Introduction to Nonlinear Dynamics
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Module -06
Lecture-19
2-Dimensional Flows, Linear Systems, Lecture 3

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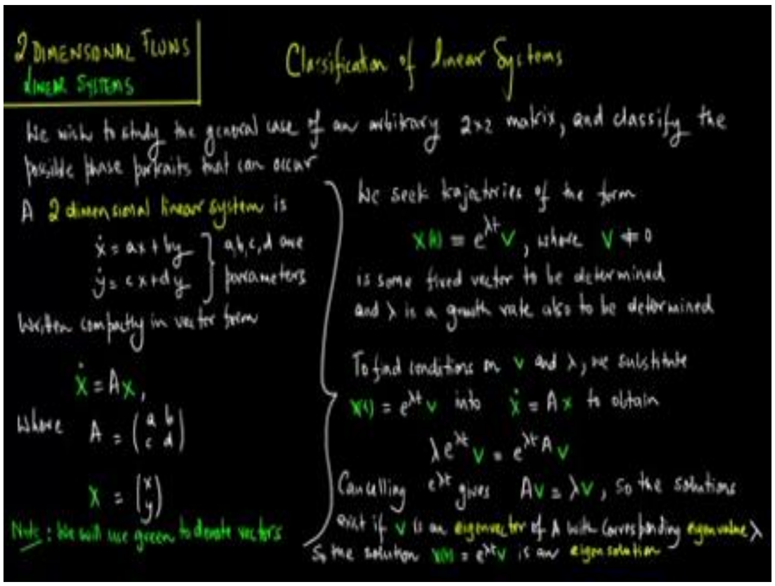

In this lecture we deal with classification of linear systems. We wish to study the general case of an arbitrary 2 by 2 matrix and classify the possible phase portraits that can occur. A two dimensional linear system is x dot = ax + by and y dot = cx + dy where a, b, c, d are parameters. So written compactly in vector form, we get x dot = a times x, where A is a b c d and x is x and y. Note that we will be using green to denote vectors.

So we seek trajectories of the form x of t =e to the lambda t times v, where v is not equal to zero. Is some fixed vector to be determined and lambda is a growth rate, which is also to be determined. To find conditions on v and lambda, we substitute x of t = e to the lambda t times v into x dot = A times x to obtain lambda e to the lambda t times v = e to the lambda t times A times v.

So cancelling e to the lambda t gives A times v = lambda times v. So the solutions exists, if v is an Eigen vector of A with corresponding Eigen value lambda. So the solution x of t = e to the lambda t times v is an Eigen solution.

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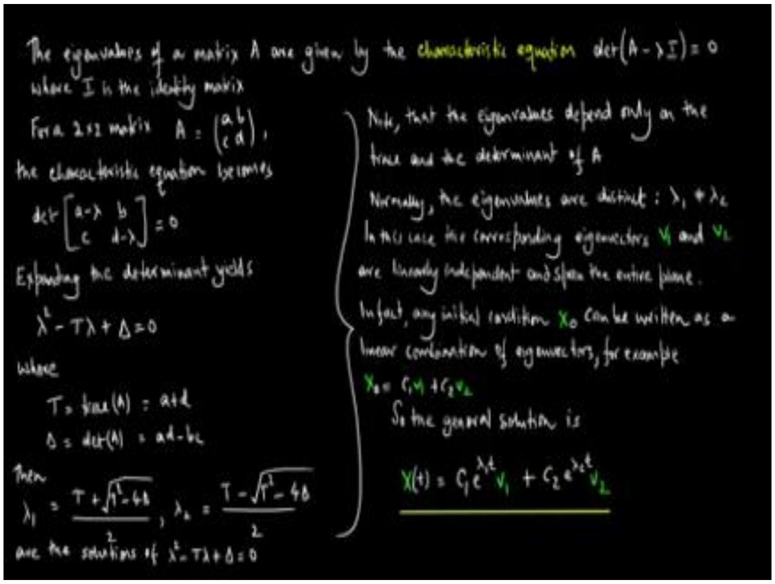

The Eigen values of a matrix A are given by the characteristic equation determinant of A - lambda I = 0, where I is the identity matrix. So for a 2 by 2 matrix Awith entries a b c and d, the associated characteristic equation becomes the determinant of a - lambda b c d - lambda =0. Expanding the determinants yields lambda squared - tau lambda + delta =0. Where tau is the trace of A which is equal to a + d and delta is the determinants of A = ad – bc.

