Introduction to Nonlinear Dynamics
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Module -06
Lecture-17
2-Dimensional Flows, Linear Systems, Lecture 1

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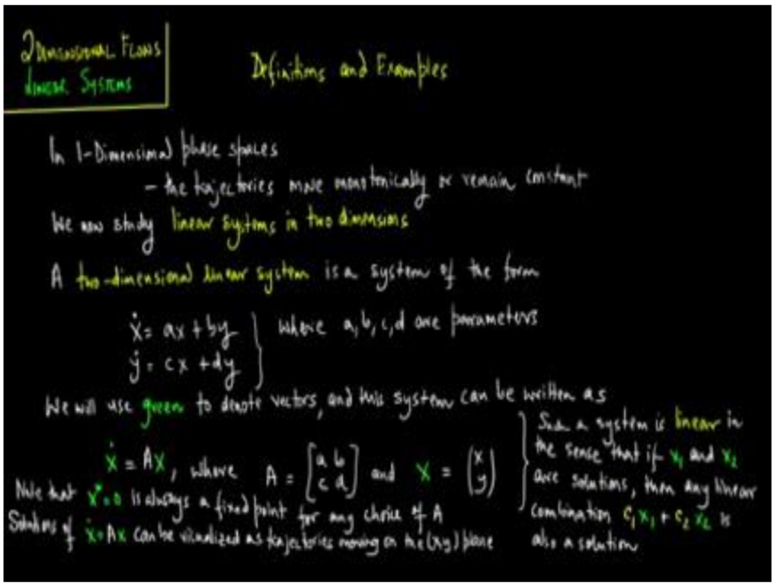

Now we move on to study of two dimensional flows and we start the study with linear systems. This lecture is centred around some definitions and some examples. In one dimensional phase spaces the trajectories move monotonically or they remain constant. We now study the linear systems in two dimensions. A two-dimensional linear system is a system of the form x dot = ax + by and y dot = cx + dy where a, b, c, d are all parameters.

We will use green to actually denote vectors and this system can be written as x dot = ax, where a is a, b, c, d and x is composed of x and y. Now such a system is linear in the sense that if x1 and x2 are solutions. Then any linear combination c1 x1 + c2 x2 is also a solution. Note that x star = 0 is always a fixed point for any choice of capital A. So solutions of x dot = ax, can in fact be visualised as trajectories moving on the xy plane, which is called the phase plane.

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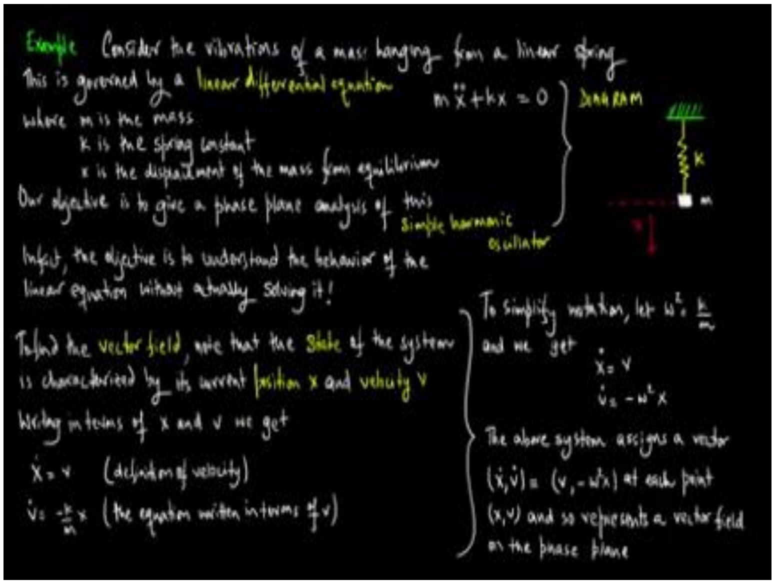

Let us consider an example, let us consider the vibrations of a mass hanging from a linear spring. This is governed by a linear differential equation mx double dot + kx = 0, where m is the mass, k is the spring constant and x is the displacement of the mass from equilibrium. Now you construct a simple diagram to show this. So, we have a spring, at the bottom of the spring there is a mass m and x is the displacement of the mass from equilibrium. Our objective is to give a phase plane analysis of this simple harmonic oscillator.

In fact, the objective is to understand the behaviour of the linear equation without actually solving it. To find the vector field, note that the state of the system is characterised by its current positions x and velocity v. So, writing in terms of x and v, we get x dot = v, this is just from the definition of the velocity v dot = - k on m x. Then the equation is written in terms of v. To simplify notation, we let omega square = k on m and we get x dot = v and v dot = - omega squared x.

The above system assigns a vector x dot v dot to v - omega squared x at each point xv and so represents a vector field on the phase plane.

