Introduction to Nonlinear Dynamics
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Module -06
Lecture-23
2-Dimensional Flows, Phase Plane, Lecture 3

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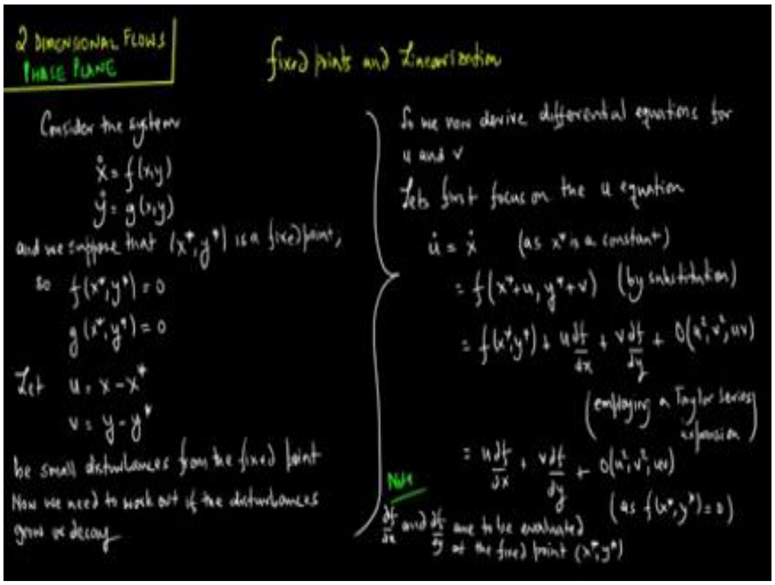

In this lecture, we deal with fixed points and linerazation. So, consider the system x dot = f of xy, y dot = g of xy. And we suppose that x*, y* is a fixed point, so f of x* y* = 0 and gs of x* and y = 0. So let u = x - x* or v = y -y*, be small disturbances from the fixed point, now we need to work out, if the disturbances grow or decay. So, we now derive differential equations for u and v.

So, let us first focus on the u equation, so u dot = x dot and that is as x* is a constant, this is = f of x* + u and y* + v and this is by the simple substitution and this expands to f of x* y* + u times df dx + v times df dy + terms that are order u square v square and uv. And this comes by employing a Taylor series expansion and this is = u df dx + v df dy + terms which are order u square v square uv, as f of x* y* = 0. Now note that df dx and df dy are to be evaluated at the fixed point x*t y*.

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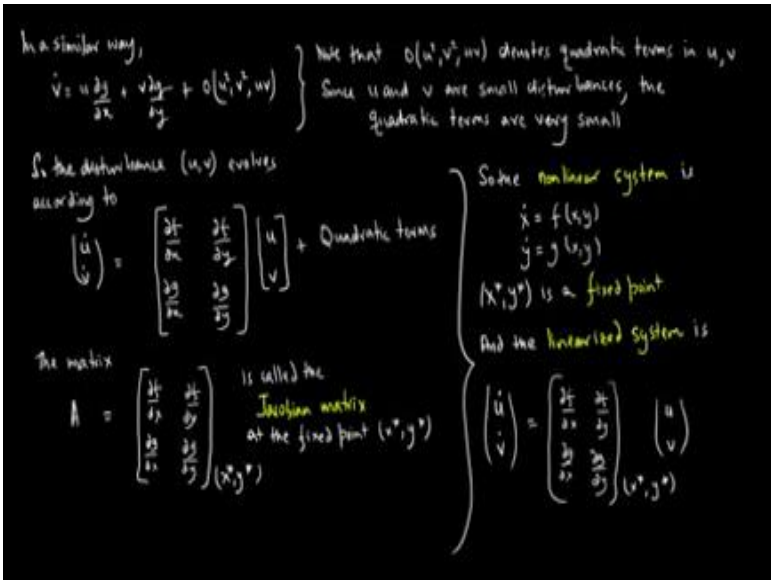

So in a similar way v dot = u dg dx + v dg dy + terms which are order u square v square and uv. Note that order u square v square uv, denotes quadratic terms in u and v and since u and v are small disturbances. The quadratic terms are in fact very small. So, the disturbance u v evolves according to u dot v dot = df dx, df dy, dg dx, dg dy times uv + quadratic terms.