Then lambda one = tau + tau squared - 4 delta square root / 2 and lambda two is tau - tau squared - 4 delta square root / 2. Then lambda 1 and lambda 2 are the solutions of lambda squared - tau lambda + delta =0. Note that the Eigen values depend only on the trace and the determinant of A, normally the Eigen values are distinct, so lambda 1 is not equal to lambda 2. In this case the corresponding Eigen vectors v1 and v2 are linearly independent and span the entire plane.

In fact any initial condition x not can be written as a linear combination of Eigen vectors. For example x of not = c1 v1 + c2 v2, so the general solution is x of t = c1 e to the lambda1 t times v1 + c2 e to the lambda2 t times v2.

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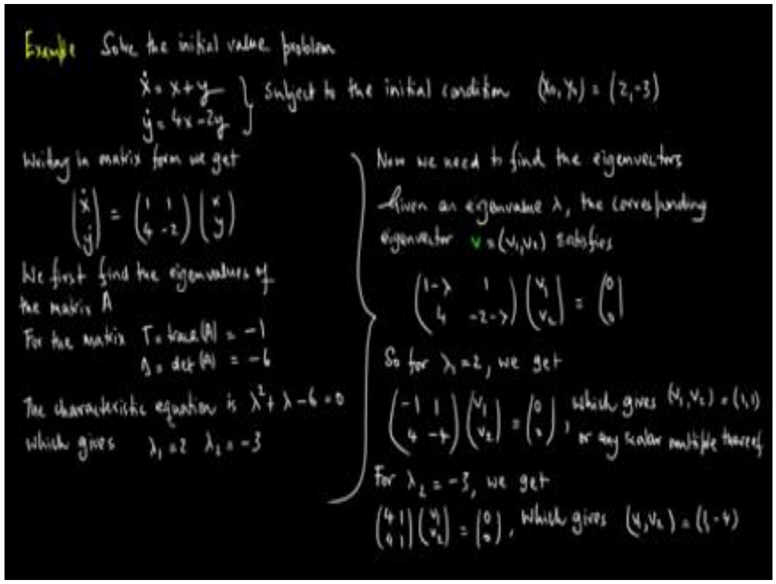

So let us consider an example, solve the initial value problem x dot =x + y and y dot =4x -2y. Subject to the initial condition x dot y dot =2-3. So writing in matrix form we get x dot y dot = 1 1 4 -1 times xy. So we first find the Eigen values of the matrix A, for the matrix the trace of A = -1 and the determinant of A =-6.

The characteristic equation is lambda squared + lambda - 6 =0, which gives lambda1 =2, lambda2 =-3, as solutions. Now we need to find the Eigen vectors, so given an Eigen value lambda the corresponding Eigen vector v, suppose to be v1 and v2 satisfies 1- lambda 1 4 -2 - lambda times v1v2 =0 0.

So for lambda1 = 2, we get -1 1 4 -4 times v1 v2 = 0 which gives v1 v2 =1 1 or in fact any scalar multiple there off. For lambda2 = -3, we get 4 1 4 1times v1 v2 =0, which gives us v1 v2 =1 – 4.

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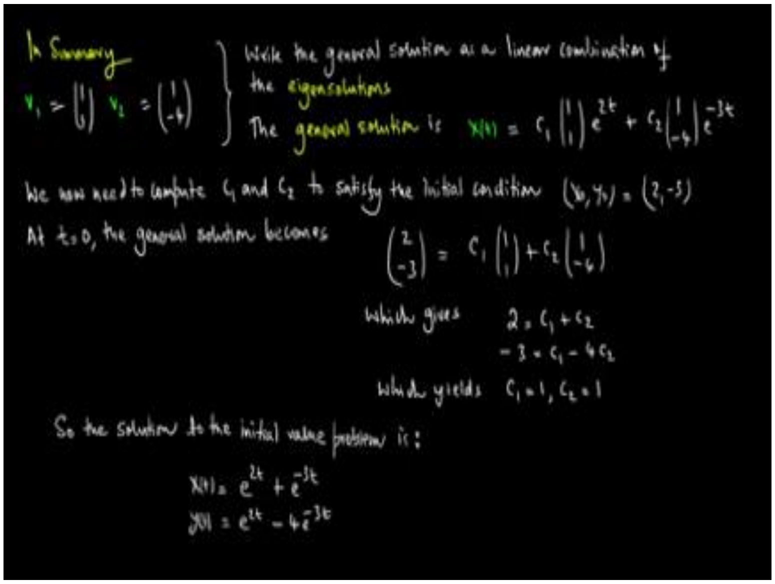