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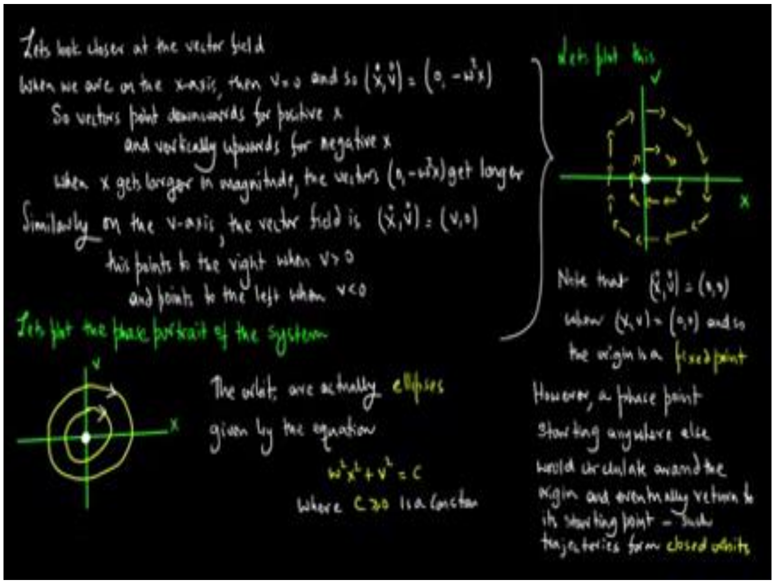

Now let us look closer at the vector field, when we are on the x axis, then v = 0 and so x dot v dot = 0 - omega squared x. So, vectors point downwards for positive x and vertically upwards for negative x. When x gets larger in magnitude, the vectors 0 - omega squared x, get longer and similarly on the v axis the vector field is x dot v dot = v0 and this points to the right when v is greater than zero and points to the left when v is less than zero.

So, let us go ahead and actually plot this information; so, we have a plot v versus x. Note that x dot v dot = 0 0, when xv = 0 0. And so, the origin is a fixed point, however a phase point starting anywhere else would actually circulate around the origin and eventually written to its starting point, so such trajectories form closed orbits. So, let us plot the phase portrait of this system, that is the plot of v versus x, so we have one closed orbit and that is another closed orbit and so on.

The orbits are actually ellipses given by the equation omega squared x squared + v squared = c, where c greater than or equal to zero, is a constant.

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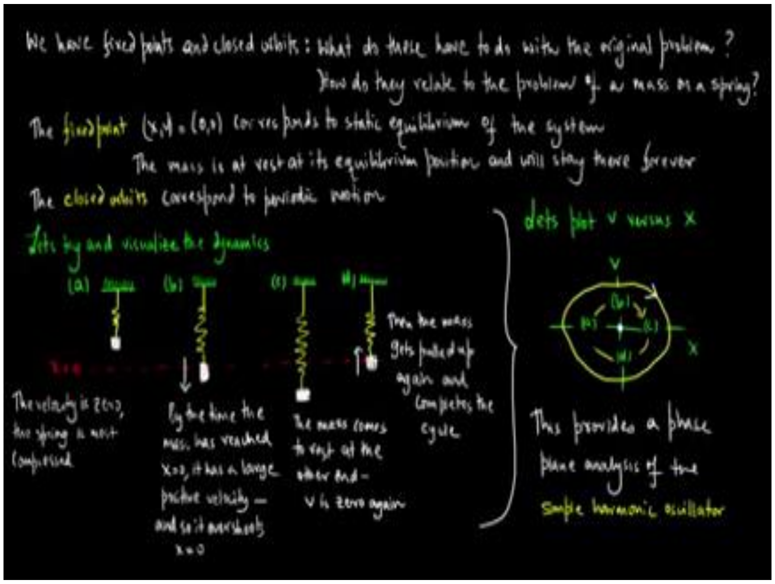

Now we have fixed points and closed orbits, so the question is what do these have to do with the original problem or how do they actually relate to the problem of a mass on a spring. The fixed point xv =00 corresponds to static equilibrium of the system; the mass is at rest at its equilibrium position and will stay there forever. The closed orbits correspond to periodic motion. Let us try and visualise the dynamics. Here the velocity is zero, so the spring is most compressed. By the time mass has reached x =0.

It has a large positive velocity and so it overshoot x =0. The mass now comes to rest at the other end where v is zero again. Then the mass gets pulled up again and completes the cycle. Now let us plot v versus x. We have the four different positions a, b, c and d and we get a closed orbit, so this provides a phase plane analysis of the simple harmonic oscillator.

