issue is that find to get them to work in practice can be slightly tricky because there is no real systematic procedure on for example you might be struck Lyapunov function. So, it is nice to know that this method exists, that it is also nice keep in mind that in practice, they sometimes can be little tricky to use and to actually show for your real-world model none the less they are extremely powerful methods which you should be aware off.