So, the matrix A = df dx, df dy, dg dx, dg dy evaluated at x* y* is called the Jacobian matrix at the fixed points x*, y*. So, the nonlinear system is x dot = f of xy, y dot = g of xy, where x*, y* is a fixed point and the associated linearized system is u dot v dot = df dx, df dy, dg dx, dg dy evaluated at x* y* times u v.

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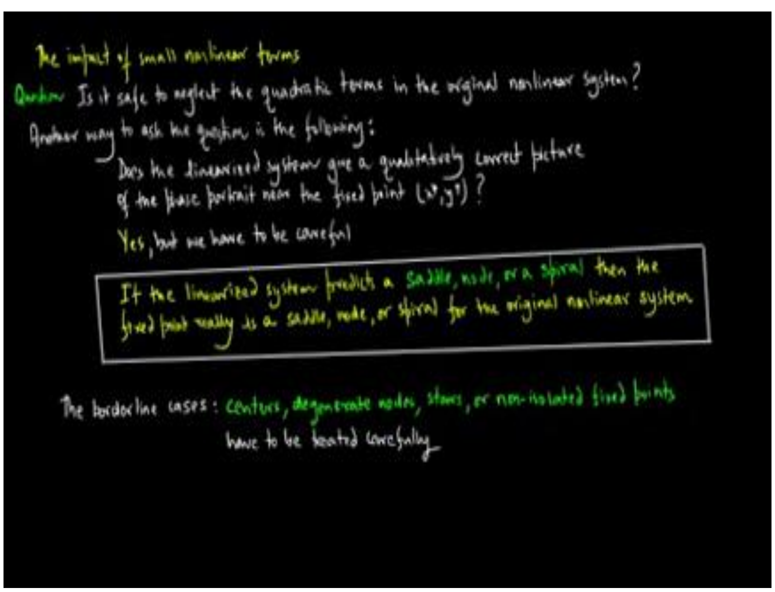

Now let us consider the impact of small nonlinear terms. So, the question we have is the following: Is it really safe to neglect the quadratic terms in the original nonlinear system? So, another way to ask the question is the following; does the linearized system give a qualitatively correct picture of the phase portrait near the fixed point x* y*? The short answer is yes, but we have to be very careful.

So if the linearized system predicts a saddle, a node or a spiral when the fixed point really is a saddle, node or a spiral for the original nonlinear system, the border line cases that is centers, degenerates nodes, stars or non isolated fixed points have to be treated much more carefully.
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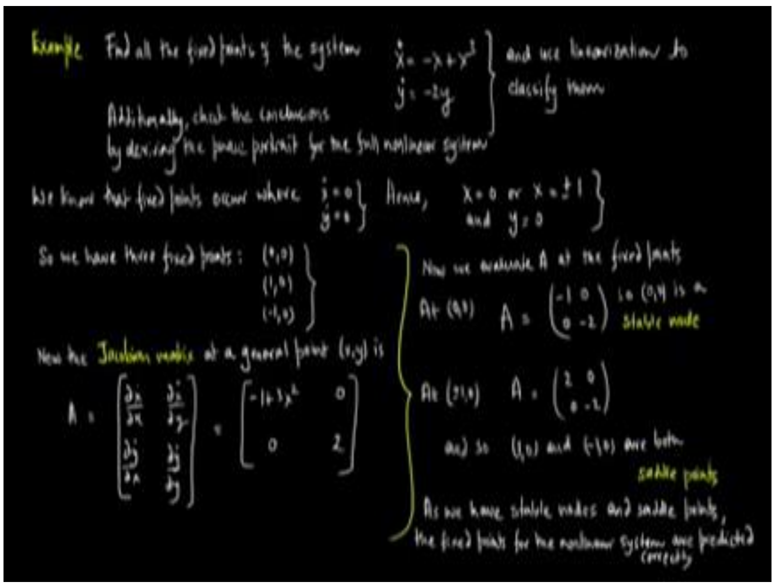

So, let us consider an example, find all the fixed points of the system x dot = - x + x cube and y dot = -dy and use the technique of linearization to classify them. Additionally, check the conclusions by deriving the phase portrait for the full nonlinear system. We know that the fixed points occur where x dot and y dot are equal to zero. And hence x = 0 or x = plus minus 1 and y = 0 are the fixed points.