In summery v1 = 1 and 1 and v2 =1 -4. So we can write the general solution as a linear combination of the Eigen solutions. The general solution is x of t = c1 times 1 1 e to the 2t + c2 times 1 -4 e to the -3t. We now also need to compute c1 and c2 to satisfy the initial condition x not y not = 2 -3. At t = 0, the general solution becomes 2 -3 = c1 times 1 1 + c2 times 1 -4. Which ends up giving 2 -3= c1 + c2 -3 = c1 -4 c2, which yields c1 is 1 and c2 is 1. So finally we have the solution to the original initial value problem, which is x of t = e to the 2t + e to the -3t, y of t = e to the 2t -4 e to the -3t.

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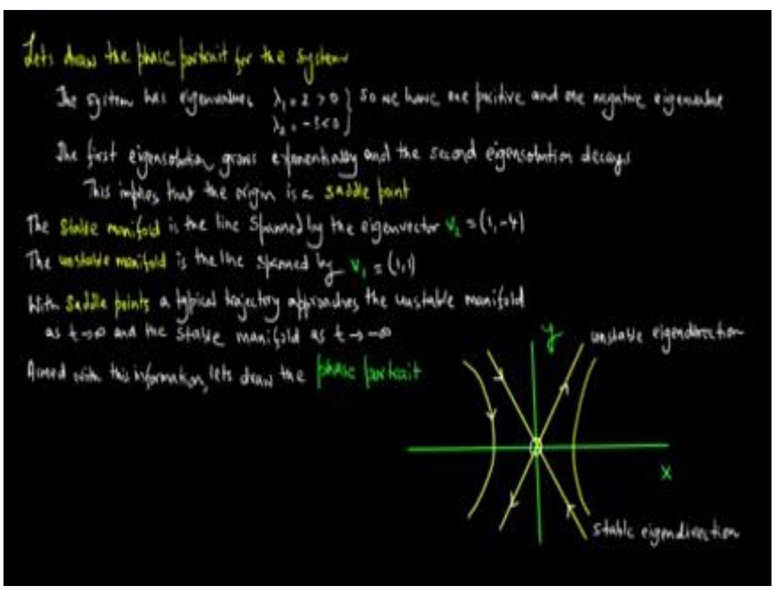

Now let us draw the phase portrait for the system. The system has two Eigen values 2 and -3, so we have one positive and one negative Eigen value. The first Eigen solution grows exponentially and second Eigen solution actually decays. So this implies that the origin is a saddle point. The stable manifold is the line spanned by the Eigen vector v2 = 1 -4 and the unstable manifold is the line spanned by the Eigen vector v1 = 1 1.

With saddle points a typical trajectory approaches the unstable manifold as t tend to infinity and the stable manifold as t tends to minus infinity. So now armed with all of this information, let us draw the phase portrait. So we plot y versus x, we first highlight the stable Eigen direction and then highlight the unstable Eigen direction. And we now go ahead and complete the phase portrait and that is the completed phase portrait.

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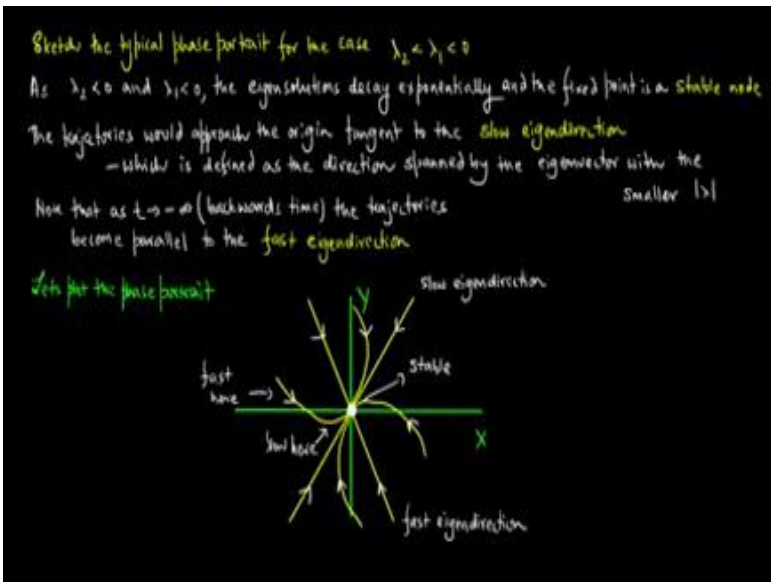