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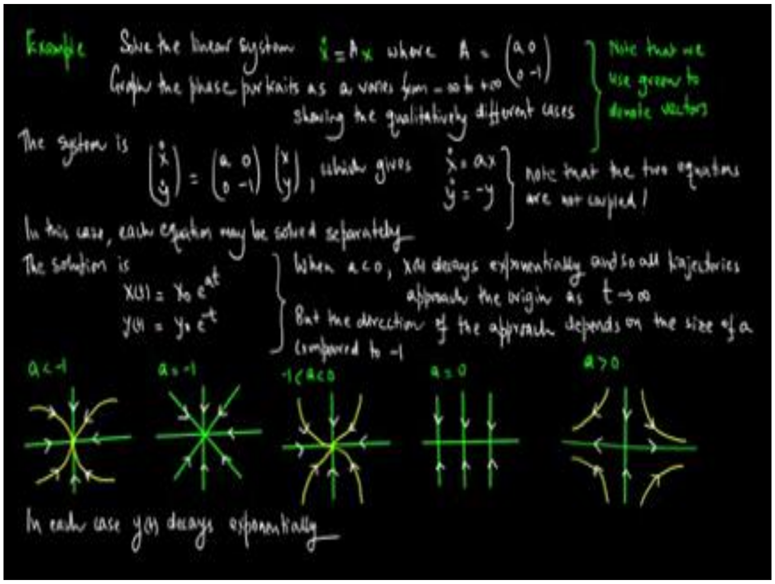

We consider another example, solve the linear system x dot = ax, where A is a 0 0 -1. And graph the phase portrait as a varies from minus infinity to plus infinity, showing the qualitatively different cases. Note that we use green to denote vectors. The system is x dot y dot = a 0 0 -1 times xy which gives x dot = ax and y dot= -y. Note that the two equations are not coupled. In this case, each equation can actually be solved separately.

The solutions is x of t = x not e to the a t, y of t = y not e to the -t. So, a less than zero, x of t decays exponentially and so all trajectories approach the origin as t tends to infinity. But the direction of the approach depends on the size of a compared to -1. So, you consider numerous cases and begin with a less than -1. And then plot a = -1 and then consider a less than 0 and greater than -1 and consider the case a = 0 and finally the case a is greater than zero, in each case y of t actually decays exponentially.

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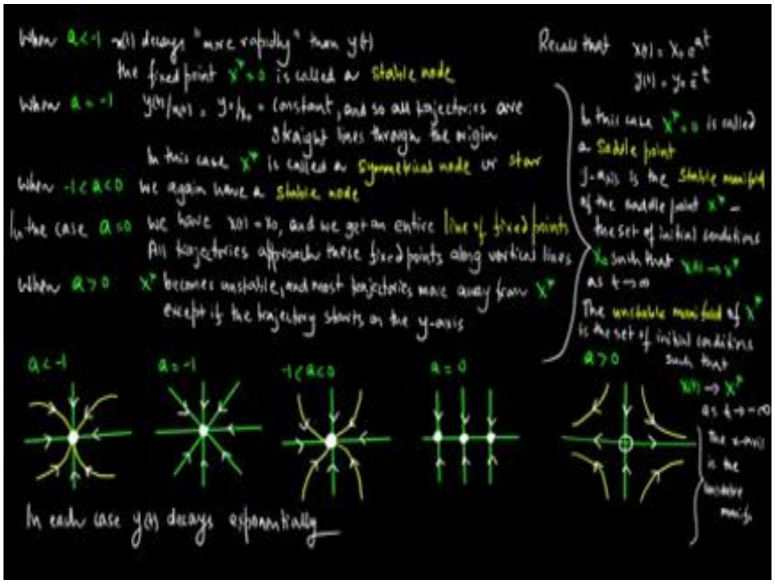

When a is less than -1, x of t decays more rapidly than y of t, recall that x of t = x not e to the a t y of t = y not e to the -t. The fixed point x star =0 is called a stable node, when a = -1 y to t / x of t =y not / x not = constant and so all the trajectories are straight lines through the origin. In this case x star is called a symmetrical node or star, when a is less than 0 and greater than -1, we again have a stable node, in the case of a = 0, we have x of t = x not and we get an entire line of fixed points.

All trajectories approach these fixed points along vertical lines. And when a is greater than zero, x star becomes unstable and most trajectories move away from x star except, if the trajectory starts on the y axis. In this case x star = 0 is called a saddle point, y axis is the stable manifold of the saddle point x star, which is a set of initial condition x not, such that x of t tends to x star as t tends to infinity. The unstable manifold of x star is the set of initial conditions such that x of t tends to x star as t tends minus infinity, so the x axis is the unstable manifold.

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Now in this lecture, we started with two dimensional flows and in particular our initial focus is going to be on linear systems. So, in one dimensional flows, we are dealing with equations of the form x dot = f of x, where f could be nonlinear. But when we are starting with linear systems with two dimensional flows, we are dealing with equations of form x dot = ax + by and y dot = cx + dy where a b c and d are parameters.

To motivate an example, you could actually look at the vibrations of a mass hanging from a linear spring and that is an example of a linear two-dimensional system. Now in that particular system hope we found is we would either have a fixed point or we could have a closed orbits that represented periodic motion. So, a feel that initial comes up, when you going for one dimensional flows to two dimensional flows and in particular, even with linear systems, we find that, we can have periodic motion that comes up very naturally in two dimensional systems.