So, we have three fixed points 0 0 1 0 -1 0. Now the Jacobian matrix at a general point xy is A = dx dot dx, dx dot dy, dy dot dx, dy dot dy = -1 + 3x square 0 0 and 2. Now we evaluate A at the fixed points at 0 0, A = -1 0 0 -2. And so 0 0 is a stable node. At plus minus 1 0 A = 2 0 0 -2 and so 1 0 and -1 0 are both saddle points. As we have stable nodes and saddle points the fixed points for the nonlinear system are in fact predicted correctly.

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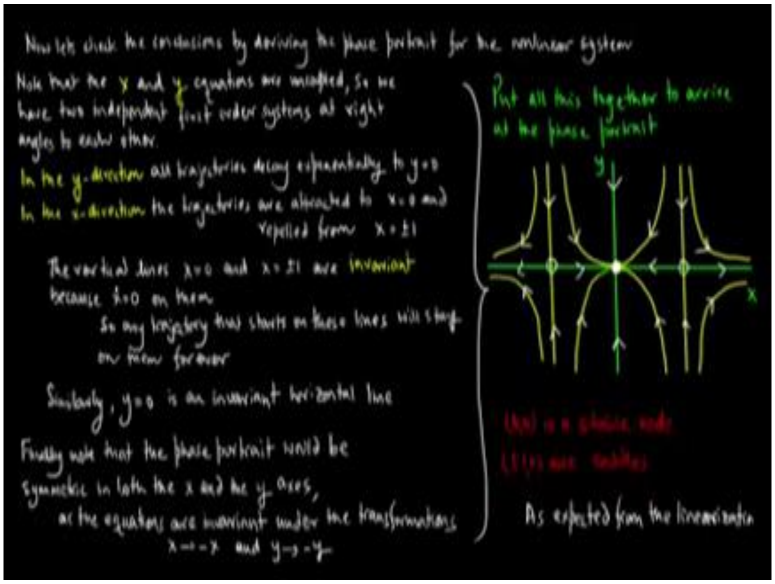

Now let us check the conclusions by deriving the phase portrait for the original nonlinear system. Note that the x and y equations are uncoupled, so we have two independent first order systems at right angles to each other. In the y direction, all the trajectories decay exponentially to y = 0. In the x direction, the trajectories are attracted to x=0 and repelled from x=plus minus 1. The vertical lines x = 0 and x = plus minus 1 or in variant because x dot =0 on them.

So, any trajectory that starts on these lines will stay on them forever. Similarly, y = 0 is an invariant horizontal line. Finally note that the phase portrait would be symmetric in both the x and the y axis as the equations are invariant under the transformations x to -x and y to -y. So, we now put this together to arrive at the phase portrait. So that is one fixed point, that is the second fixed point, and that is the third fixed point and so we go ahead and fill out the rest of phase portrait for this nonlinear system.

Note that 0 0 is a stable node plus minus 1 0 are saddles and this is exactly as expected from the linearization.

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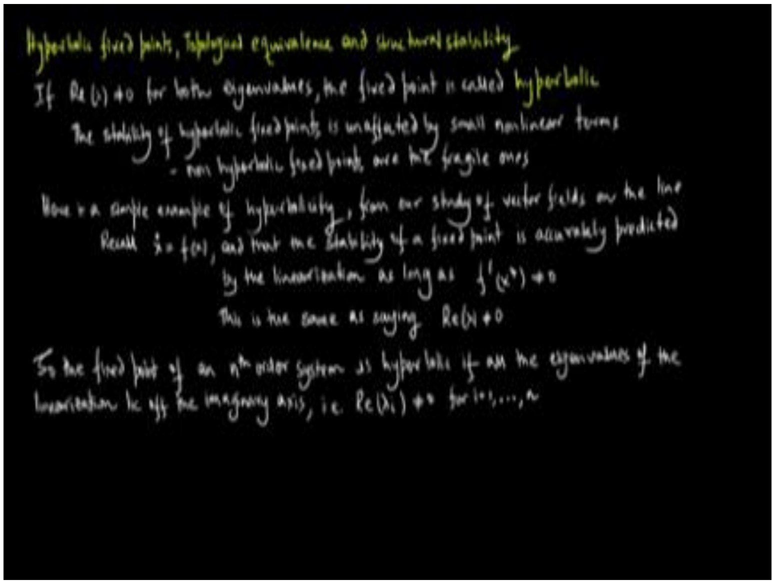

We now offer some comments on hyperbolic fixed points, topological equivalence and structural stability. If the real part of the lambda is not equal to zero for both Eigen values, then the fixed points are called hyperbolic. The stability of hyperbolic fixed points is unaffected by small nonlinear terms. Non-hyperbolic fixed points are the fragile ones. So here is a simple example of hyperbolicity from our study of vector fields on the line.