Now let us sketch the typically phase portrait for the case lambda2 less than lambda1 is less than zero. As lambda2 is less than zero and lambda1 is less than zero. The Eigen solutions decay exponentially and the fixed point is a stable node. The trajectories would approach the origin, tangent to the slow Eigen direction, which is defined as the direction spanned by the Eigen vector with the smaller absolute value of lambda.

Note that as t tends to minus infinity that is going backwards in time the trajectories become parallel to the fast Eigen direction. So now let us plot the typical phase portrait. So we plot y versus x we highlight the slow Eigen direction. Then we highlight the faster Eigen direction that is the stable node, that is were it could be fast, that is were it would be slow. And now we go ahead and complete the phase portrait.

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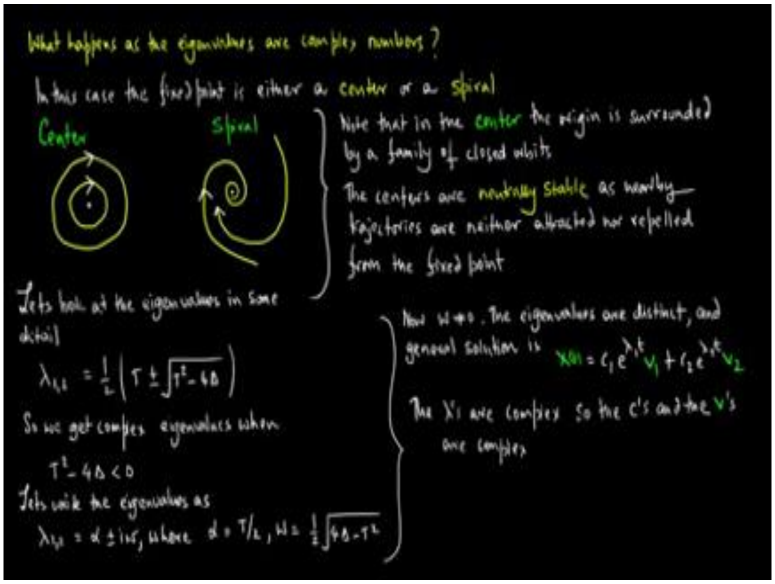

Now what happens in the case if the Eagan values are complex numbers? In this case the fixed point are either a center or a spiral. Now we go ahead and plot the center, so that is what typical centers would look like. Now we plot the spiral, essentially the trajectories spiral inwards. Note that in the center, the origin is surrounded by family of closed orbits. The centers are neutrally stable as nearby trajectories are neither attracted nor repelled from the fixed point.

Now let us look at the Eigen values in some detail. So lambda1 2 = 1/2 trace plus or minus trace squared - 4delta square root. So we get complex Eigen values when trace squared -4delta is less than zero. So let us write the Eigen values as lambda1 2 = alpha plus minus i omega, where alpha is equal to trace by 2 and omega = 1/2 of square root 4delta - trace squared.

Now omega is not equal to zero, so the Eigen value are distinct and the general solution is x of t = c1 e to the lambda1 t times v1 + c2 e to the lambda2 t times v2. Now note that the lambdas are complex and so C's and the V's are also complex.

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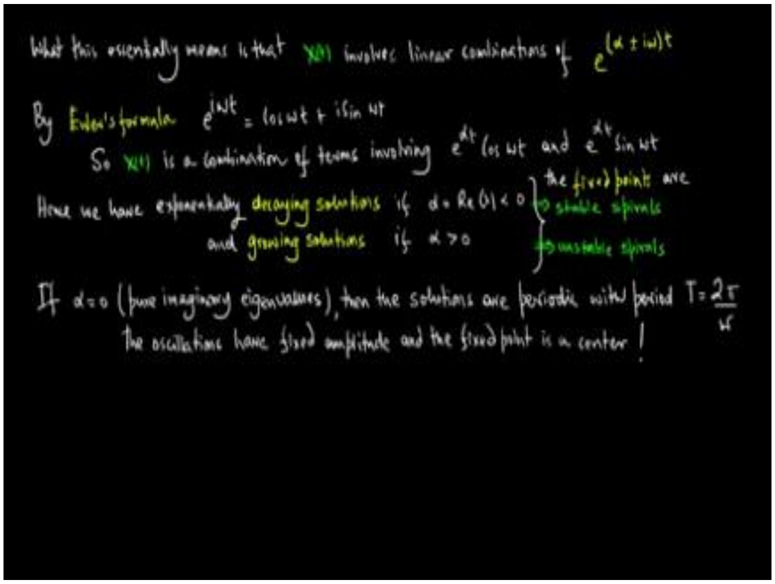

Now what this essentially means is that x of t involves linear combinations of e to the alpha + i omega times t. By Euler's formula e to the i omega t + omega t + i sine omega t, so x of t is a combination of terms involving e to the alpha t cos omega t and e to the alpha t sine omega t. Hence we have exponentially decaying solutions, if alpha which is the real part of lambda is less than zero and growing solutions, if alpha is greater than zero.

So the fixed points are stable spirals, if alpha is less than zero and unstable spirals, if alpha is greater than zero. If alpha = 0, ie we have pure imaginary Eigen values. Then the solutions are periodic with period capital T = 2pi by omega. The oscillations have fixed amplitude and the fixed point is the center.

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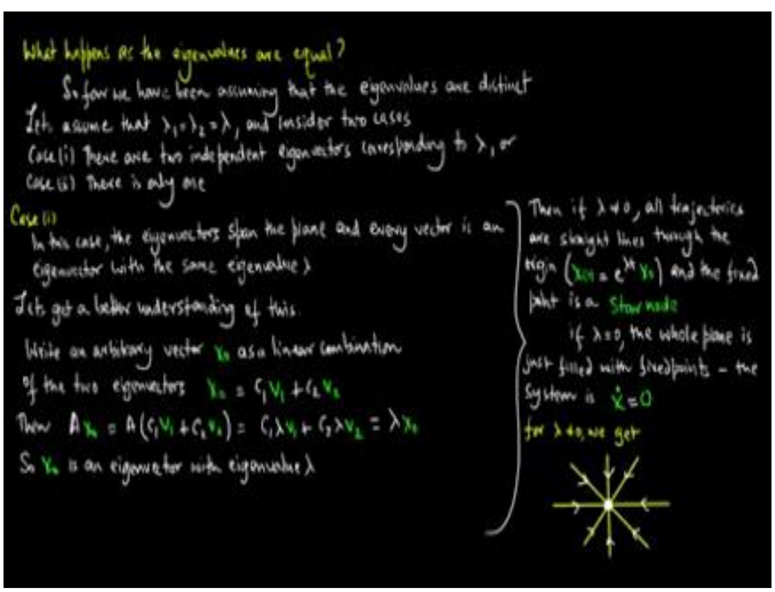

Now what happens as the Eigen values are equal, so far we have been assuming that the Eigen values are distinct? So let us assume that lambda1 = lambda2 = lambda and consider two cases. Case1, there are two independent Eigen vectors corresponding to lambda or case2 there is only one. So now let us consider case1 in some detail. In this case the Eigen vectors spanned the plane and every vector is an Eigen vector with the same Eigen value lambda.

So let us get a better understanding of this, write an arbitrary vector x not as a linear combination of the two Eigen vectors. So x not = c1 v1 + c2 v2, then A times x not = A times = c1 v1 + c2 v2 = c1 lambda v1 + c2 lambda v2 = lambda times x not. And so x not is an Eigen vector with Eigen value lambda. Then if lambda is not equal to zero, all trajectories are straight lines through origin x of t = e to the lambda t times x not and the fixed point is a star mode.

If lambda = 0, the whole plane is simply filled with fixed points. The system is x dot = 0, for lambda is not equal to zero. We get the following phase portrait, where we essentially have straight lines that are following through the origin and point is the stack mode.