Recall x dot = f of x and that the stability of a fixed point is accurately predicted by the linearization as long as f prime of x* is not equal to zero. Now this is the same as saying that the real part of the lambda is not equal to zero. So, the fixed points of an nth order system is hyperbolic, if all the Eigen values of the linearization lie off the imaginary axis ie real part of lambda i is not equal to zero for i = 1 to n.

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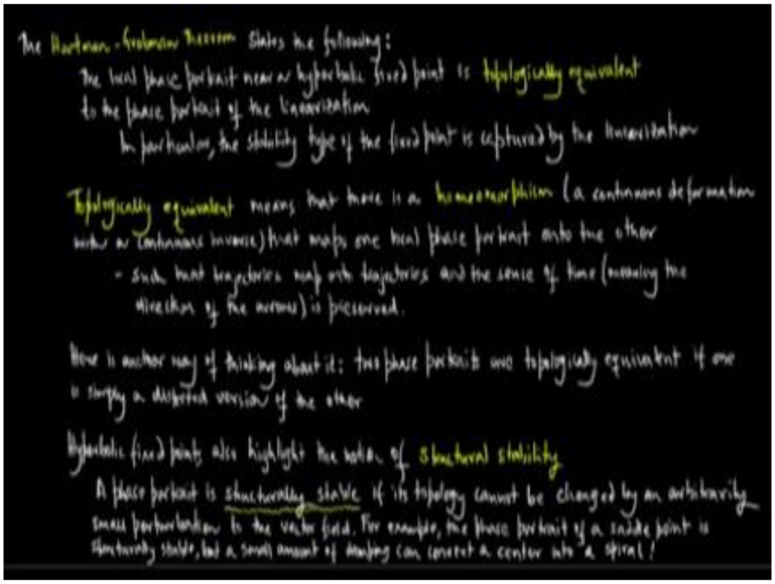

The Hartman Grobmen theorem states the following; the local phase portrait near a hyperbolic fixed point is topologically equivalent to the face portrait of the linearization. In particular, the stability type of the fixed points is captured by the linearization. Topologically equivalent essentially means that there a homeomorphism which is a continuous deformation with a continuous inverse that maps one local phase portrait on to the other.

Such that trajectories map on to trajectories and the sense of time meaning the direction of the arrows is actually preserved. Here is another way of thinking about it, two phase portraits are topologically equivalent. If one is simply a distorted version of the other, hyperbolic fixed points also highlight the notion of the structural stability. A phase portrait is structurally stable if its topology cannot be changed by an arbitrarily small perturbation to the vector field.

For example, the phase portrait of a saddle point is structurally stable but a small amount of damping can actually convert a center into a spiral.

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Now in this lecture, we dealt with a very important topic called linearization. So you can start with a two dimensional flow of the form x dot = f of xy and y dot = gs of xy, where this nonlinear system as a equilibrium point denoted as x* and y*, so then what one can do is the following; We introduced small disturbances u and v around the equilibrium point. We introduced this into the original nonlinear system and we take a Taylor series expansion around the equilibrium.

In the Taylor series expansion, we only retain the linear terms. The quadratic and whole higher order terms are discarded. So, the resulting equation would be linear and so essentially we have a linearized equation associated with the original nonlinear system around the equilibrium x* y*. So fundamental question that you really want to know as the following; to what extent does the linearized version give a qualitatively correct picture of the phase portrait around the equilibrium.

So to some extent how much can we trust anything that we get out of this linearized and the answer is that the if the linearized version predicts a saddle, a node or a spiral. Then the fixed point really is a saddle, a node or a spiral for the original nonlinear system. So, in the sense this technique of linearization can be very, very powerful to get qualitative aspect about the phase portrait of the original nonlinear system. And we can trust this as long as we have a saddle, a node or a spiral.