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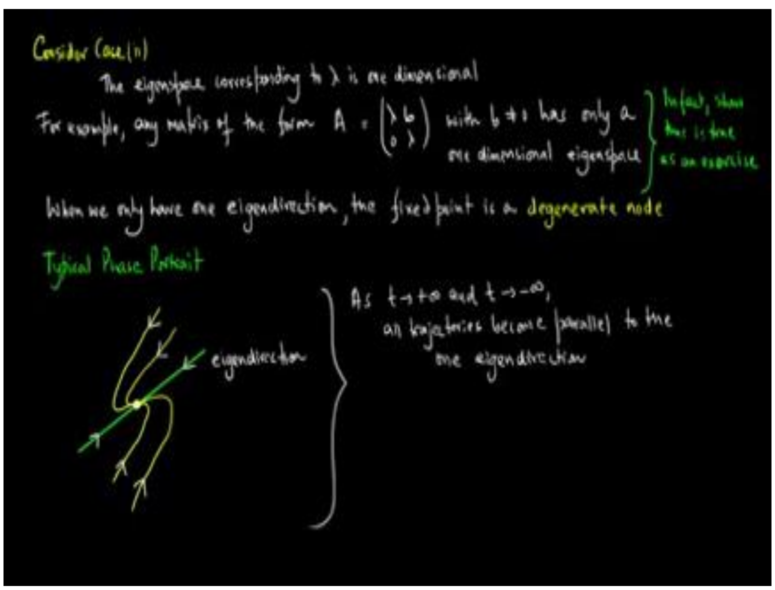

Now and let us consider case2 the Eigen space corresponding to lambda is one dimensional. So for example any matrix of the form A = lambda b 0 lambda, with b not equal to zero has only a one dimensional Eigen space. In fact we request that, to show this is true as an exercise. When we have only one Eigen direction the fixed point is a degenerate mode.

So now lets us plot the typical phase portrait that is the only Eigen direction that completes the typical phase portrait as t tends to positive infinity and t tends to negative infinity all the trajectories become parallel to the one Eigen direction.

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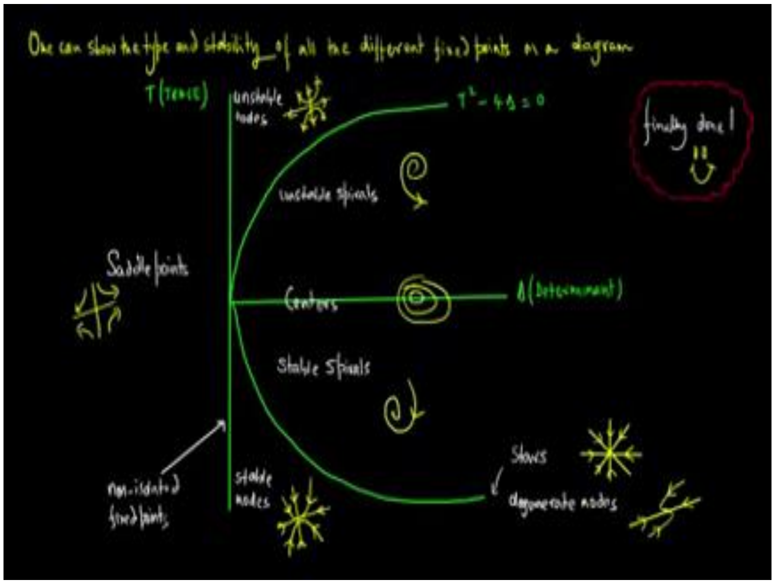

One can now show the type and stability of all the different fixed points on a single diagram. So we plot the trace versus the determinant delta and we also plot the curve t squared -4delta =0. So that is were the saddle points live, that is were the unstable nodes are, that is region for unstable spirals, centers, stable spirals, stable nodes and along with curve you will have stars and degenerate nodes. And finally this is the point for non isolated fixed points.

So now we will plot the simple caricatures, for the typical phase portraits. We have unstable spiral, a center, stable spiral, stable nodes, stars and finally degenerate nodes. So we are finally, finally done that is hard work I think we deserves smile face at the end of it all.

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The intention of this lecture was to look at the classification of two dimensional linear systems in terms of the qualitatively different phase portraits that could have come from such systems. So when you are looking at a two dimensional system you have equations of the form x dot = ax + by and y dot = cx +dy where a, b, c and d are parameters. And in the last question of the form, what happens when you have negative Eigen values that they are distinct.

What happens when you have complex Eigen values? What happens when the Eigen values are in fact the same? So essentially what we did in the lecture was we systematically went through all the numerous cases and then eventually put it all together in a plot, while you plotted the trace versus determinant, highlighting qualitatively different phase portraits that could occur depending on where you are in the trace versus determinant plot.